Statistical Investigation about the SDRE Optimality for Satellites with Nonlinearities

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Abstract: - The Attitude Control System (ACS) of a Satellite can be designed with success by linear control theory if the satellite has slow angular motions. However, for fast maneuvers, the linearized models are not able to represent the effects of the nonlinear terms. One candidate technique to design the ACS of the satellite under fast maneuvers is the State-Dependent Riccati Equation (SDRE), which provides an effective algorithm for synthesizing nonlinear feedback control taking into account the nonlinearities. Nonetheless, much criticism has been leveled against the SDRE method since it does not assure global asymptotic stability (GAS), because there are situations in which global asymptotic stability cannot be achieved (e.g., systems with multiple equilibrium points). One way to study the GAS is by estimating the region of attraction (ROA) which in turn is fundamental to investigate the performance of the controller designed by the SDRE method The Brazilian National Institute for Space Research (INPE) is responsible to build a Remote-Sensing Satellites, called Amazonia-1. The ACS of the Amazonia-1 satellite must be stabilized in three axes so that the optical payload can point to the desired target. In this paper, one investigates the performance of the Amazonia-1 ACS using a statistical test (unpaired t-test) that compares the optimality inside the ROA controlled by LQR and SDRE. By several simulations running a full Monte Carlo perturbation satellite model one observes a significant difference between the optimality of the two controllers in favor of SDRE when nonlinearities are accounted for.

Key-Words: - Nonlinear, control, SDRE, LQR, Region of Attraction, t-test

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1 Introduction

The design of the Attitude Control System (ACS) of a satellite that involves plant uncertainties, large-angle manoeuvres, and fast attitude control following a stringent pointing, requires nonlinear control techniques in order to satisfy performance and robustness requirements. An example is the Amazonia-1 satellite mission built by the Brazilian National Institute for Space Research (INPE), in which the ACS must stabilize the satellite in three axes so that the optical payload can point to the desired target with few arcsecs of pointing accuracy. One candidate technique to deal with nonlinear dynamics and design ACS control law is the State-Dependent Riccati Equation (SDRE), which is based on the arrangement of the system model in a form known as State-Dependent Coefficient (SDC) matrices, [1]. Accordingly, a suboptimal control law is carried out by a real-time solution of an Algebraic Riccati Equation (ARE) using the SDC matrices by means of a numerical algorithm. SDRE was proposed by [1], and then explored in detail by [2]. A good survey of the SDRE technique can be found in [3], and its systematic application to deal with a nonlinear plant in [4]. The SDRE technique was applied by [5], [6], for controlling a nonlinear six-degree-of-freedom satellite model.

The SDRE method was applied considering a locally asymptotically stable equilibrium point, [5], [6]. On the other hand, the knowledge of the ROA is fundamental mainly when the local stability considers the presence of nonlinearities. Indeed, much criticism has been leveled against the SDRE technique since it does not provide assurance of global asymptotic stability. However, empirical experience shows that in many cases the ROA may be as large as the domain of interest, [3]. Moreover, there are situations in which global asymptotic stability cannot be achieved (for example, systems with multiple equilibrium points). Therefore, estimating the ROA is an alternative to investigating the performance of the satellite's ACS. Obtaining a good estimate of such a ROA, especially of a higher-order system, is a challenging task in itself. Analytical ROA for nonlinear systems with dimensions greater than two is usually unavailable, [6]. Such a task becomes even more difficult since the closed-loop matrix of SDRE is usually not available, [7].

In this paper, one investigates the performance of the Amazonia-1 ACS using a statistical test (unpaired t-test) that compares the optimality inside the ROA of the controller designed by LQR and SDRE techniques using a full Monte Carlo perturbation satellite model. The simulation results showed a significant difference between the optimality of the two controllers in favor of SDRE when nonlinearities are accounted for. Recall LQR guarantees global stability for linear systems (afterward, the linearization process), nevertheless, such property is lost when nonlinearities are present.

The statistical test is focused on two mutually exclusive hypotheses:

- **H_0** - the means of a cost function for the two controllers are similar.
- **H_1** - the means of the cost function for the two controllers are not similar.

2 Problem Description

The SDRE technique entails factorization of the nonlinear dynamics into the state vector and the product of a matrix-valued function that depends on the state itself. In doing so, SDRE brings the nonlinear system to a nonunique linear structure having SDC matrices given by

$$\dot{x} = A(x)x + B(x)u$$

$$y = Cx \tag{1}$$

where $x \in R^n$ is the state vector and $u \in R^m$ is the control vector. The SDC form has the same structure as a linear system, but with matrices A and B being functions of the state vector x. The no uniqueness of the SDC matrices creates extra degrees of freedom, which can be used to enhance controller performance, however, it poses challenges since not all SDC matrices fulfill the SDRE requirements. The system model in Eq. (1) is subject to the cost function described given by

$$J(x_0, u) = \frac{1}{2}\int_0^\infty (x^T Q(x)x + u^T R(x)u)dt \tag{2}$$

where $Q(x) \in R^{n\times n}$ and $R(x) \in R^{m\times m}$ are the state-dependent weighting matrices. To ensure local stability, $Q(x)$ is required to be positive semi-definite and $R(x)$ to be positive definite, [6].

The SDRE controller linearizes the plant about the current operating point and creates constant state space matrices so that the LQR can be used. This process is repeated in all samplings steps, resulting in a linear model from a non-linear model, so that an ARE is solved and a proportional gain is computed in each step. Therefore, according to LQR theory, [4], applied to Eq. (1) and Eq. (2), one obtains the state-feedback control law in each sampling step given by $u = -K(x)x$, where the state-dependent gain $K(x)$ is

$$K(x) = R^{-1}(x)B^T(x)P(x) \tag{3}$$

and P(x) is the unique, symmetric, positive definite solution of the ARE given by

$$P(x)A(x) + A^T(x)P(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) + Q(x) = 0 \tag{4}$$

We are interested in the ROA, such that the region of the state space in which the initial conditions of the trajectories lie to attain stable behavior, [5], [6], [8]. An equilibrium point, if exists, must lie in the ROA. A nonlinear system can have an infinite number of equilibrium points, and each one can be stable or unstable. Such equilibrium points, if exist, lie in the ROA, which is defined by its attractor. One particular type of attractor is the fixed point, i.e., stable equilibrium point. A fixed-point attractor, [9], can be defined by

$$A = \{x_0 \in \mathcal{R}^n : \lim_{t\to+\infty} x(t, x_0) = 0\} \tag{5}$$

3 Satellite Model

Considering that Amazonia-1, ACS must be stabilized in three axes so that the optical payload can point to the desired target. In the sequence, it presents the kinematics and the rotational dynamics of the satellite attitude studied.

Given the inertial reference frame (F_i) and the body reference frame with origin at the satellite center of mass (F_b), then a rotation matrix, [5], represented by a unit quaternion $Q = [q_1\ q_2\ q_3 \mid q_4]^T$ can define the attitude of the satellite. Defining the satellite angular velocity $\omega = [\omega_1\ \omega_2\ \omega_3]^T$, the kinematics can be modeled using the Gibbs vector $Q = [g \mid q_4]$ given by

$$\dot{Q} = -\frac{1}{2}\begin{bmatrix}\omega^{\times} \\ \omega^{T}\end{bmatrix}\begin{bmatrix}q_1 \\ q_2 \\ q_3\end{bmatrix} + \frac{1}{2}q_4\begin{bmatrix}1_{3\times 3} \\ 0\end{bmatrix}\omega \tag{6}$$

where $\omega^{\times}$ is the skew-symmetric matrix and 1 is the identity matrix.

The satellite has a set of three reaction wheels, each one aligned with its principal axes of inertia. As a result, the angular momentum of the satellite is defined by

$$\vec{h} = (\boldsymbol{I} - \sum_{n=1}^{3} I_{n,s} a_n a_n^T)\vec{\omega} + \sum_{n=1}^{3} h_{w,n}\vec{a_n} \tag{7}$$

where I is the inertia tensor, $I_{n,s}$ is the inertia moment of the n reaction wheel in its symmetry axis $\vec{a}_n$, $h_{w,n}$ is the angular momentum of the n reaction wheel about its center of mass ($h_{w,n} = I_{n,s} a^T{}_n \omega + I_{n,s}\omega_n$) and ω_n is the angular velocity of the n reaction wheel.

Deriving Eq. 7 one obtains the rotational dynamics of the satellite given by

$$I_b\dot{\omega}^b = g_{cm} - \omega^{\times}(I_b\omega + \sum_{n=1}^{3} h_{w,n}a_n) - \sum_{n=1}^{3} g_n a_n \tag{8}$$

where I_b is the satellite inertia moment, g_{cm} is the net external torque and g_n are the torques generated by the reaction wheels.

4 Controller Design and ROA Region

Two satellite dynamics states must be controlled: the first is the attitude of the quaternions Q and the second is the satellite angular velocity ω. Although Eq. (6) can be used to linearize the system around a stationary point, assuming there is no net external torque. In reference, [10], it was shown that this linearized model with all quaternion components was not stabilizable, meaning that LQR is not applicable. As a result, one option is to model the state of the system without the scalar q_4, which leads to the following equation

$$\begin{bmatrix}x_3 \\ x_2\end{bmatrix} = \begin{bmatrix}q_1 \\ q_2 \\ q_3 \\ \omega\end{bmatrix}$$

$$\begin{bmatrix}\dot{x_3} \\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{2}I_{3x3} \\ 0 & 0\end{bmatrix}\begin{bmatrix}x_3 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ -I_b^{-1}\end{bmatrix}[u_1]$$
$$[y] = I\begin{bmatrix}x_3 \\ x_2\end{bmatrix} \tag{9}$$

Assuming that there are no external torques, the state space satellite model can be defined using Eq. (6), which leads to the following equation

$$\begin{bmatrix}\dot{x_1} \\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\begin{bmatrix}\omega^{\times} \\ \omega^{T}\end{bmatrix} & 0 & \begin{bmatrix}\frac{1}{2}q_4 I_{3\times 3} \\ 0\end{bmatrix} \\ 0 & 0 & -I_b^{-1}\omega^{\times}I_b + I_b^{-1}(\sum_{n=1}^{3} h_{w,n}a_n)^{\times}\end{bmatrix}\begin{bmatrix}x_0 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ -I_b^{-1}\end{bmatrix}[u_1]$$
$$[y] = I\begin{bmatrix}x_0 \\ x_2\end{bmatrix} \tag{10}$$

It was shown that Eq. (10) satisfies the SDRE conditions, [5], [6], except for the region on which the angular velocity is close to zero and, consequently, the *pair (A, B)* loses rank in such a region. In practical problems, regarding such a region, one approach is to switch to another SDC parametrization.

The ROA definition is associated with the fixed-point attractor presented in Eq. (5) which is restricted by an explicit final time t_f and a numerical error ϵ. Besides, the ROA region is restricted by the selection of sole angular velocities x_2 in Eq. (9) and Eq. (10), according to the following equation:

$$A = \{x_0 \in \mathcal{R}^n : \lim_{t \to t_f} ||x_2(t, x_0)||_2 < \epsilon\} \tag{11}$$

5 Statistical Test Procedure

The simulation results were computed using the nonlinear Amazonia-1 satellite model and the control laws of the LQR and SDRE considering the nonlinearities in the reaction wheels, i.e., maximum torque and maximum angular velocity. As for the initial angular velocities, they are defined by independent uniform distributions based on the maximum angular velocity of the satellite that is controllable by the reaction wheels. The maximum angular momentum of the set of reaction wheels was computed and then the corresponding maximum angular velocity of the satellite was found.

The cost functional defined by Eq. (2) was applied in the statistical test for the different ROAs according to the following equation

$$J_m(x_0, u) = \frac{1}{2}\int_0^{t_f} x_2^T Q x_2 \, dt + \frac{1}{2}\int_0^{t_f} u^T R u \, dt \tag{12}$$

In summary, the approach for optimality evaluation for the different ROAs can be summarized as:

a) Compute attitude initial conditions for the Monte Carlo perturbation model considering the independent uniform distributions of the Euler angles. Subsequently, compute the angular velocities based on the maximum angular velocity of the satellite that is controllable by the reaction wheels.

b) Perform the time-domain simulation until the predefined t_f.

c) Compute ROAs for each controller, i.e., the initial conditions that satisfied Eq. (11).

d) Calculate and collect the J_m, Eq. (12), for those simulations inside ROA for each controller.

e) Perform the t-test between the collected measures J_m for each controller.

The simulations are based on the satellite Amazonia-1 data shown in Table 1. The parameter values used in the Monte Carlo perturbation model were $Q = R = I$, $\epsilon = 0.0001 rad/s$, samples 200 for each controller, $t_f = 3600s$, and fixed step size = $0.05s$.

The simulation results are shown in Fig. 1 where each component of J_m for LQR (in blue, labeled as Proportional Linear Quaternions Partial LQR Controller) and for SDRE (in red, labeled as Proportional Non-Linear Quaternion SDRE Gibbs Controller) are plotted. One observes that the number of samples that converged, Eq. (11), was less than the initial conditions, therefore, the global asymptotic stability property was not present in LQR nor SDRE.

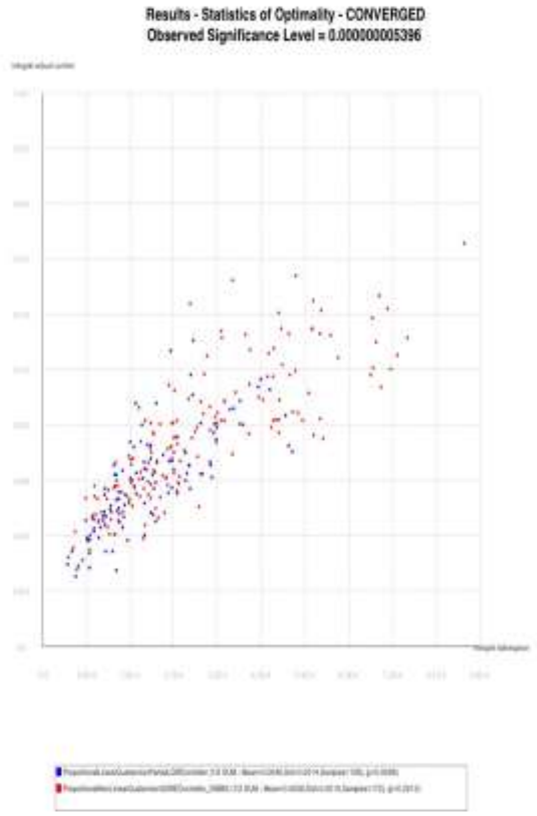

Fig. 1. Optimality with nonlinearities

Table 1. Satellite characteristics and references

Name	Value
Satellite Characteristics	
Inertia tensor (kg.m^2), I	$\begin{bmatrix} 310.0 & 1.11 & 1.01 \\ 1.11 & 360.0 & -0.35 \\ 1.01 & -0.35 & 530.7 \end{bmatrix}$
Actuators Characteristics - Reaction Wheels	
Inertia tensor (kg.m^2)	diag(0.01911,0.01911,0.01911)
Maximum torque (N.m)	0.075
Maximum angular velocity (RPM)	6000
References for the controller	
Solar vector in the body (XYZ)	$[1\ 0\ 0]^T$
Angular velocity (radians/second)	$[0\ 0\ 0]^T$

5 Conclusions

The ROA of the SDRE was larger since more initial conditions (172 versus 108 for LQR) converged for t_f. The unpaired t-test was performed at the *5%* level of significance ($\alpha = 0.05$) and the observed significance level obtained was $p = 0.000000005396$. Therefore, as $p < \alpha$, the $\mathbf{H_0}$ (the means of J_m for the two controllers are similar) was rejected. Consequently, the $\mathbf{H_1}$ (the means of the cost function for the two controllers are not similar) was accepted.

As the results are based on analysis through simulations, they are neither valid for general cases nor scenarios out of the range of the Monte Carlo perturbation models due to the underlying nonlinear dynamics.

The statistical investigation based on an unpaired t-test provides evidence that SDRE improves optimality for Amazonia-1 when nonlinearities are accounted for. In general, the approach applied here, which is ROA's definition through simulation, provides an alternative to the evaluation of the performance of the satellite's ACS when analytical ROA is unavailable.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Both authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution. Moreover, Dr. Alessandro G Romero carried out the simulation of the ACS, and Dr. Luiz Carlos G Souza was responsible for the derivation of the dynamics of the satellite.

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Conflict of Interest

The authors have no conflicts of interest to declare.