Numerical Estimation Method for the Generalized Weibull Distribution Parameters

M. MASWADAH
Department of Mathematics, Faculty of Science,
Aswan University, Aswan,
EGYPT

Abstract: - In this study, a new estimation method using the Runge-Kutta iteration technique is presented to improve point estimation methods. The improved method has been applied to the generalized Weibull distribution, which is a member of a family of distributions (T-X family). The estimates of the generalized Weibull model parameters were derived using the Runge-Kutta and Bayesian estimation methods based on the generalized progressive hybrid censoring scheme, via a Monte Carlo simulation. The simulation results indicated that the Runge-Kutta estimation method is highly efficient and outperforms the Bayesian estimation method based on the informative and kernel priors. Finally, two real data sets were studied to ensure the Runge-Kutta estimation method can be used more effectively than the most popular estimation methods in fitting and analyzing real lifetime data.

Key-Words: - Bayesian inference, Generalized progressive hybrid censoring scheme, Informative prior, Kernel prior, Rung-Kutta method, Weibull model

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1 Introduction

The maximum likelihood estimation (MLE) method is the most popular point estimation method in statistical inference, and it is widely used in the social sciences and psychology, though it is biased in situations where sample sizes are small or data is heavily censored. These biases can mislead subsequent inferences, and it is not as effective as the Bayesian estimation method. Furthermore, in some distributions, it contains nonlinear equations that require numerical techniques. Therefore, the main objective of this study is to introduce an improvement in point estimation methods such as the MLE method by using the Runge–Kutta iteration technique. The simulation results indicated that the improved estimation method is highly efficient and outperforms the Bayesian estimation method based on the informative and kernel priors for different loss functions. Thus, the statistical significance of this method is its efficiency compared to the most popular point estimation methods in statistical inference and it is reliable and easy to apply, especially for social sciences and psychology researchers.

In the last decade, extensive efforts have been made to present new models in distribution theory and related statistical applications. Some of the new distributions were developed as generalizations or modifications of the Weibull distribution and have been extensively used for data modelling in many fields, such as engineering and medical science. For a review of some generalized Weibull distributions, one may refer to [8]. [5], created a new class of distributions known as the (T-X family), which is defined as, for given a random variable $\mathbf{X}$ with a cumulative distribution function $\mathbf{G}(\mathbf{x})$ and a generator random variable $\mathbf{T}$ defined on $[\mathbf{0}, \infty]$ with $\mathbf{h}(\mathbf{t})$ and $\mathbf{H}(\mathbf{t})$ as the probability density function "PDF" and the cumulative distribution function "CDF," respectively. As a result, the generalised (T-X family) CDF is given by:

$$\mathbf{F}(\mathbf{x}) = \mathbf{H}(-\log(\mathbf{1} - \mathbf{G}(\mathbf{x}))).$$

Some lifetime distributions are extended by this family, including the generalised Weibull, Weibull, Weibull extension, Lomax, Logistic, and Log-Logistic distributions. Several new distributions within this family, including the Weibull-Pareto distribution, have been introduced and studied in [6], the Gamma-Pareto distribution in [4], and the Gamma-Normal distribution in [7]. For a review of this family and other distributions, [3].

For deriving the generalized Weibull distribution, let $h(t)$ be the PDF of the Weibull distribution, which is defined as follows:

$$\mathbf{h(t) = \alpha\beta t^{\alpha-1} exp(-\beta t^{\alpha})}, \quad \mathbf{t \geq 0}, \quad \boldsymbol{\alpha, \beta} > 0.$$

Then, we have the Weibull (T-X family) with CDF defined as follows:

$$\mathbf{F(x) = 1 - exp[-\beta(-log(1 - G(x)))^{\alpha}]}.$$

Letting $\mathbf{G(x) = 1 - exp(-(e^{\gamma x} - 1))}$, be the

CDF of the exponential extension model. Thus, the CDF of the generalized Weibull distribution (GWD) can be derived as follows:

$$\mathbf{F(x) = 1 - exp[-\beta(e^{x\gamma} - 1)^{\alpha}]}, \quad \mathbf{x \geq 0}, \quad \boldsymbol{\alpha, \beta, \gamma} > 0. \tag{1}$$

The corresponding PDF is given as

$$f(x) = \alpha\beta\gamma(e^{x\gamma} - 1)^{\alpha-1} \exp[x\gamma - \beta(e^{x\gamma} - 1)^{\alpha}], \quad \mathbf{x \geq 0}, \boldsymbol{\alpha, \beta, \gamma} > 0 \tag{2}$$

where β and γ are scale parameters, and α is a shape parameter.

This distribution has higher skewness compared with the Weibull, inverse Weibull, and Log-Logistic distributions and therefore it is more suitable to model heavily skewed data that often arise in reliability and survival analysis, [3], [8]. The GWD has some special cases when the random variable $Y = e^{x\gamma} - 1$, we get $\mathbf{Y}$ has the two-parameter Weibull distribution with β and α are the scale and shape parameters respectively. When $\alpha = 1$, we get the exponential extension model, which is a special case of the Weibull extension model, [32]. For comparison between the two estimation methods, the generalized Weibull distribution parameters have been estimated using the Runge-Kutta (R-K) and Bayesian estimation methods based on the informative and informative kernel priors with different loss functions based on the generalized progressive hybrid censoring scheme.

In reliability analysis, the progressive Type-II censoring scheme is the most applicable in life test experiments, it is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at various stages before testing is terminated. [9], [10], presented comprehensive studies on the topic of progressive censoring and its applications. However, the trial time can be quite long due to some highly reliable units. Thus, [20], recently proposed a censoring scheme called the Type-II progressive hybrid censoring scheme.

However, the progressive hybrid censoring scheme has the disadvantage of having very few failures before time point T. To provide a guarantee of the number of failures observed as well as the time to complete the test, [15], [16], proposed the generalized progressive hybrid censoring scheme (GPHCS) that modifies the progressive hybrid censoring scheme. If the number of failures is less than a specified number of m, it allows the experiment to continue beyond time to observe at least the number of failures k. The GPHCS has been described in [24], [25].

Thus, given a generalized progressive hybrid censored sample, the likelihood function can be written in a unified form as follows:

$$L(\bar{X};\theta) = C\prod_{i=1}^{N} f(x_i)\,[1 - F(x_i)]^{R_i}[1 - F(T)]^{\delta R_T^*}, \tag{3}$$

$$N = \begin{cases} m, & \delta = 0, \quad \text{if} \quad X_{k:m:N} < X_{m:m:N} < T \\ k, & \delta = 0, \quad \text{if} \quad T < X_{K:m:N} < X_{m:m:N}, \\ J, & \delta = 1, \quad \text{if} \quad X_{k:m:N} < T < X_{m:m:N} \end{cases}$$

where $\underline{X} = (X_1, X_2, \ldots, X_N)$, N is the sample size and $R_T{}^*$ is the number of surviving units that are removed at the stopping time

$$\mathbf{T^* = max\{X_{k:m:N}, min\{X_{m:m:N}, T\}\}}.$$

The GPHCS has been applied to some distributions such as the Weibull distribution, [16], the inverse Weibull distribution, [24], [27], the exponential distribution, [15], [17], the Rayleigh distribution, [14], the shape-scale family, [25], and the generalized shape-scale family, [26].

2 Estimation Methods

2.1 Runge-Kutta Method

The MLE $\hat{\theta} = \hat{\theta}(\underline{X})$ of θ is the solution of the stationary equation, $\frac{\partial \mathbf{H}(\underline{X};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}|_{\theta=\hat{\theta}} = \mathbf{0}$, which is a function of $\underline{X}$ and $\hat{\theta}(\bar{x})$, where $H(\underline{X};\bar{\theta})$ is the log-likelihood function that depends on the unknown parameter $\hat{\theta} = (\alpha, \beta, \gamma)$ and the data $\underline{X} = (x_1, x_2, \ldots, x_N)$. Applying the implicit function theorem to the stationary equation by considering all partial derivatives, as well as the total derivatives, are assumed to be evaluated at some known value of $\hat{\theta}(x) = \theta_0$, say. Taking the total derivative for the stationary equation with respect to $x \in \underline{X}$, [28], we obtain.

$$\frac{d}{dx}\left(\frac{\partial H(\underline{X};\theta)}{\partial\theta}\right)|_{\theta=\hat{\theta}} = \left(\frac{\partial^2 H(\underline{X};\theta)}{\partial\theta\partial x}|_{\theta=\hat{\theta}}\right) + \left(\frac{\partial^2 H(\underline{X};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)\frac{d\hat{\theta}}{dx} = \mathbf{0}. \quad (4)$$

Solving (4) we obtain the first derivative with respect to x for $\hat{\theta}$ at $\theta = \hat{\theta}$ as follows:

$$\frac{d\hat{\theta}}{dx} = -\left(\frac{\partial^2 H(\underline{X};\theta)}{\partial\theta^2}|_{\theta=\hat{\theta}}\right)^{-1}\frac{\partial^2 H(\underline{X};\theta)}{\partial\theta\partial x}|_{\theta=\hat{\theta}}. \quad (5)$$

Thus, we can write (5) as the first-order ordinary differential equation in the maximum likelihood estimator $\hat{\theta}(\underline{X})$ as

$$\frac{d\hat{\theta}}{dx} = \mathbf{f}(\mathbf{x},\hat{\boldsymbol{\theta}}), \quad \hat{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}_0. \quad (6)$$

where $\mathbf{f}(\mathbf{x},\hat{\boldsymbol{\theta}})$ and $\frac{df(x,\hat{\theta})}{d\hat{\theta}}$ are defined and continuous functions at all points $(\underline{X},\hat{\theta})$, which ensures the existence of a unique solution for (6). Using any numerical technique, such as the fourth-order Runge-Kutta, we can find the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities concerning which the fit is optimized.

Thus, the R-K recurrence solution for (6) can be obtained as

$$\boldsymbol{\theta}^*_{i+1} = \boldsymbol{\theta}^*_i + \mathbf{h}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6, \quad (7)$$

for $i = 0,1,2,\ldots$,
where

$$\mathbf{K}_1 = \mathbf{f}(\mathbf{x}_i,\boldsymbol{\theta}^*_i), \quad \mathbf{K}_2 = \mathbf{f}(\mathbf{x}_i + \mathbf{h}/2, \boldsymbol{\theta}^*_i + \mathbf{K}_1/2),$$
$$\mathbf{K}_3 = \mathbf{f}(\mathbf{x}_i + \mathbf{h}/2, \boldsymbol{\theta}^*_i + \mathbf{K}_2/2),$$
$$\mathbf{K}_4 = \mathbf{f}(\mathbf{x}_i + \mathbf{h}, \boldsymbol{\theta}^*_i + \mathbf{K}_3).$$

Here h is a small known value (say, 1E-02) and $\theta = \theta_0$, is the initial value for θ.

For the generalized Weibull lifetime model (1) and (2), the log-likelihood function based on the GPHCS (3) can be derived as follows:

$$\mathbf{H}(\underline{\mathbf{x}};\boldsymbol{\theta}) = \mathbf{K}\ln(\boldsymbol{\alpha\beta\gamma}) + \boldsymbol{\gamma}\sum_{i=1}^{N}\mathbf{x}_i + (\boldsymbol{\alpha}-\mathbf{1})\sum_{i=1}^{N}\ln(\mathbf{e}^{\boldsymbol{\gamma}\mathbf{x}_i}-\mathbf{1})$$
$$-\boldsymbol{\beta}\left[\sum_{i=1}^{N}(\mathbf{R}_i+\mathbf{1})(\mathbf{e}^{\mathbf{x}_i\boldsymbol{\gamma}}-\mathbf{1})^{\boldsymbol{\alpha}} + \boldsymbol{\delta}\mathbf{R}^*(\mathbf{e}^{\mathbf{T}\boldsymbol{\gamma}}-\mathbf{1})^{\boldsymbol{\alpha}}\right], \quad (8)$$

K is the normalizing constant. The derivatives of (8) have been derived in Appendix A.

Thus, using (6) with the corresponding derivatives, we can find the point estimates for θ as $\hat{\theta} = (\hat{\alpha},\hat{\beta},\hat{\gamma})$ with the fourth-order Runge–Kutta method using the recurrence relations (7).

2.2 Bayesian Method

In this section, the Bayes estimations will be derived using gamma and kernel prior distributions based on two loss functions:

Firstly, the squared error loss function (SLF), $\mathbf{L}(\boldsymbol{\theta},\boldsymbol{\theta}^*) = (\boldsymbol{\theta}-\boldsymbol{\theta}^*)^2$. For this loss function, the Bayes estimator that minimizes the risk function is given by $\boldsymbol{\theta}^* = \mathbf{E}(\boldsymbol{\theta}|\mathbf{x})$.

Secondly, the compound LINEX loss function is defined as follows:

$$\mathbf{L}(\boldsymbol{\Delta}) = \mathbf{L}_{\boldsymbol{\delta}}(\boldsymbol{\Delta}) + \mathbf{L}_{-\boldsymbol{\delta}}(\boldsymbol{\Delta}) = \mathbf{e}^{\boldsymbol{\delta\Delta}} + \mathbf{e}^{\boldsymbol{\delta\Delta}} - \mathbf{2}, \quad \boldsymbol{\delta} > 0$$

It is named LINEX-based loss function, [31], where
$\mathbf{L}_{\boldsymbol{\delta}}(\boldsymbol{\Delta}) = \mathbf{exp}[\boldsymbol{\delta\Delta}] - \boldsymbol{\delta\Delta} - \mathbf{1}$, $\boldsymbol{\Delta} = \boldsymbol{\theta}^* - \boldsymbol{\theta}$, $\boldsymbol{\delta} \neq \mathbf{0}$.

is the LINEX loss function (LLF) that has been introduced in [30], [32]. The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\boldsymbol{\theta}^*_L = \frac{1}{2\delta}\ln\left[\frac{E(e^{\delta\theta}|x)}{E(e^{-\delta\theta}|x)}\right].$$

22.1 Informative Prior

We consider the unknown parameters α, β and γ to have independent gamma prior distributions with the joint probability density function, which is given by:

$$\mathbf{h}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) \propto \boldsymbol{\alpha}^{a-1}\boldsymbol{\beta}^{c-1}\boldsymbol{\gamma}^{e-1}\mathbf{e}^{-b\alpha-d\beta-f\gamma}, \quad (9)$$

where the hyper-parameter $a, b, c, d, e,$ and f are assumed to be known, positive, and chosen to reflect the prior belief about the unknown parameters.

2.2.2 Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function $\mathbf{g}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ with support on $(0,\infty)$, which is defined as

$$\hat{\mathbf{g}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \frac{1}{\mathbf{Nh_1h_2h_3}}\sum_{i=1}^{N}\mathbf{K}\left(\frac{\alpha-\hat{\alpha}_i}{h_1},\frac{\beta-\hat{\beta}_i}{h_2},\frac{\gamma-\hat{\gamma}_i}{h_3}\right), \quad (10)$$

$h_i, i = 1,2,3$ are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $Nh_i \to \infty$ as $N \to \infty$, where N is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i, which minimizes the mean squared errors given by $\mathbf{h_i = 1.06}\boldsymbol{S_i}\mathbf{N^{-0.2}}$, and S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the trivariate standard normal distribution for the parameters α,β, and γ. The basic elements associated with the kernel density function have been studied extensively by [18, 19]. Based on the properties of the MLEs of the parameters, which are converging in probability to the original parameters, the kernel prior estimate has been derived, [26]. It is worthwhile to mention that this kernel prior has been used for some distributions, [2], [21], [22], [23], [24], [25].

Thus, using the joint priors (9) and (10) with the likelihood function of the GPHCS (3) the posterior density for the parameters α, β, and γ can be written in a unified form as follows:

$\mathbf{f}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\underline{\mathbf{x}}) = \boldsymbol{K}\boldsymbol{l}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\mathbf{L}(\bar{\mathbf{X}};\boldsymbol{\theta})$, where

$$\boldsymbol{l}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \mathbf{h}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\hat{\mathbf{g}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$$

$$= \hat{\mathbf{g}}_1^{p_1}(\boldsymbol{\alpha})\,\hat{\mathbf{g}}_2^{p_2}(\boldsymbol{\beta})\,\hat{\mathbf{g}}_3^{p_3}(\boldsymbol{\gamma})\boldsymbol{\alpha}^{a-1}\boldsymbol{\beta}^{c-1}\boldsymbol{\gamma}^{e-1}\mathbf{e}^{-b\alpha-d\beta-f\gamma}$$

is the general prior distribution function with $p_1 = p_2 = p_3 = 0$ for the informative prior (9), and $p_1 = p_2 = p_3 = 1$, $a = c = e = 1$, and $b = d = f = 0$ for the kernel prior (10).
Thus, the posterior density can be written as

$$\mathbf{f}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\underline{\mathbf{x}}) = \mathbf{K}\,\hat{\mathbf{g}}_1^{p_1}(\boldsymbol{\alpha})\,\hat{\mathbf{g}}_2^{p_2}(\boldsymbol{\beta})\,\hat{\mathbf{g}}_3^{p_3}(\boldsymbol{\gamma})\,\boldsymbol{\alpha}^{D+a-1}\,\boldsymbol{\beta}^{D+c-1}\,\boldsymbol{\gamma}^{D+e-1}$$

$$\times\,\mathbf{exp}(-\boldsymbol{\beta}[\mathbf{d}+\textstyle\sum_{i=1}^{N}(\mathbf{R_i}+\mathbf{1})\,(\mathbf{e}^{x_i\gamma}-\mathbf{1})^{\alpha}+\boldsymbol{\delta}\mathbf{R}_T^*(\mathbf{e}^{T\gamma}-\mathbf{1})^{\alpha}]$$

$$-\boldsymbol{\gamma}(\mathbf{f}-\textstyle\sum_{i=1}^{N}\mathbf{x_i})-\boldsymbol{\alpha}\mathbf{b}+(\boldsymbol{\alpha}-\mathbf{1})\textstyle\sum_{i=1}^{N}\mathbf{ln}(\mathbf{e}^{x_i\gamma}-\mathbf{1})]. \quad (11)$$

Thus, based on (11) we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [29], introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For details, let $q(\alpha,\beta,\gamma)$ be a smooth, positive function in the parameter space. The posterior expectation of $q(\alpha,\beta,\gamma)$ can be obtained as

$$\mathbf{q}^* = \mathbf{E}(\mathbf{q}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})|\mathbf{x}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \mathbf{e}^{NH^*(\alpha,\beta,\gamma)}\mathbf{d\alpha d\beta d\gamma}}{\int_0^\infty\int_0^\infty\int_0^\infty e^{NH(\alpha,\beta,\gamma)}\boldsymbol{d\alpha d\beta d\gamma}}, \quad (12)$$

where $\mathbf{H} = \mathbf{ln\,f}\,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\mathbf{x})/\mathbf{N}$, and
$\mathbf{H}^* = \mathbf{H} + \mathbf{ln\,q}\,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})/\mathbf{N}$.

For (α,β,γ) the Bayes estimator using Tierney and Kadane approximation for $q(\alpha,\beta,\gamma)$ can be obtained as

$$\mathbf{q}^* = \sqrt{|\sum^*|/|\sum|}\,\mathbf{exp}[\,\mathbf{N}[\mathbf{H}^*(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) - \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})],$$

where $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ maximize the $H(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $H^*(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$, respectively

$$|\sum| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1}, \text{ and}$$

$$|\sum*| = \begin{vmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ at $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ respectively.
Thus, we can define $H = H(\alpha,\beta,\gamma)$, with the log likelihood of the posterior (11) as follows:

$$\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}|\underline{\mathbf{x}}) = [\mathbf{p_1ln}\hat{\mathbf{g}}_1(\boldsymbol{\alpha}) + \mathbf{p_2ln}\hat{\mathbf{g}}_2(\boldsymbol{\beta}) + \mathbf{p_3ln}\hat{\mathbf{g}}_3(\boldsymbol{\gamma})$$

$$+(\mathbf{N}+\mathbf{a}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\alpha} + (\mathbf{N}+\mathbf{c}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\beta} + (\mathbf{N}+\mathbf{e}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\gamma}$$

$$+(\mathbf{N}+\mathbf{a}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\alpha} + (\mathbf{N}+\mathbf{c}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\beta} + (\mathbf{N}+\mathbf{e}-\mathbf{1})\,\mathbf{ln}\,\boldsymbol{\gamma}$$

$$-\boldsymbol{\beta}[\mathbf{d} + \sum_{i=1}^{N}(\mathbf{R_i}+\mathbf{1})\,(\mathbf{e}^{x_i\gamma}-\mathbf{1})^{\alpha} + \boldsymbol{\delta}\mathbf{R}_T^*(\mathbf{e}^{T\gamma}-\mathbf{1})^{\alpha}]$$

$$+\boldsymbol{\gamma}\textstyle\sum_{i=1}^{N}\mathbf{x_i} + (\boldsymbol{\alpha}-\mathbf{1})\textstyle\sum_{i=1}^{N}\mathbf{ln}(\mathbf{e}^{x_i\gamma}-\mathbf{1}).$$

The derivatives of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ have been derived in Appendix B.

3 Simulation Study

The purpose of the simulation study is to compare the performance of the estimates using the Runge-Kutta and Bayes methods based on the informative and the informative kernel priors with two different loss functions, through two criteria: the average bias (AVB) and the root mean squared error (RMSEs) as given by:

$\mathbf{AVB} = \frac{1}{L}\sum_{i=1}^{L} |\hat{\theta}_i - \theta|$ and $\boldsymbol{RMSE} = \sqrt{\sum_{i=1}^{L}\left(\hat{\theta}_i - \theta\right)^2 / L}$, $\hat{\theta}$ is the estimate of θ and L is the number of replications.

In the simulation study, we choose different combinations for the hyperparameters of α and β as: $a = (2, 4)$, $b = (8, 7)$, $c = (3, 5)$, $d = (8, 10)$, $e = (4, 7)$ and $f = (8, 9)$. Thus, we can generate from the gamma distribution two values for each parameter $\alpha = (0.59, 1.11)$, $\beta = (0.79, 0.92)$, and $\gamma = (0.97, 1.32)$. Using the above parameter values for generating different samples from the generalized Weibull distribution with sizes N = 20, 40, and 60 to represent small, moderate, and large sizes. To assess the performance of these estimates, the average Bias (AVB) and the RMSEs for each were calculated using 1000 replicates.

An algorithm for generating the generalized progressive hybrid censoring scheme has been written in [24], [25]. Some of the points are quite clear based on these estimates from the simulation results in Table 2, Table 3, Table 4, Table 5, Table 6, and Table 7, and the others have been summarized in the following main points:

i. It is clear that generally, for both parameters β and γ the average Bias values based on the Runge-Kutta method outperform the corresponding values based on Bayes' method for the different loss functions. However, for the parameter α, the Runge-Kutta and Bayes methods have almost the same average bias especially based on the LINEX-based loss function.

ii. In terms of the RMSE values, we can easily see that the R-K method has the smallest RMSE values compared with their counterparts that are based on the Bayes' method, but for the parameter α, the Runge-Kutta and Bayes methods have the same RMSEs approximately.

iii. It is evident that the estimated AVB and RMSE values decrease with increasing the hyperparameters, the termination time of the experiment T, and the sample sizes as expected for all methods.

iv. For the parameter α, the estimated RMSE values increase with increasing the value of α, while decreasing as the value of β and γ increase.

v. For the parameters β and γ, the estimated RMSE values increase with increasing values of β and γ while decreasing as the value of α increases

vi. In general, the estimated RMSE values for the Bayes method based on the LINEX-based loss function are less than those based on the squared error loss function.

In conclusion, the R-K estimates compete and outperform the Bayesian estimation method based on the informative and kernel priors.

4 Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the generalized Weibull model, which is suitable for fitting several types of data and can be adapted to fit the data set with the monotone hazard rate function. It also demonstrates that the GW distribution can be used in many applications in reliability engineering and new fields such as biomedical sciences and survival analysis to describe the age of specific mortality rates and failure rates. As a result, for a significance level of 0.05, we fitted these datasets using the goodness of fit tests such as the Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) tests. [12], [13], presented a comprehensive study of these tests.

4.1 Vinyl Chloride Data Application

Since vinyl chloride is a known human carcinogen, exposure to this compound should be avoided to the maximum practicable extent, and levels should be kept as low as technically possible. Whereas, it is known that the concentration of vinyl chloride in drinking water of 0.5 mg/liter was calculated to be associated with an increased risk of liver and brain tumors for exposure starting from adulthood and would double the risk of developing cancer from continuous exposure from birth. Therefore, we consider the dataset used by [11], which represents 34 data points in mg/L of vinyl chloride obtained from clean upgrade monitoring wells, as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

We found that the generalized Weibull model is a good fit for this dataset as shown in Figure (1 a). Also, the goodness of fit tests such as the K-S test has a value of 0.7349, which is less than the critical value of 0.8579 with a P-value of 0.2389. The A-D test has a value of 0.7170, which is less than the critical value of 0.7464 with a P-value of 0.1583.

To study the concentration of vinyl chloride in the water of these wells based on this dataset, we find the estimates of the parameters that represent the scale and shape of the concentration using our model to determine the average concentration in the water. We observed that the R-K and Bayes estimates α are very close to one, indicating that this dataset is right-skewed, which means that the concentration decreases with increasing time, see Figure (1 b). Also, the R-K and Bayes estimates for β and γ are close to 2, ensuring that the dataset is right-skewed, which means the vinyl chloride concentration will decrease with increasing time, so monitoring these wells is very important.

4.2 Leukemia Data Application

In healthcare, Leukemia affects blood status and can be detected with a Blood Cell Counter (CBC). Mostly, leukemia patients undergo chemotherapy. Therefore, we study the effect of this treatment on leukemia patients based on a dataset collected by the Ministry of Health Hospital in Saudi Arabia and used in [1], which indicates the lifetimes in days for forty-three blood patients with leukemia after chemotherapy:
115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1025, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

We found that the generalized Weibull model is more fitting for this dataset than the Vinyl chloride data as shown in Figure (2 a). Also, the goodness of fit tests such as the K-S test has a value of 0.5402, which is less than the critical value of 0.8666 with a P-value of 0.6512. The A-D test has a value of 0.4421, which is less than the critical value of 0.7462 with a P-value of 0.3044.

To study the effect of chemotherapy on patients based on this dataset, we estimated the distribution parameters, which represent the scale and shape of the lifetime. We observed that the estimates of the R-K and Bayes methods for β and γ are greater than one and close to 2, while for α is less than one in most cases, which indicates a decreasing hazard rate and the graph is approximately symmetric, see Figure (2 b). Thus, the parameter estimates indicate the decreasing hazard rate for cancer and that means the longer the patient survives, the more likely they are to reach the upper limit of their natural lifespan. So overall, this dataset indicates that the patient's lifespan is more stable and lives longer due to the chemotherapy dose, and it is highly effective in giving patients more antibodies against cancer.

Figure (1 a) and Figure (2 a) display the empirical CDF and the CDF of the GWD distribution for these data sets, which confirm the goodness of fit tests. The results in Table 1 indicated that the R-K estimates of the parameters have AVB and MSEs have values lower than the Bayesian estimates based on the gamma prior and almost as close as to those based on the informative kernel prior. For both sets of data the MSE values for the parameters α , β and γ decrease as the T values increase.

5 Conclusion

In this study, we applied the Runge-Kutta estimation and Bayesian estimation methods for estimating the generalized Weibull distribution parameters, as a new lifetime distribution. The simulation results indicated that the average Bias and RMSEs of the parameters based on the Runge-Kutta method are more efficient than the Bayesian estimation method based on the informative and kernel priors using two different loss functions, based on the generalized progressive hybrid censored scheme. However, Bayes estimates based on the informative kernel prior are more efficient than those based on the informative prior and are close to those based on the Runge-Kutta estimates. Thus, the statistical significance of the Runge-Kutta method is its efficiency compared to most popular methods of estimation, it is a viable point estimation method for effectively any lifetime model and is reliable and easy to apply especially for medical, biological, social, psychological, and engineering researchers.

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Appendix A:

The log-likelihood function of (8) can be derived as

$$H(\underline{x};\theta) = N\,ln(\alpha\beta\gamma) + \gamma\sum_{i=1}^{N} x_i + (\alpha-1)\sum_{i=1}^{N} ln\,(e^{\gamma x_i}-1)$$
$$-\beta[\textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha} + \delta R^*(e^{T\gamma}-1)^{\alpha}],$$

$$\frac{\partial H}{\partial\alpha} = \frac{N}{\alpha} + \sum_{i=1}^{N} ln\,(e^{\gamma x_i}-1)$$
$$-\beta[\textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha}\ln(e^{x_i\gamma}-1)$$
$$+\delta R^*(e^{T\gamma}-1)^{\alpha}\ln(e^{T\gamma}-1)],$$

$$\frac{\partial^2 H}{\partial\alpha^2} = -N/\alpha^2 - \beta[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha}(\ln(e^{x_i\gamma}-1))^2$$
$$+\delta R^*(e^{T\gamma}-1)^{\alpha}(ln(e^{T\gamma}-1))^2],$$

$$\frac{\partial^2 H}{\partial\alpha\,\partial x} = \sum_{i=1}^{N}\frac{\gamma e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$
$$-\gamma\beta\alpha\left[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}\ln(e^{x_i\gamma}-1)e^{x_i\gamma}\right]$$
$$-\beta\gamma[\textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}e^{x_i\gamma}],$$

$$\frac{\partial H}{\partial\beta} = N/\beta - [\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha} + \delta R^*(e^{T\gamma}-1)^{\alpha}]$$

$$\frac{\partial^2 H}{\partial\beta^2} = -N/\beta^2,$$

$$\frac{\partial^2 H}{\partial\beta\,\partial x} = -\alpha\gamma\textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}e^{x_i\gamma}.$$

$$\frac{\partial H}{\partial\gamma} = N/\gamma + \sum_{i=1}^{N} x_i + (\alpha-1)\sum_{i=1}^{N}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$
$$-\beta\alpha[\textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}x_i e^{x_i\gamma}$$
$$+\delta R^*(e^{T\gamma}-1)^{\alpha-1}Te^{T\gamma}],$$

$$\frac{\partial^2 H}{\partial\gamma^2} = -\frac{N}{\gamma^2} + (\alpha-1)\sum_{i=1}^{N}\frac{x_i^2 e^{x_i\gamma}(e^{x_i\gamma}-1) - (x_i e^{x_i\gamma})^2}{(e^{x_i\gamma}-1)^2}$$
$$-\beta\alpha(\alpha-1)[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-2}(x_i e^{x_i\gamma})^2$$
$$+\delta R^*(e^{T\gamma}-1)^{\alpha-2}(Te^{T\gamma})^2]$$
$$-\alpha\beta[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}x_i^2 e^{x_i\gamma}$$
$$+\delta R^*(e^{T\gamma}-1)^{\alpha-1}T^2 e^{T\gamma}],$$

$$\frac{\partial^2 H}{\partial\gamma\partial x} = N + (\alpha-1)\sum_{i=1}^{N}\frac{(e^{x_i\gamma}-1)(1+\gamma x_i)e^{x_i\gamma} - \gamma x_i e^{2x_i\gamma}}{(e^{x_i\gamma}-1)^2}$$
$$-\beta\alpha\gamma(\alpha-1)[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-2}x_i e^{2x_i\gamma}]$$
$$-\beta\alpha[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha-1}e^{x_i\gamma}(1+\gamma x_i)].$$

Appendix B:

The log of the posterior density function (9) can be derived as

$$H(\alpha,\beta,\gamma|\underline{x}) = [p_1 ln\hat{g}_1(\alpha) + p_2 ln\hat{g}_2(\beta) + p_3 ln\hat{g}_3(\gamma)$$
$$+(N+a-1)\ln\alpha + (N+c-1)\ln\beta + (N+e-1)\ln\gamma$$
$$-\beta[d + \textstyle\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha} + \delta R_T^*(e^{T\gamma}-1)^{\alpha}]$$
$$+\gamma\textstyle\sum_{i=1}^{N}x_i + (\alpha-1)\sum_{i=1}^{N}\ln(e^{x_i\gamma}-1)]/N$$

$$H_1 = \frac{\partial H}{\partial\alpha} = [p_1\frac{\hat{g}_1^{/}(\alpha)}{\hat{g}_1(\alpha)} + (N+a-1)/\alpha$$
$$+\sum_{i=1}^{N}\ln(e^{x_i\gamma}-1) - \beta[\sum_{i=1}^{N}(1+R_i))^{\alpha}\,ln(e^{x_i\gamma}-1)$$
$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}\,ln(e^{T\gamma}-1)]]/N$$

$$H_{12} = \frac{\partial^2 H}{\partial\alpha\partial\beta} = -[\sum_{i=1}^{N}(1+R_i)(e^{x_i\gamma}-1)^{\alpha}\,ln(e^{x_i\gamma}-1)$$
$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}\,ln(e^{T\gamma}-1)]]/N$$

$$H_{13} = \frac{\partial^2 H}{\partial\alpha\partial\gamma} = [\sum_{i=1}^{N}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$
$$-\beta[\textstyle\sum_{i=1}^{N}(1+R_i)(e^{x_i\gamma}-1)^{\alpha}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$
$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}\frac{Te^{T\gamma}}{(e^{T\gamma}-1)}]$$
$$-\alpha\beta[\textstyle\sum_{i=1}^{N}(1+R_i)x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}\,ln(e^{x_i\gamma}-1)$$
$$+\delta R_T^* Te^{T\gamma}(e^{T\gamma}-1)^{\alpha-1}\,ln(e^{T\gamma}-1)]]/N$$

$$H_{11} = \frac{\partial^2 H}{\partial\alpha^2} = [p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{//}(\alpha) - \hat{g}^{/2}_1(\alpha)}{\hat{g}_1^2(\alpha)} - (N+a-1)/\alpha^2$$
$$-\beta[\sum_{i=1}^{N}(1+R_i)(e^{x_i\gamma}-1)^{\alpha}(ln(e^{x_i\gamma}-1))^2$$
$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}(ln(e^{T\gamma}-1))^2]]/N$$

$$H_2 = \frac{\partial H}{\partial\beta} = [p_2\frac{\hat{g}_2^{'}(\beta)}{\hat{g}_2(\beta)} + (N+c-1)/\beta$$
$$-[d + \sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha} + \delta R_T^*(e^{T\gamma}-1)^{\alpha}]]/N$$

$$H_{22} = \frac{\partial^2 H}{\partial\beta^2} = [p_2\frac{\hat{g}_2(\beta)g_2^{//}(\beta) - \hat{g}_2^{/2}(\beta)}{\hat{g}_2{}^2(\beta)}$$
$$-(N+c-1)/\beta^2]/N.$$

$$H_{12} = \frac{\partial^2 H}{\partial\alpha\,\partial\beta} = -[\sum_{i=1}^{N}(R_i+1)\,(e^{x_i\gamma}-1)^{\alpha}\ln(e^{x_i\gamma}-1)$$

$$+\delta R_T^*(e^{T\gamma}-1)^{\alpha}\ln e^{T\gamma}-1)]/N$$

$$H_{23} = \frac{\partial^2 H}{\partial\beta\,\partial\gamma} = [-\alpha[\sum_{i=1}^{N}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/N$$

$$H_3 = \frac{\partial H}{\partial\gamma} = [p_3\frac{\hat{g}_3'(\gamma)}{\hat{g}_3(\gamma)} + (N+e-1)/\gamma$$

$$+\sum_{i=1}^{N}x_i + (\alpha-1)\sum_{i=1}^{N}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$

$$-\alpha\beta[\sum_{i=1}^{N}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/N$$

$$H_{31} = \frac{\partial H}{\partial\gamma\partial\alpha} = [\sum_{i=1}^{N}\frac{x_i e^{x_i\gamma}}{(e^{x_i\gamma}-1)}$$

$$-\beta[\sum_{i=1}^{N}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]$$

$$-\alpha\beta[\sum_{i=1}^{N}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}\ln(e^{x_i\gamma}-1)$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}\ln(e^{T\gamma}-1)]]/N,$$

$$H_{32} = \frac{\partial H}{\partial\gamma\partial\beta} = [-\alpha[\sum_{i=1}^{N}(R_i+1)\,x_i e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+\delta R_T^* T e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}]]/N.$$

$$H_{33} = \frac{\partial^2 H}{\partial\gamma^2} = [p_3\frac{\hat{g}_3(\gamma)g_3^{//}(\gamma) - \hat{g}_3^{/^2}(\gamma)}{\hat{g}_3{}^2(\gamma)}$$

$$-(N+e-1)/\gamma^2 + \sum_{i=1}^{N}x_i$$

$$+(\alpha-1)\sum_{i=1}^{N}\frac{(e^{x_i\gamma}-1)x_i^2 e^{x_i\gamma} - (x_i e^{x_i\gamma})^2}{(e^{x_i\gamma}-1)^2}$$

$$-\alpha\beta[\sum_{i=1}^{N}(R_i+1)\,[x_i^2 e^{x_i\gamma}(e^{x_i\gamma}-1)^{\alpha-1}$$

$$+(\alpha-1)(x_i e^{x_i\gamma})^2(e^{x_i\gamma}-1)^{\alpha-2}$$

$$+\delta R_T^*[T^2 e^{T_i\gamma}(e^{T\gamma}-1)^{\alpha-1}$$

$$+(\alpha-1)(Te^{T\gamma})^2(e^{T\gamma}-1)^{\alpha-2}]]/N,$$

where the r^{th} derivative of the kernel density estimation can be defined as

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r} = \hat{g}_1{}^r(\alpha) = \frac{1}{Nh_1^{r+1}}\sum_{i=1}^{N}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right), \qquad (*)$$

where r=0,1,2,3,-----.

Using the Gaussian kernel and (*), we have

$$\hat{g}_1(\alpha) = \frac{1}{Nh_1\sqrt{2\pi}}\sum_{i=1}^{N}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha) = -\frac{1}{Nh_1^2\sqrt{2\pi}}\sum_{i=1}^{N}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1''(\alpha) = \frac{1}{Nh_1^3\sqrt{2\pi}}\sum_{i=1}^{N}[(\frac{\alpha-\alpha_i}{h_1})^2-1]\,e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

Table 1. The estimate, AVB, and the MSEs in parentheses for the parameters α, β, and γ based on the R-K and Bayes methods using the gamma and kernel priors under the squared error loss function based on the GHPCS for $m = n/2$, k = m/2. The hyperparameters are: $a = 2, b = 4, c = 8, d = 4, e = 2, f = 4$.

Samples	T	Par.	R-K Estimate		Gamma Prior		Kernel Prior	
			Estimate	AVB(MSE)	Estimate	AVB (MSE)	Estimate	AVB (MSE)
The vinyl Chloride data N=34	0.75	α	0.6189	0.065(0.0043)	0.5965	0.088(0.0078)	0.5956	0.089(0.0079)
		β	0.8503	0.101(0.0101)	0.8512	0.099(0.0099)	0.8473	0.103(0.0108)
		γ	0.2964	0.036(0.0013)	0.2958	0.036(0.0013)	0.2951	0.037(0.0014)
	3.5	α	0.6179	0.067(0.0045)	0.6071	0.078(0.0061)	0.6066	0.078(0.0061)
		β	0.8527	0.099(0.0096)	0.8494	0.101(0.0103)	0.8479	0.103(0.0106)
		γ	0.2982	0.034(0.0012)	0.2979	0.034(0.0012)	0.2979	0.034(0.0012)
The Leukemia Data N=43	500	α	0.6157	0.069(0.0048)	0.6159	0.069(0.0048)	0.6159	0.069(0.0048)
		β	0.8440	0.106(0.0113)	0.8110	0.139(0.0195)	0.7729	0.178(0.0316)
		γ	0.2985	0.034(0.0012)	0.2985	0.034(0.0011)	0.2985	0.034(0.0011)
	850	α	0.6161	0.069(0.0047)	0.6161	0.069(0.0047)	0.6161	0.069(0.0047)
		β	0.8445	0.106(0.0113)	0.7764	0.174(0.0303)	0.7522	0.206(0.0423)
		γ	0.2988	0.033(0.0011)	0.2987	0.033(0.0011)	0.2988	0.033(0.0011)

Table 2. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for GWD parameter $\boldsymbol{\alpha}$ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\boldsymbol{\delta = 2}$ for LINEX loss function.

N	m	k	α	β	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.027(0.0414)	0.070(0.0895)	0.034(0.0401)	0.065(0.0806)	0.034(0.0393)
				0.92	1.32	0.035(0.0544)	0.076(0.1285)	0.034(0.0432)	0.068(0.1050)	0.033(0.0411)
			1.11	0.79	0.97	0.032(0.0358)	0.136(0.1432)	0.084(0.0861)	0.126(0.1306)	0.087(0.0891)
				0.92	1.32	0.028(0.0342)	0.127(0.1364)	0.076(0.0792)	0.115(0.1220)	0.080(0.0824)
		8	0.59	0.79	0.97	0.025(0.0389)	0.062(0.0861)	0.035(0.0404)	0.059(0.0765)	0.035(0.0398)
				0.92	1.32	0.030(0.0460)	0.061(0.1003)	0.031(0.0384)	0.057(0.0841)	0.031(0.0372)
			1.11	0.79	0.97	0.035(0.0394)	0.127(0.1310)	0.088(0.0892)	0.122(0.1248)	0.090(0.0918)
				0.92	1.32	0.028(0.0344)	0.115(0.1214)	0.078(0.0808)	0.111(0.1151)	0.082(0.0841)
	15	8	0.59	0.79	0.97	0.026(0.0420)	0.066(0.1003)	0.036(0.0426)	0.062(0.0855)	0.036(0.0417)
				0.92	1.32	0.028(0.0428)	0.058(0.0932)	0.031(0.0373)	0.055(0.0811)	0.030(0.0364)
			1.11	0.79	0.97	0.035(0.0373)	0.126(0.1294)	0.088(0.0894)	0.122(0.1242)	0.091(0.0922)
				0.92	1.32	0.030(0.0397)	0.120(0.1364)	0.080(0.0832)	0.114(0.1215)	0.084(0.0858)
		11	0.59	0.79	0.97	0.020(0.0276)	0.055(0.0611)	0.036(0.0397)	0.054(0.0592)	0.036(0.0400)
				0.92	1.32	0.023(0.0344)	0.052(0.0611)	0.032(0.0358)	0.050(0.0581)	0.032(0.0357)
			1.11	0.79	0.97	0.036(0.0388)	0.120(0.1228)	0.091(0.0920)	0.119(0.1205)	0.093(0.0944)
				0.92	1.32	0.030(0.0331)	0.111(0.1146)	0.083(0.0846)	0.110(0.1126)	0.086(0.0877)
40	20	10	0.59	0.79	0.97	0.022(0.0329)	0.048(0.0576)	0.028(0.0323)	0.046(0.0549)	0.028(0.0323)
				0.92	1.32	0.027(0.0376)	0.042(0.0538)	0.023(0.0279)	0.040(0.0507)	0.023(0.0276)
			1.11	0.79	0.97	0.029(0.0321)	0.111(0.1139)	0.084(0.0851)	0.108(0.1108)	0.086(0.0874)
				0.92	1.32	0.022(0.0251)	0.101(0.1050)	0.076(0.0776)	0.099(0.1019)	0.079(0.0804)
		15	0.59	0.79	0.97	0.022(0.0309)	0.043(0.0501)	0.028(0.0316)	0.042(0.0484)	0.028(0.0317)
				0.92	1.32	0.026(0.0372)	0.040(0.0539)	0.024(0.0290)	0.039(0.0510)	0.024(0.0287)
			1.11	0.79	0.97	0.030(0.0325)	0.105(0.1079)	0.084(0.0859)	0.105(0.1070)	0.087(0.0883)
				0.92	1.32	0.024(0.0277)	0.097(0.1007)	0.076(0.0783)	0.097(0.0996)	0.080(0.0811)
	30	15	0.59	0.79	0.97	0.021(0.0300)	0.043(0.0526)	0.028(0.0321)	0.042(0.0507)	0.028(0.0323)
				0.92	1.32	0.026(0.0373)	0.040(0.0543)	0.024(0.0292)	0.039(0.0513)	0.024(0.0289)
			1.11	0.79	0.97	0.032(0.0337)	0.108(0.1105)	0.087(0.0879)	0.108(0.1094)	0.089(0.0901)
				0.92	1.32	0.024(0.0271)	0.097(0.0999)	0.077(0.0786)	0.096(0.0992)	0.080(0.0814)
		23	0.59	0.79	0.97	0.014(0.0186)	0.044(0.0467)	0.034(0.0367)	0.044(0.0466)	0.035(0.0371)
				0.92	1.32	0.017(0.0241)	0.037(0.0413)	0.026(0.0298)	0.037(0.0409)	0.027(0.0302)
			1.11	0.79	0.97	0.037(0.0388)	0.108(0.1087)	0.092(0.0928)	0.108(0.1086)	0.094(0.0944)
				0.92	1.32	0.029(0.0315)	0.099(0.1003)	0.084(0.0847)	0.099(0.1008)	0.086(0.0869)
			0.59	0.79	0.97	0.019(0.0269)	0.037(0.0442)	0.024(0.0279)	0.036(0.0431)	0.025(0.0282)
				0.92	1.32	0.027(0.0347)	0.033(0.0408)	0.019(0.0237)	0.032(0.0393)	0.020(0.0236)

60	30	15	1.11	0.79	0.97	0.028(0.0297)	0.101(0.1036)	0.084(0.0846)	0.101(0.1027)	0.086(0.0866)
				0.92	1.32	0.020(0.0231)	0.091(0.0942)	0.075(0.0761)	0.091(0.0934)	0.077(0.0785)
		23	0.59	0.79	0.97	0.018(0.0244)	0.036(0.0408)	0.026(0.0291)	0.036(0.0404)	0.026(0.0295)
				0.92	1.32	0.023(0.0311)	0.031(0.0387)	0.020(0.0238)	0.031(0.0377)	0.020(0.0240)
			1.11	0.79	0.97	0.029(0.0310)	0.099(0.1009)	0.085(0.0861)	0.100(0.1011)	0.087(0.0880)
				0.92	1.32	0.022(0.0241)	0.090(0.0924)	0.077(0.0781)	0.091(0.0930)	0.079(0.0805)
	45	23	0.59	0.79	0.97	0.018(0.0242)	0.035(0.0402)	0.025(0.0285)	0.035(0.0398)	0.026(0.0289)
				0.92	1.32	0.024(0.0325)	0.032(0.0392)	0.020(0.0241)	0.031(0.0381)	0.020(0.0242)
			1.11	0.79	0.97	0.029(0.0315)	0.099(0.1011)	0.085(0.0861)	0.100(0.1012)	0.087(0.0881)
				0.92	1.32	0.021(0.0235)	0.088(0.0911)	0.076(0.0772)	0.090(0.0917)	0.079(0.0796)
		34	0.59	0.79	0.97	0.012(0.0162)	0.039(0.0415)	0.031(0.0340)	0.039(0.0416)	0.032(0.0344)
				0.92	1.32	0.016(0.0211)	0.032(0.0360)	0.025(0.0275)	0.032(0.0360)	0.025(0.0280)
			1.11	0.79	0.97	0.036(0.0375)	0.103(0.1035)	0.091(0.0919)	0.103(0.1038)	0.093(0.0933)
				0.92	1.32	0.028(0.0299)	0.094(0.0951)	0.083(0.0839)	0.095(0.0959)	0.085(0.0857)

Table 3. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for GWD parameter $\boldsymbol{\alpha}$ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and $\boldsymbol{\delta = 2}$ for LINEX loss function.

N	m	k	α	β	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.024(0.0366)	0.060(0.0776)	0.034(0.0398)	0.058(0.0712)	0.035(0.0395)
				0.92	1.32	0.032(0.0485)	0.062(0.0874)	0.031(0.0393)	0.058(0.0783)	0.031(0.0378)
			1.11	0.79	0.97	0.034(0.0387)	0.128(0.1332)	0.086(0.0883)	0.122(0.1259)	0.089(0.0910)
				0.92	1.32	0.029(0.0340)	0.119(0.1265)	0.079(0.0811)	0.113(0.1184)	0.082(0.0844)
		8	0.59	0.79	0.97	0.024(0.0357)	0.058(0.0738)	0.035(0.0398)	0.056(0.0682)	0.035(0.0396)
				0.92	1.32	0.029(0.0466)	0.063(0.1841)	0.032(0.0400)	0.058(0.1259)	0.031(0.0387)
			1.11	0.79	0.97	0.036(0.0389)	0.123(0.1265)	0.088(0.0900)	0.120(0.1223)	0.091(0.0926)
				0.92	1.32	0.029(0.0329)	0.113(0.1178)	0.080(0.0822)	0.111(0.1140)	0.084(0.0856)
	15	8	0.59	0.79	0.97	0.025(0.0381)	0.061(0.0773)	0.036(0.0406)	0.059(0.0715)	0.036(0.0401)
				0.92	1.32	0.028(0.0445)	0.060(0.0903)	0.031(0.0387)	0.056(0.0780)	0.031(0.0376)
			1.11	0.79	0.97	0.035(0.0387)	0.124(0.1297)	0.087(0.0892)	0.120(0.1237)	0.091(0.0919)
				0.92	1.32	0.029(0.0346)	0.116(0.1219)	0.079(0.0814)	0.112(0.1159)	0.083(0.0847)
		11	0.59	0.79	0.97	0.022(0.0315)	0.056(0.0693)	0.036(0.0402)	0.055(0.0647)	0.036(0.0401)
				0.92	1.32	0.025(0.0384)	0.054(0.0724)	0.032(0.0380)	0.052(0.0665)	0.032(0.0375)
			1.11	0.79	0.97	0.037(0.0396)	0.120(0.1224)	0.090(0.0919)	0.118(0.1198)	0.093(0.0942)
				0.92	1.32	0.030(0.0335)	0.111(0.1144)	0.082(0.0842)	0.110(0.1124)	0.086(0.0874)
40	20	10	0.59	0.79	0.97	0.019(0.0281)	0.044(0.0534)	0.030(0.0335)	0.043(0.0515)	0.030(0.0337)
				0.92	1.32	0.025(0.0358)	0.040(0.0522)	0.025(0.0295)	0.039(0.0495)	0.025(0.0293)
			1.11	0.79	0.97	0.032(0.0340)	0.107(0.1092)	0.087(0.0879)	0.107(0.1084)	0.089(0.0901)
				0.92	1.32	0.025(0.0290)	0.097(0.1000)	0.077(0.0791)	0.097(0.0993)	0.080(0.0818)
		15	0.59	0.79	0.97	0.018(0.0259)	0.043(0.0491)	0.029(0.0328)	0.042(0.0481)	0.030(0.0331)
				0.92	1.32	0.023(0.0329)	0.039(0.0468)	0.024(0.0283)	0.038(0.0451)	0.024(0.0284)
			1.11	0.79	0.97	0.032(0.0340)	0.106(0.1083)	0.087(0.0881)	0.106(0.1079)	0.089(0.0903)
				0.92	1.32	0.025(0.0277)	0.097(0.0996)	0.078(0.0801)	0.097(0.0994)	0.081(0.0828)
	30	15	0.59	0.79	0.97	0.018(0.0254)	0.042(0.0479)	0.029(0.0327)	0.042(0.0470)	0.030(0.0331)
				0.92	1.32	0.025(0.0348)	0.038(0.0479)	0.023(0.0279)	0.037(0.0459)	0.024(0.0279)
			1.11	0.79	0.97	0.031(0.0330)	0.105(0.1075)	0.085(0.0866)	0.105(0.1070)	0.088(0.0889)
				0.92	1.32	0.024(0.0276)	0.096(0.0995)	0.077(0.0790)	0.097(0.0990)	0.080(0.0818)
		23	0.59	0.79	0.97	0.015(0.0199)	0.045(0.0477)	0.034(0.0368)	0.045(0.0475)	0.035(0.0372)
				0.92	1.32	0.018(0.0252)	0.037(0.0427)	0.027(0.0304)	0.037(0.0422)	0.027(0.0307)
			1.11	0.79	0.97	0.037(0.0387)	0.108(0.1089)	0.092(0.0929)	0.108(0.1088)	0.094(0.0944)
				0.92	1.32	0.030(0.0320)	0.099(0.1006)	0.084(0.0850)	0.099(0.1010)	0.086(0.0871)
	30	15	0.59	0.79	0.97	0.015(0.0210)	0.036(0.0402)	0.027(0.0306)	0.036(0.0401)	0.028(0.0310)
				0.92	1.32	0.023(0.0313)	0.032(0.0394)	0.021(0.0250)	0.031(0.0384)	0.021(0.0251)
			1.11	0.79	0.97	0.030(0.0322)	0.100(0.1013)	0.086(0.0871)	0.100(0.1016)	0.088(0.0889)
				0.92	1.32	0.022(0.0247)	0.089(0.0918)	0.077(0.0785)	0.091(0.0926)	0.080(0.0808)
		23	0.59	0.79	0.97	0.016(0.0227)	0.037(0.0416)	0.028(0.0307)	0.037(0.0413)	0.028(0.0311)
				0.92	1.32	0.023(0.0301)	0.031(0.0372)	0.021(0.0247)	0.031(0.0365)	0.021(0.0249)
			1.11	0.79	0.97	0.031(0.0325)	0.100(0.1016)	0.087(0.0877)	0.101(0.1019)	0.089(0.0895)
				0.92	1.32	0.023(0.0255)	0.090(0.0919)	0.077(0.0789)	0.091(0.0927)	0.080(0.0812)

60	45	23	0.59	0.79	0.97	0.014(0.0194)	0.038(0.0418)	0.029(0.0319)	0.038(0.0417)	0.030(0.0324)
				0.92	1.32	0.022(0.0299)	0.031(0.0382)	0.021(0.0252)	0.031(0.0374)	0.022(0.0254)
			1.11	0.79	0.97	0.032(0.0336)	0.101(0.1024)	0.088(0.0887)	0.102(0.1027)	0.090(0.0904)
				0.92	1.32	0.023(0.0261)	0.091(0.0928)	0.079(0.0797)	0.092(0.0936)	0.081(0.0819)
		34	0.59	0.79	0.97	0.012(0.0151)	0.039(0.0421)	0.032(0.0345)	0.040(0.0422)	0.033(0.0350)
				0.92	1.32	0.015(0.0204)	0.032(0.0357)	0.024(0.0275)	0.032(0.0358)	0.025(0.0280)
			1.11	0.79	0.97	0.036(0.0377)	0.103(0.1035)	0.091(0.0920)	0.103(0.1039)	0.093(0.0933)
				0.92	1.32	0.028(0.0300)	0.095(0.0959)	0.084(0.0844)	0.096(0.0967)	0.086(0.0862)

Table 4. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for the GWD parameter $\boldsymbol{\beta}$ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\boldsymbol{\delta = 2}$ for LINEX loss function.

N	m	K	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.058(0.0582)	0.141(0.1432)	0.095(0.0955)	0.131(0.1331)	0.093(0.0932)
				0.92	1.32	0.083(0.0837)	0.181(0.1832)	0.125(0.1256)	0.165(0.1664)	0.119(0.1194)
			1.11	0.79	0.97	0.053(0.0528)	0.145(0.1468)	0.090(0.0901)	0.130(0.1316)	0.088(0.0885)
				0.92	1.32	0.069(0.0695)	0.181(0.1828)	0.112(0.1118)	0.159(0.1597)	0.108(0.1081)
		8	0.59	0.79	0.97	0.050(0.0505)	0.106(0.1071)	0.089(0.0885)	0.103(0.1038)	0.087(0.0874)
				0.92	1.32	0.067(0.0669)	0.133(0.1338)	0.110(0.1103)	0.127(0.1279)	0.107(0.1075)
			1.11	0.79	0.97	0.049(0.0492)	0.106(0.1073)	0.087(0.0868)	0.102(0.1025)	0.086(0.0858)
				0.92	1.32	0.063(0.0628)	0.133(0.1338)	0.106(0.1057)	0.125(0.1258)	0.104(0.1036)
	15	8	0.59	0.79	0.97	0.050(0.0504)	0.106(0.1075)	0.088(0.0885)	0.103(0.1040)	0.087(0.0874)
				0.92	1.32	0.067(0.0666)	0.133(0.1336)	0.110(0.1100)	0.127(0.1277)	0.107(0.1073)
			1.11	0.79	0.97	0.049(0.0491)	0.108(0.1085)	0.087(0.0867)	0.103(0.1034)	0.086(0.0858)
				0.92	1.32	0.063(0.0628)	0.133(0.1337)	0.106(0.1057)	0.125(0.1256)	0.104(0.1036)
		11	0.59	0.79	0.97	0.047(0.0474)	0.092(0.0923)	0.086(0.0857)	0.091(0.0911)	0.085(0.0851)
				0.92	1.32	0.061(0.0608)	0.112(0.1128)	0.105(0.1047)	0.110(0.1106)	0.103(0.1031)
			1.11	0.79	0.97	0.047(0.0473)	0.093(0.0931)	0.085(0.0851)	0.091(0.0913)	0.085(0.0845)
				0.92	1.32	0.059(0.0595)	0.114(0.1141)	0.103(0.1027)	0.111(0.1108)	0.101(0.1013)
40	20	10	0.59	0.79	0.97	0.057(0.0573)	0.114(0.1147)	0.095(0.0948)	0.111(0.1110)	0.093(0.0933)
				0.92	1.32	0.079(0.0796)	0.146(0.1459)	0.122(0.1219)	0.139(0.1394)	0.118(0.1182)
			1.11	0.79	0.97	0.052(0.0520)	0.112(0.1120)	0.089(0.0893)	0.107(0.1070)	0.088(0.0883)
				0.92	1.32	0.068(0.0679)	0.141(0.1409)	0.110(0.1104)	0.132(0.1324)	0.108(0.1080)
		15	0.59	0.79	0.97	0.052(0.0522)	0.098(0.0979)	0.090(0.0902)	0.096(0.0965)	0.089(0.0893)
				0.92	1.32	0.069(0.0695)	0.123(0.1228)	0.113(0.1128)	0.120(0.1199)	0.111(0.1106)
			1.11	0.79	0.97	0.049(0.0495)	0.097(0.0970)	0.087(0.0871)	0.095(0.0949)	0.086(0.0864)
				0.92	1.32	0.064(0.0635)	0.120(0.1198)	0.106(0.1064)	0.116(0.1161)	0.105(0.1048)
	30	15	0.59	0.79	0.97	0.052(0.0521)	0.097(0.0976)	0.090(0.0900)	0.096(0.0962)	0.089(0.0891)
				0.92	1.32	0.069(0.0696)	0.122(0.1226)	0.113(0.1128)	0.120(0.1197)	0.111(0.1106)
			1.11	0.79	0.97	0.050(0.0496)	0.096(0.0967)	0.087(0.0871)	0.094(0.0946)	0.086(0.0865)
				0.92	1.32	0.063(0.0635)	0.120(0.1201)	0.106(0.1063)	0.116(0.1163)	0.105(0.1048)
		23	0.59	0.79	0.97	0.048(0.0478)	0.086(0.0864)	0.086(0.0861)	0.086(0.0860)	0.086(0.0857)
				0.92	1.32	0.061(0.0614)	0.105(0.1049)	0.105(0.1054)	0.104(0.1041)	0.104(0.1042)
			1.11	0.79	0.97	0.047(0.0471)	0.086(0.0859)	0.085(0.0849)	0.085(0.0854)	0.084(0.0845)
				0.92	1.32	0.059(0.0591)	0.103(0.1034)	0.102(0.1024)	0.102(0.1025)	0.101(0.1015)
60	30	15	0.59	0.79	0.97	0.056(0.0559)	0.104(0.1042)	0.094(0.0935)	0.102(0.1023)	0.092(0.0925)
				0.92	1.32	0.078(0.0782)	0.135(0.1346)	0.121(0.1207)	0.131(0.1307)	0.118(0.1179)
			1.11	0.79	0.97	0.052(0.0517)	0.103(0.1034)	0.089(0.0891)	0.100(0.1004)	0.088(0.0883)
				0.92	1.32	0.067(0.0675)	0.128(0.1283)	0.110(0.1100)	0.123(0.1234)	0.108(0.1082)
		23	0.59	0.79	0.97	0.052(0.0517)	0.093(0.0934)	0.090(0.0897)	0.093(0.0926)	0.089(0.0890)
				0.92	1.32	0.069(0.0688)	0.117(0.1167)	0.112(0.1122)	0.115(0.1149)	0.110(0.1105)
			1.11	0.79	0.97	0.049(0.0493)	0.092(0.0923)	0.087(0.0869)	0.091(0.0911)	0.086(0.0864)
				0.92	1.32	0.063(0.0632)	0.113(0.1134)	0.106(0.1061)	0.111(0.1113)	0.105(0.1049)
	45	23	0.59	0.79	0.97	0.052(0.0517)	0.093(0.0933)	0.090(0.0897)	0.092(0.0924)	0.089(0.0890)
				0.92	1.32	0.069(0.0689)	0.117(0.1169)	0.112(0.1123)	0.115(0.1152)	0.111(0.1106)
			1.11	0.79	0.97	0.049(0.0493)	0.092(0.0923)	0.087(0.0869)	0.091(0.0912)	0.086(0.0864)
				0.92	1.32	0.063(0.0632)	0.113(0.1133)	0.106(0.1061)	0.111(0.1112)	0.105(0.1049)
			0.59	0.79	0.97	0.048(0.0479)	0.085(0.0855)	0.086(0.0863)	0.085(0.0852)	0.086(0.0859)

		34		0.92	1.32	0.062(0.0616)	0.104(0.1039)	0.105(0.1055)	0.103(0.1033)	0.105(0.1046)
			1.11	0.79	0.97	0.047(0.0471)	0.085(0.0848)	0.085(0.0850)	0.085(0.0845)	0.085(0.0846)
				0.92	1.32	0.059(0.0592)	0.102(0.1021)	0.102(0.1025)	0.102(0.1015)	0.102(0.1017)

Table 5. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for GWD parameter $\boldsymbol{\beta}$ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and $\boldsymbol{\delta = 2}$ for LINEX loss function.

N	m	k	α	β	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.050(0.0503)	0.105(0.1067)	0.088(0.0884)	0.102(0.1034)	0.087(0.0873)
				0.92	1.32	0.070(0.0705)	0.146(0.1469)	0.114(0.1136)	0.137(0.1384)	0.110(0.1102)
			1.11	0.79	0.97	0.050(0.0500)	0.116(0.1174)	0.088(0.0876)	0.109(0.1102)	0.086(0.0865)
				0.92	1.32	0.064(0.0645)	0.144(0.1454)	0.107(0.1072)	0.133(0.1342)	0.105(0.1047)
		8	0.59	0.79	0.97	0.049(0.0490)	0.099(0.1000)	0.087(0.0872)	0.097(0.0978)	0.086(0.0863)
				0.92	1.32	0.064(0.0642)	0.124(0.1247)	0.108(0.1078)	0.120(0.1205)	0.106(0.1056)
			1.11	0.79	0.97	0.048(0.0484)	0.101(0.1020)	0.086(0.0861)	0.098(0.0985)	0.085(0.0853)
				0.92	1.32	0.061(0.0615)	0.125(0.1255)	0.104(0.1045)	0.119(0.1196)	0.103(0.1027)
	15	8	0.59	0.79	0.97	0.049(0.0495)	0.101(0.1019)	0.088(0.0877)	0.099(0.0994)	0.087(0.0867)
				0.92	1.32	0.067(0.0667)	0.133(0.1337)	0.110(0.1100)	0.127(0.1278)	0.107(0.1074)
			1.11	0.79	0.97	0.049(0.0491)	0.108(0.1088)	0.087(0.0867)	0.103(0.1037)	0.086(0.0858)
				0.92	1.32	0.063(0.0628)	0.133(0.1338)	0.106(0.1057)	0.125(0.1258)	0.104(0.1036)
		11	0.59	0.79	0.97	0.047(0.0474)	0.092(0.0923)	0.086(0.0858)	0.091(0.0912)	0.085(0.0851)
				0.92	1.32	0.061(0.0608)	0.112(0.1127)	0.105(0.1048)	0.110(0.1105)	0.103(0.1031)
			1.11	0.79	0.97	0.047(0.0473)	0.094(0.0939)	0.085(0.0851)	0.092(0.0920)	0.084(0.0845)
				0.92	1.32	0.060(0.0595)	0.113(0.1137)	0.103(0.1027)	0.110(0.1105)	0.101(0.1013)
40	20	10	0.59	0.79	0.97	0.051(0.0506)	0.093(0.0933)	0.089(0.0887)	0.092(0.0923)	0.088(0.0879)
				0.92	1.32	0.068(0.0678)	0.119(0.1189)	0.111(0.1112)	0.116(0.1165)	0.109(0.1092)
			1.11	0.79	0.97	0.049(0.0491)	0.095(0.0948)	0.087(0.0867)	0.093(0.0930)	0.086(0.0861)
				0.92	1.32	0.063(0.0627)	0.117(0.1167)	0.106(0.1057)	0.114(0.1136)	0.104(0.1042)
		15	0.59	0.79	0.97	0.050(0.0505)	0.093(0.0928)	0.089(0.0886)	0.092(0.0918)	0.088(0.0879)
				0.92	1.32	0.066(0.0665)	0.115(0.1156)	0.110(0.1100)	0.114(0.1136)	0.108(0.1082)
			1.11	0.79	0.97	0.049(0.0487)	0.092(0.0925)	0.086(0.0864)	0.091(0.0911)	0.086(0.0858)
				0.92	1.32	0.062(0.0620)	0.114(0.1138)	0.105(0.1050)	0.111(0.1112)	0.104(0.1037)
	30	15	0.59	0.79	0.97	0.050(0.0503)	0.092(0.0924)	0.088(0.0884)	0.091(0.0915)	0.088(0.0877)
				0.92	1.32	0.068(0.0680)	0.119(0.1192)	0.111(0.1114)	0.117(0.1168)	0.109(0.1094)
			1.11	0.79	0.97	0.049(0.0491)	0.095(0.0948)	0.087(0.0867)	0.093(0.0931)	0.086(0.0861)
				0.92	1.32	0.063(0.0627)	0.116(0.1164)	0.106(0.1057)	0.113(0.1134)	0.104(0.1042)
		23	0.59	0.79	0.97	0.048(0.0478)	0.086(0.0864)	0.086(0.0861)	0.086(0.0861)	0.086(0.0856)
				0.92	1.32	0.061(0.0615)	0.105(0.1049)	0.105(0.1054)	0.104(0.1041)	0.104(0.1043)
			1.11	0.79	0.97	0.047(0.0471)	0.086(0.0857)	0.085(0.0849)	0.085(0.0853)	0.084(0.0845)
				0.92	1.32	0.059(0.0591)	0.103(0.1035)	0.102(0.1023)	0.103(0.1025)	0.101(0.1014)
60	30	15	0.59	0.79	0.97	0.050(0.0499)	0.089(0.0889)	0.088(0.0880)	0.088(0.0885)	0.087(0.0875)
				0.92	1.32	0.067(0.0670)	0.113(0.1132)	0.110(0.1105)	0.112(0.1118)	0.109(0.1090)
			1.11	0.79	0.97	0.049(0.0488)	0.090(0.0902)	0.086(0.0865)	0.089(0.0893)	0.086(0.0860)
				0.92	1.32	0.062(0.0622)	0.110(0.1102)	0.105(0.1052)	0.109(0.1086)	0.104(0.1041)
		23	0.59	0.79	0.97	0.050(0.0501)	0.089(0.0894)	0.088(0.0882)	0.089(0.0889)	0.088(0.0877)
				0.92	1.32	0.066(0.0661)	0.112(0.1116)	0.110(0.1097)	0.110(0.1104)	0.108(0.1083)
			1.11	0.79	0.97	0.049(0.0486)	0.089(0.0892)	0.086(0.0863)	0.088(0.0884)	0.086(0.0858)
				0.92	1.32	0.062(0.0618)	0.109(0.1089)	0.105(0.1048)	0.107(0.1074)	0.104(0.1038)
	45	23	0.59	0.79	0.97	0.049(0.0495)	0.088(0.0884)	0.088(0.0877)	0.088(0.0879)	0.087(0.0872)
				0.92	1.32	0.065(0.0654)	0.110(0.1102)	0.109(0.1090)	0.109(0.1090)	0.108(0.1077)
			1.11	0.79	0.97	0.048(0.0484)	0.088(0.0883)	0.086(0.0861)	0.088(0.0877)	0.086(0.0856)
				0.92	1.32	0.061(0.0615)	0.108(0.1078)	0.105(0.1045)	0.106(0.1065)	0.104(0.1035)
		34	0.59	0.79	0.97	0.048(0.0479)	0.085(0.0855)	0.086(0.0862)	0.085(0.0853)	0.086(0.0858)
				0.92	1.32	0.062(0.0615)	0.104(0.1039)	0.105(0.1055)	0.103(0.1033)	0.105(0.1046)
			1.11	0.79	0.97	0.047(0.0471)	0.085(0.0849)	0.085(0.0849)	0.085(0.0846)	0.085(0.0846)
				0.92	1.32	0.059(0.0592)	0.102(0.1022)	0.102(0.1025)	0.102(0.1016)	0.102(0.1018)

Table 6. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for the GWD parameter $\boldsymbol{\gamma}$ using the R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\boldsymbol{\delta} = \mathbf{2}$ for LINEX loss function.

N	m	k	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.077(0.0772)	0.101(0.1015)	0.123(0.1233)	0.103(0.1030)	0.113(0.1127)
				0.92	1.32	0.139(0.1392)	0.138(0.1435)	0.200(0.2000)	0.146(0.1464)	0.161(0.1608)
			1.11	0.79	0.97	0.060(0.0602)	0.103(0.1032)	0.107(0.1074)	0.103(0.1028)	0.106(0.1057)
				0.92	1.32	0.091(0.0906)	0.144(0.1446)	0.154(0.1540)	0.143(0.1431)	0.148(0.1480)
		8	0.59	0.79	0.97	0.062(0.0616)	0.101(0.1011)	0.109(0.1086)	0.101(0.1011)	0.105(0.1051)
				0.92	1.32	0.098(0.0979)	0.141(0.1412)	0.160(0.1605)	0.141(0.1407)	0.147(0.1473)
			1.11	0.79	0.97	0.056(0.0561)	0.102(0.1016)	0.104(0.1035)	0.101(0.1012)	0.103(0.1028)
				0.92	1.32	0.082(0.0818)	0.141(0.1411)	0.146(0.1456)	0.140(0.1400)	0.143(0.1428)
	15	8	0.59	0.79	0.97	0.062(0.0616)	0.101(0.1011)	0.109(0.1086)	0.101(0.1010)	0.105(0.1051)
				0.92	1.32	0.097(0.0975)	0.141(0.1415)	0.160(0.1602)	0.141(0.1407)	0.147(0.1471)
			1.11	0.79	0.97	0.056(0.0561)	0.101(0.1014)	0.104(0.1035)	0.101(0.1011)	0.103(0.1028)
				0.92	1.32	0.082(0.0818)	0.141(0.1412)	0.146(0.1457)	0.140(0.1401)	0.143(0.1429)
		11	0.59	0.79	0.97	0.056(0.0556)	0.100(0.0999)	0.103(0.1031)	0.100(0.0997)	0.102(0.1018)
				0.92	1.32	0.083(0.0833)	0.139(0.1388)	0.147(0.1471)	0.138(0.1379)	0.141(0.1415)
			1.11	0.79	0.97	0.054(0.0541)	0.100(0.1004)	0.102(0.1017)	0.100(0.1002)	0.101(0.1013)
				0.92	1.32	0.078(0.0776)	0.139(0.1387)	0.142(0.1417)	0.138(0.1380)	0.140(0.1401)
40	20	10	0.59	0.79	0.97	0.075(0.0755)	0.107(0.1070)	0.122(0.1218)	0.107(0.1066)	0.114(0.1144)
				0.92	1.32	0.130(0.1298)	0.152(0.1540)	0.191(0.1910)	0.152(0.1521)	0.164(0.1643)
			1.11	0.79	0.97	0.059(0.0592)	0.104(0.1040)	0.107(0.1065)	0.104(0.1036)	0.106(0.1055)
				0.92	1.32	0.089(0.0886)	0.146(0.1461)	0.152(0.1521)	0.144(0.1445)	0.148(0.1484)
		15	0.59	0.79	0.97	0.064(0.0642)	0.104(0.1042)	0.111(0.1113)	0.104(0.1037)	0.108(0.1080)
				0.92	1.32	0.103(0.1033)	0.148(0.1479)	0.166(0.1659)	0.146(0.1458)	0.153(0.1532)
			1.11	0.79	0.97	0.056(0.0565)	0.102(0.1022)	0.104(0.1039)	0.102(0.1019)	0.103(0.1033)
				0.92	1.32	0.083(0.0827)	0.143(0.1427)	0.147(0.1465)	0.142(0.1417)	0.144(0.1443)
	30	15	0.59	0.79	0.97	0.064(0.0641)	0.104(0.1043)	0.111(0.1111)	0.104(0.1037)	0.108(0.1078)
				0.92	1.32	0.103(0.1033)	0.148(0.1480)	0.166(0.1659)	0.146(0.1459)	0.153(0.1532)
			1.11	0.79	0.97	0.056(0.0565)	0.102(0.1023)	0.104(0.1039)	0.102(0.1020)	0.103(0.1034)
				0.92	1.32	0.083(0.0827)	0.143(0.1426)	0.146(0.1465)	0.142(0.1416)	0.144(0.1443)
		23	0.59	0.79	0.97	0.055(0.0551)	0.101(0.1006)	0.103(0.1029)	0.100(0.1004)	0.102(0.1020)
				0.92	1.32	0.082(0.0820)	0.140(0.1404)	0.146(0.1463)	0.139(0.1394)	0.143(0.1427)
			1.11	0.79	0.97	0.054(0.0537)	0.100(0.1003)	0.101(0.1014)	0.100(0.1002)	0.101(0.1011)
				0.92	1.32	0.077(0.0767)	0.139(0.1388)	0.141(0.1410)	0.138(0.1383)	0.140(0.1401)
60	30	15	0.59	0.79	0.97	0.073(0.0728)	0.108(0.1086)	0.119(0.1192)	0.108(0.1076)	0.114(0.1139)
				0.92	1.32	0.127(0.1268)	0.156(0.1573)	0.188(0.1882)	0.154(0.1544)	0.166(0.1664)
			1.11	0.79	0.97	0.059(0.0590)	0.104(0.1040)	0.106(0.1063)	0.104(0.1037)	0.106(0.1055)
				0.92	1.32	0.088(0.0880)	0.147(0.1466)	0.152(0.1515)	0.145(0.1452)	0.149(0.1486)
		23	0.59	0.79	0.97	0.063(0.0635)	0.105(0.1048)	0.111(0.1105)	0.104(0.1042)	0.108(0.1080)
				0.92	1.32	0.102(0.1018)	0.149(0.1497)	0.164(0.1645)	0.147(0.1472)	0.154(0.1543)
			1.11	0.79	0.97	0.056(0.0563)	0.102(0.1022)	0.104(0.1038)	0.102(0.1020)	0.103(0.1033)
				0.92	1.32	0.082(0.0824)	0.143(0.1428)	0.146(0.1462)	0.142(0.1419)	0.145(0.1445)
	45	23	0.59	0.79	0.97	0.064(0.0636)	0.105(0.1049)	0.111(0.1106)	0.104(0.1042)	0.108(0.1080)
				0.92	1.32	0.102(0.1020)	0.149(0.1490)	0.165(0.1647)	0.147(0.1469)	0.154(0.1544)
			1.11	0.79	0.97	0.056(0.0564)	0.102(0.1022)	0.104(0.1038)	0.102(0.1020)	0.103(0.1033)
				0.92	1.32	0.082(0.0824)	0.143(0.1428)	0.146(0.1462)	0.142(0.1419)	0.145(0.1445)
		34	0.59	0.79	0.97	0.055(0.0554)	0.101(0.1010)	0.103(0.1032)	0.101(0.1007)	0.102(0.1024)
				0.92	1.32	0.082(0.0825)	0.141(0.1413)	0.147(0.1468)	0.140(0.1403)	0.144(0.1437)
			1.11	0.79	0.97	0.054(0.0538)	0.100(0.1004)	0.101(0.1015)	0.100(0.1003)	0.101(0.1013)
				0.92	1.32	0.077(0.0770)	0.139(0.1390)	0.141(0.1412)	0.139(0.1386)	0.140(0.1404)

Table 7. The Average Bias (AVB) and Root Mean Squared Errors (RMSEs) in parentheses for the GWD parameter $\boldsymbol{\gamma}$ using The R-K and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5 and $\boldsymbol{\delta = 2}$ for LINEX loss function.

N	m	k	α	β	γ	R-K Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.59	0.79	0.97	0.061(0.0614)	0.101(0.1011)	0.108(0.1084)	0.101(0.1010)	0.105(0.1049)
				0.92	1.32	0.106(0.1065)	0.140(0.1412)	0.169(0.1686)	0.142(0.1418)	0.150(0.1504)
			1.11	0.79	0.97	0.057(0.0571)	0.102(0.1020)	0.104(0.1045)	0.102(0.1016)	0.104(0.1035)
				0.92	1.32	0.084(0.0839)	0.142(0.1421)	0.148(0.1476)	0.141(0.1409)	0.144(0.1442)
		8	0.59	0.79	0.97	0.059(0.0587)	0.100(0.1005)	0.106(0.1060)	0.100(0.1004)	0.104(0.1035)
				0.92	1.32	0.091(0.0915)	0.140(0.1406)	0.155(0.1546)	0.140(0.1396)	0.145(0.1448)
			1.11	0.79	0.97	0.055(0.0553)	0.101(0.1010)	0.103(0.1028)	0.101(0.1008)	0.102(0.1022)
				0.92	1.32	0.080(0.0801)	0.140(0.1402)	0.144(0.1441)	0.139(0.1393)	0.142(0.1418)
	15	8	0.59	0.79	0.97	0.060(0.0597)	0.101(0.1007)	0.107(0.1069)	0.101(0.1006)	0.104(0.1041)
				0.92	1.32	0.097(0.0975)	0.141(0.1415)	0.160(0.1601)	0.141(0.1407)	0.147(0.1471)
			1.11	0.79	0.97	0.056(0.0561)	0.101(0.1014)	0.104(0.1035)	0.101(0.1011)	0.103(0.1027)
				0.92	1.32	0.082(0.0818)	0.141(0.1411)	0.146(0.1456)	0.140(0.1400)	0.143(0.1428)
		11	0.59	0.79	0.97	0.056(0.0556)	0.100(0.0998)	0.103(0.1032)	0.100(0.0997)	0.102(0.1018)
				0.92	1.32	0.083(0.0834)	0.139(0.1387)	0.147(0.1471)	0.138(0.1378)	0.141(0.1415)
			1.11	0.79	0.97	0.054(0.0541)	0.100(0.1003)	0.102(0.1017)	0.100(0.1001)	0.101(0.1013)
				0.92	1.32	0.078(0.0776)	0.139(0.1388)	0.142(0.1417)	0.138(0.1381)	0.140(0.1401)
40	20	10	0.59	0.79	0.97	0.061(0.0609)	0.103(0.1029)	0.108(0.1081)	0.102(0.1025)	0.106(0.1058)
				0.92	1.32	0.099(0.0989)	0.147(0.1469)	0.162(0.1618)	0.145(0.1448)	0.151(0.1511)
			1.11	0.79	0.97	0.056(0.0560)	0.102(0.1019)	0.103(0.1035)	0.102(0.1016)	0.103(0.1030)
				0.92	1.32	0.082(0.0817)	0.142(0.1420)	0.146(0.1455)	0.141(0.1410)	0.144(0.1436)
		15	0.59	0.79	0.97	0.061(0.0607)	0.103(0.1031)	0.108(0.1080)	0.103(0.1026)	0.106(0.1057)
				0.92	1.32	0.095(0.0955)	0.145(0.1456)	0.159(0.1587)	0.144(0.1438)	0.150(0.1495)
			1.11	0.79	0.97	0.056(0.0556)	0.102(0.1016)	0.103(0.1031)	0.101(0.1014)	0.103(0.1026)
				0.92	1.32	0.081(0.0807)	0.141(0.1414)	0.145(0.1447)	0.141(0.1405)	0.143(0.1429)
	30	15	0.59	0.79	0.97	0.060(0.0603)	0.103(0.1027)	0.108(0.1076)	0.102(0.1023)	0.105(0.1055)
				0.92	1.32	0.099(0.0991)	0.146(0.1465)	0.162(0.1620)	0.145(0.1447)	0.151(0.1513)
			1.11	0.79	0.97	0.056(0.0560)	0.102(0.1019)	0.103(0.1035)	0.102(0.1016)	0.103(0.1030)
				0.92	1.32	0.082(0.0816)	0.142(0.1420)	0.146(0.1455)	0.141(0.1411)	0.144(0.1436)
		23	0.59	0.79	0.97	0.055(0.0551)	0.101(0.1006)	0.103(0.1029)	0.100(0.1004)	0.102(0.1020)
				0.92	1.32	0.082(0.0821)	0.140(0.1403)	0.146(0.1465)	0.139(0.1394)	0.143(0.1428)
			1.11	0.79	0.97	0.054(0.0537)	0.100(0.1003)	0.101(0.1014)	0.100(0.1002)	0.101(0.1011)
				0.92	1.32	0.077(0.0767)	0.139(0.1387)	0.141(0.1410)	0.138(0.1382)	0.140(0.1400)
60	30	15	0.59	0.79	0.97	0.059(0.0595)	0.103(0.1032)	0.107(0.1069)	0.103(0.1027)	0.105(0.1053)
				0.92	1.32	0.097(0.0968)	0.147(0.1473)	0.160(0.1599)	0.145(0.1454)	0.152(0.1517)
			1.11	0.79	0.97	0.056(0.0557)	0.102(0.1018)	0.103(0.1032)	0.102(0.1016)	0.103(0.1029)
				0.92	1.32	0.081(0.0811)	0.142(0.1419)	0.145(0.1450)	0.141(0.1411)	0.144(0.1435)
		23	0.59	0.79	0.97	0.060(0.0600)	0.103(0.1034)	0.107(0.1073)	0.103(0.1029)	0.106(0.1057)
				0.92	1.32	0.094(0.0946)	0.147(0.1467)	0.158(0.1578)	0.145(0.1448)	0.150(0.1505)
			1.11	0.79	0.97	0.055(0.0555)	0.102(0.1016)	0.103(0.1030)	0.101(0.1014)	0.103(0.1026)
				0.92	1.32	0.080(0.0805)	0.142(0.1415)	0.144(0.1444)	0.141(0.1408)	0.143(0.1431)
	45	23	0.59	0.79	0.97	0.059(0.0588)	0.103(0.1027)	0.106(0.1062)	0.102(0.1023)	0.105(0.1048)
				0.92	1.32	0.093(0.0926)	0.146(0.1459)	0.156(0.1560)	0.144(0.1441)	0.149(0.1494)
			1.11	0.79	0.97	0.055(0.0552)	0.101(0.1015)	0.103(0.1028)	0.101(0.1013)	0.102(0.1024)
				0.92	1.32	0.080(0.0800)	0.141(0.1412)	0.144(0.1440)	0.141(0.1405)	0.143(0.1427)
		34	0.59	0.79	0.97	0.055(0.0553)	0.101(0.1010)	0.103(0.1031)	0.101(0.1008)	0.102(0.1024)
				0.92	1.32	0.082(0.0824)	0.141(0.1413)	0.147(0.1468)	0.140(0.1402)	0.144(0.1436)
			1.11	0.79	0.97	0.054(0.0538)	0.100(0.1004)	0.101(0.1015)	0.100(0.1003)	0.101(0.1013)
				0.92	1.32	0.077(0.0770)	0.139(0.1390)	0.141(0.1412)	0.139(0.1386)	0.140(0.1404)

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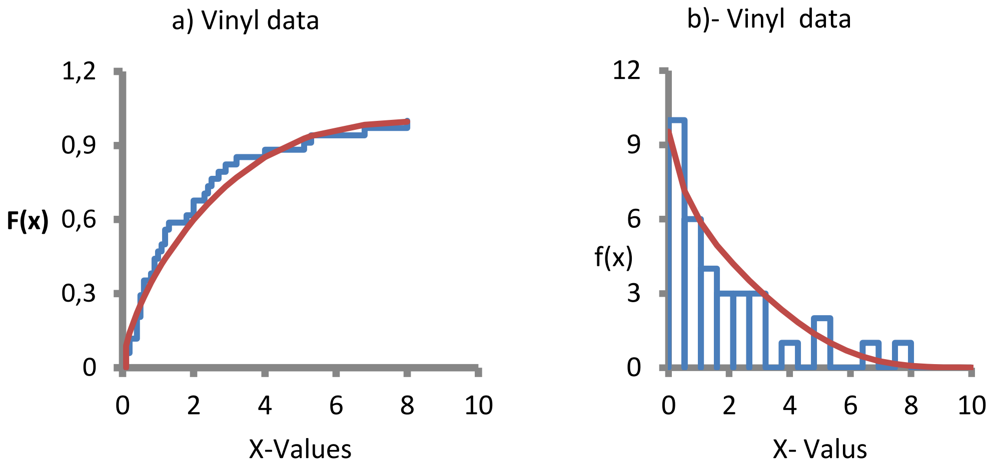

Fig. 1: a) The Empirical CDF and the fitted CDF for the Vinyl Chloride Data.
b) The Histogram and the fitted PDF for the Vinyl Chloride Data.

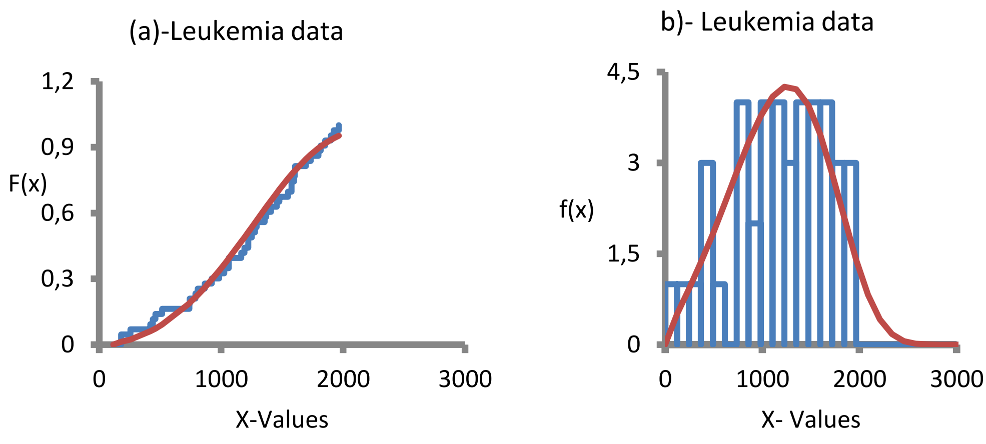

Fig. 2: a) The Empirical CDF and the fitted CDF for the Leukemia Data.
b) The Histogram and the fitted PDF for the Leukemia Data.