Conditional Inference on the Generalized Shape-Scale Family

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Abstract: In parameter estimation techniques, there are many methods for estimating the distribution parameters in life data analyses. However, most of them are less efficient than the Bayesian method, despite its subjectivity. Thus, the main objective of this study is to present the conditional inference method as an alternative and efficient method for estimating the generalized shape-scale family parameters and comparing them with the Bayesian estimates. A comparison between these estimators is provided by using an extensive Monte Carlo simulation study based on two criteria, namely, the absolute average bias and mean squared error based on the generalized progressive hybrid censoring scheme. The simulation results indicated that the conditional inference is highly efficient, which provides better estimates and outperforms the Bayesian inference. Finally, two real dataset analyses are presented to illustrate the efficiency of the proposed methods.

Keywords: Bayes estimation, Burr type-II distribution, Conditional inference, Generalized progressive hybrid, Censoring scheme, kernel prior
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1 Introduction

In statistical inference, Bayesian inference is the most usable method that has been used for estimating the distribution parameters. However, the disadvantage of this method is the subjectivity of prior information, which can lead to different posterior distributions that worry some statisticians when making undesirable decisions. Therefore, the challenge in this paper is presenting the conditional inference for the first time in the literature for estimating the three-parameter distributions. To demonstrate this, the generalized shape-scale family was applied to the three-parameter Burr type-XII distribution introduced in the literature in survivorship applications. [5] and [6] showed that, if one chooses the parameters appropriately, the three-parameter Burr type-XII distribution covers many of the curve-shaped characteristics of types I, IV, and VI in the Pearson family of distributions. The cumulative distribution function (CDF) and the probability density function (PDF) for the generalized shape-scale family are given, respectively, by:

$$F(x) = 1 - [1 + \beta g^{\alpha}(x)]^{-\gamma},\ 0 \le x, \quad \alpha, \beta, \gamma > 0. \tag{1}$$

$$f(x) = \alpha\beta\gamma g^{\alpha-1}(x) g'(x)[1 + \beta g^{\alpha}(x)]^{-\gamma-1}, \quad \alpha, \beta, \gamma, x > 0. \tag{2}$$

The parameters $\alpha, \gamma > 0$ are the shape parameters and $\beta > 0$ is the scale parameter.

For convenience, we assume $g(x)$ to be differentiable as well as a strictly increasing function of x, $g(0^{+}) = 0$ and $g(x) \to \infty \quad as \quad x \to \infty$.

This family includes the most popular parametric models in lifetime distributions, such as the generalised Weibull distribution, generalised gamma distribution, Pareto distribution, Lomax distribution, three-parameter Burr type-XII distribution, generalised Pareto distribution, and generalised Gamma distribution, based on the values of $g(x)$. Because of its flexibility in describing life test data, we will study the three-parameter Burr-XII distribution, which has the CDF and the PDF as given respectively by:

$$F(\alpha, \beta, \gamma | \underline{X}) = 1 - [1 + \beta x^{\alpha}]^{-\gamma},\ x, \alpha, \beta, \gamma > 0 \tag{3}$$

$$f(\alpha, \beta, \gamma | \underline{X}) = \alpha\beta\gamma x^{\alpha-1}[1 + \beta x^{\alpha}]^{-\gamma-1}, \quad x, \alpha, \beta, \gamma > 0\ , \tag{4}$$

where α and γ are form factors and beta is the scale factor.

Many authors have studied inferences on the Burr type-XII distribution, [34, 35, 36] and [37] have described methods for fitting the Burr type-XII distribution on life test data based on type-II censoring using the maximum likelihood estimation (MLE) method. [32] derived the MLE based on censored samples. [24] studied Bayesian inference based on various loss functions. [28, 29] derived the MLE and Bayes estimators for some lifetime parameters and the Burr type-XII model parameters based on progressive type-II censored samples. [37] proposed pivotal quantities for testing the shape parameter and established the confidence interval for the shape parameter for the two-parameter Burr type-XII distribution under censored samples. [38] studied the statistical inference of the Burr-XII distribution based on progressively censored samples with random removals. [39] studied the empirical estimates of the reliability of the Burr type-XII distribution using the LINEX error loss function based on progressive type-II censored samples.

In reliability analysis, the progressive Type-II censoring scheme is the most applicable in life test experiments, it is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at different stages before testing is terminated. [2] and [3] presented comprehensive studies on the topic of progressive censoring and its applications. However, the trial time can be quite long due to some highly reliable units. Thus, [12] recently proposed a censoring scheme called the type-II progressive hybrid censoring scheme. However, the progressive hybrid censoring scheme has the disadvantage of having very few failures occur before the time point **T**. In order to provide a guarantee of the number of failures observed as well as the time to complete the test, [8, 9] proposed the generalized progressive hybrid censoring scheme (GPHCS), which modifies the progressive hybrid censoring scheme. If the number of failures is less than m, it allows the experiment to run for an extended time T to observe at least K failures.The GPHCS has been described in [21. 22].

Thus, given a generalized progressive hybrid censored sample, the likelihood function can be written in a unified form as follows:

$$L(\bar{X};\theta) = C\prod_{i=1}^{N} f(x_i)\,[1-F(x_i)]^{R_i}[1-F(T)]^{R_T^*\delta}, \tag{5}$$

$$N = \begin{cases} m, & \delta = 0 \quad , \text{ if } X_{k:m:N} \le X_{m:m:N} < T \\ k, & \delta = 0 \quad , \text{ if } T < X_{K:m:N} \le X_{m:m:N}, \\ l, & \delta = 1 \quad , \text{ if } X_{k:m:N} < T < X_{m:m:N} \end{cases}$$

where $\underline{X} = (X_1, X_2, \ldots, X_N)$ and R_T^* is the number of surviving units that are removed at the stopping time $T^* = \max\{X_{k:m:N}, \min\{X_{m:m:N}, T\}\}$.

The GPHCS has been applied to some distributions such as the Weibull distribution, see [9], the inverse Weibull distribution, see [21, 23], the exponential distribution, see [8] and [12], the Rayleigh distribution, see [7], and the shape-scale family, see [22].

2 Conditional inference

This section outlines the conditional inference, which converts the standard likelihood function to one depending on pivotal quantities and ancillary statistics based on the generalized progressive hybrid censoring scheme. For more details, see [13, 14, 15] and [16, 17], who used the pivotal quantities for the parameters as tools for constructing the confidence intervals for the unknown parameters based on complete and censored samples.

For the generalized shape-scale family, we can write the likelihood function of (2) based on the GPHCS (5) as

$$L(\underline{x};\theta) = C\prod_{i=1}^{N} \alpha\beta\gamma g^{\alpha-1}(x_i)g'(x_i)(1+\beta g^{\alpha}(x_i))^{-\gamma-1}$$
$$\times (1+\beta g^{\alpha}(x_i))^{-\gamma R_i}\,(1+\beta g^{\alpha}(T))^{-\gamma\delta R_T} \tag{6}$$
$$= C(\alpha\beta\gamma)^N \prod_{i=1}^{N} \frac{g^{\alpha-1}(x_i)g'(x_i)}{(1+\beta g^{\alpha}(x_i))})$$
$$\times \exp[-\gamma[\sum_{i=1}^{N}(1+R_i)\log(1+\beta g^{\alpha}(x_i))$$
$$+\delta R_T^* \log(1+\beta g^{\alpha}(T))]]$$

Let $\beta g^{\alpha}(x) = [\beta^{\hat{\alpha}/\alpha}\hat{\beta}g^{\hat{\alpha}}(x)/\hat{\beta}]^{\frac{\alpha}{\hat{\alpha}}} = (z_2 a_i)^{Z_1}$

Thus,
$Z_1 = \frac{\alpha}{\hat{\alpha}}$, $Z_2 = \beta^{1/Z_1}/\hat{\beta}$, $Z_3 = \gamma/\hat{\gamma}$, $a_T = \hat{\beta}g^{\hat{\alpha}}(x_T)$ and $a_i = \hat{\beta}g^{\hat{\alpha}}(x_i)$ for $i = 1,2,\ldots,N$, where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are the MLEs of α , β and γ and $\underline{A} = (a_1, a_2 \ldots, a_{N-3})$ forms a set of ancillary statistics satisfies the maximum likelihood equations, thus only N-3 of which are functionally independent. The MLE of γ can be derived as:

$$\hat{\gamma} = N/[\sum_{i=1}^{N}(1+R_i)\ln\left(1+\beta g^{\alpha}(x_i)\right)$$

$+ \delta R_T^* \ln\left(1+\beta g^{\alpha}(T)\right)\Big]$

Theorem:

Let $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ be the MLEs of α , β and γ based on the generalized progressive hybrid censored sample. Thus, the joint conditional PDF of

$Z_1 = \alpha/\hat{\alpha}$, $Z_2 = \beta^{1/Z_1}/\hat{\beta}$, $Z_3 = \gamma/\hat{\gamma}$ given

$\underline{A} = (a_1, a_2 \ldots, a_{N-3})$

is of the form

$$f\left(z_1, z_2, z_3 \middle| \underline{A}\right) = D Z_1^{N-1} Z_2^{N z_1 - 1} Z_3^{N-1}$$

$$\times \frac{\exp[-N Z_3 + (Z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(Z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N} \quad (7)$$

Proof:

Make the change of variables from $(x_1, x_2, x_3, \ldots, x_N)$ that has joint density function (6) to $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, a_1, a_2 \ldots, a_{N-3})$. This transformation can be written as follows:

$g(x_i) = (a_i/\hat{\beta})^{1/\hat{\alpha}}$, $i = 1,2,\ldots, N-3$, $g(x_{N-2}) = (a_{n-2}/\hat{\beta})^{1/\hat{\alpha}}$, $g(x_{N-1}) = (a_{N-1}/\hat{\beta})^{1/\hat{\alpha}}$ and $g(x_N) = (a_N/\hat{\beta})^{1/\hat{\alpha}}$, where a_N, a_{N-1} and a_{N-2} can be expressed in terms of $a_1, a_2 \ldots, a_{N-3}$. The Jacobin of this transformation is of the form

$$|J| = \prod_{i=1}^{N} a_i^{\frac{1}{\hat{\alpha}}} \hat{\beta}^{-\frac{N}{\hat{\alpha}} - N} \hat{\alpha}^{-N} K(A, N),$$

which is independent of Z_1, Z_2 and Z_3. Therefore the joint PDF of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \underline{A}$ can be derived as

$$f\left(\hat{\alpha}.\hat{\beta}, \hat{\gamma};\ \underline{A}\right) = C(\alpha\beta\gamma)^N exp[-N\gamma/\hat{\gamma}]$$

$$\times \prod_{i=1}^{N} \frac{(a_i/\hat{\beta})^{\frac{\alpha-1}{\hat{\alpha}}} \frac{a_1{}'}{\hat{\alpha}\hat{\beta}} (a_i/\hat{\beta})^{\frac{1}{\hat{\alpha}}-1}}{\left(1 + [\beta^{\hat{\alpha}/\alpha}\hat{\beta} g^{\hat{\alpha}}(x)/\hat{\beta}]^{\frac{\alpha}{\hat{\alpha}}}\right)}.$$

Making further change of variables from $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}; \underline{A})$ to $(Z_1, Z_2, Z_3; \underline{A})$, the Jacobin of this transformation can be derived as follows:

$$|J| = \left|\frac{\partial(\hat{\alpha}, \hat{\beta}, \hat{\gamma})}{\partial(z_1, z_2, z_3)}\right| = \begin{vmatrix} -\alpha/z_1^2 & 0 & 0 \\ * & -\frac{\beta^{\frac{1}{Z_1}}}{Z_2^2} & 0 \\ 0 & 0 & -\gamma/z_3^2 \end{vmatrix}$$

$$= \frac{\hat{\alpha}\hat{\beta}\hat{\gamma}}{z_1 z_2 z_3} \propto \frac{1}{z_1 z_2 z_3}.$$

Thus,

$$f(z_1, z_2, z_3;\ \underline{A}) = C Z_1^{N-1} Z_2^{N z_1 - 1} Z_3^{N-1}$$

$$\times \frac{\exp[-N z_3 + (z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(z_2 a_T)^{Z_1})]^N}$$

where D is the normalizing constant.

Finally, the joint conditional distribution function of (Z_1, Z_2, Z_2) given $\underline{A}$ can be derived as in (7)■

Thus, based on (7), we can derive the marginal conditional distribution for each Z_1 and Z_2 given $\underline{A}$ as follows:

$$f(z_1, z_2|A) = D\Gamma(N) Z_1^{N-1} Z_2^{N z_1 - 1}$$

$$\times \frac{\exp[(Z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(Z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N}$$

$$D^{-1} = \Gamma(N) \int_0^{\infty}\int_0^{\infty} Z_1^{N-1} Z_2^{N Z_1 - 1}$$

$$\times \frac{\exp[(Z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(Z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N} dZ_1 dZ_2$$

$$f(z_1|A) = D\Gamma(N) \int_0^{\infty} Z_1^{N-1} Z_2^{N Z_1 - 1}$$

$$\times \frac{\exp[(z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N} dZ_2 \quad (8)$$

$$f(z_2|A) = D\Gamma(N) \int_0^{\infty} Z_1^{N-1} Z_2^{N Z_1 - 1}$$

$$\times \frac{\exp[(Z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(Z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N} dZ_1 \quad (9)$$

$$f(z_3|A) = D \int_0^{\infty}\int_0^{\infty} Z_1^{N-1} Z_2^{N z_1 - 1} Z_3^{N-1}$$

$$\times \frac{\exp[-N Z_3 + (Z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1+(Z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1+R_i)\ln(1+(Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1+(Z_2 a_T)^{Z_1})]^N} dZ_2 dZ_1 \quad (10)$$

The conditional estimators for the pivotal Z_1, Z_2 and Z_3 can be derived from (8), (9) and (10) and fiducially transformed to the parameters α , β and γ separately.

[13] developed the conditional inference for determining confidence intervals for unknown parameters based on pivotal quantities; thus, the statistical significance of this work is determining point estimators for the unknown parameters. As a result, when we integrate (8) with respect to Z_1 from z_{10} to z_{11}, we get the point estimator for the parameter α say. Thus,

$\mathrm{F}(\mathrm{z}_{11}|\mathrm{A}) - \mathrm{F}(\mathrm{z}_{10}|\mathrm{A}) = \mathrm{U} - \mathrm{W}(\mathrm{z}_{10})$
(11)

where $W(z_{10}) = \mathrm{D}\Gamma(\mathrm{N}) \int_0^{z_{10}} \int_0^{\infty} \mathrm{Z}_1^{\mathrm{N}-1} \mathrm{Z}_2^{\mathrm{N}Z_1-1}$

$$\times \frac{\exp[(z_1 - 1)\sum_{i=1}^{N} \ln(a_i) - \sum_{i=1}^{N} \ln(1 + (z_2 a_i)^{Z_1})]}{[\sum_{i=1}^{n}(1 + R_i)\ln(1 + (Z_2 a_i)^{Z_1}) + \delta R_T^* \ln(1 + (Z_2 a_T)^{Z_1})]^N} dZ_1 dZ_2$$

and U is a uniform random number from the uniform distribution $\mathrm{U}(0,1)$.
It is known that the second order approximation of the first derivative $\frac{dF(z_{11}|A)}{dZ_1}$, which is defined by the centered differencing, can be written as

$$\frac{dF(Z_1^*|\underline{A})}{dZ_1} = \frac{F(z_{11}|\underline{A}) - F(z_{10}|\underline{A})}{Z_{11} - Z_{10}} = f(z_1^*|\underline{A}), \qquad (12)$$

$z_{10} < z_1^* < z_{11}$.

From (11) and (12) we can derive the conditional estimator for Z_1 as:

$$z_{i+1} = z_i + C[U - W(z_i)]. \qquad (13)$$

The convergence of (13) is guaranteed by the condition $C < \frac{D}{f(z_1^*|\underline{A})}$, where D is the normalizing constant.

The iterative process is repeated for $i = 0,1,2,3, ...$ until two consecutive numerical solutions are nearly identical, that is if $|z_{i+1} - z_i| < 1E - 05$. As a result, we can get the successive approximation for the pivotal Z_1 rom (13) and fiducially transform for α as follows: $\alpha^* = \hat{\alpha}\, z_{i+1}$. Similarly, the estimators for β and γ can be derived from (9) and (10) respectively.

3 Bayes estimations

In this section, the Bayes estimators for the parameters α, β and γ will be derived using the informative gamma prior and the non-parametric kernel prior, based on two different loss functions.
Firstly, the squared error loss function (SQEL), $L(g(\theta), \hat{g}(\theta)) = (g(\theta) - \hat{g}(\theta))^2$. For this loss function the Bayes estimator that minimizes the risk function is given by $\hat{g}(\theta) = E_\theta(g(\theta)|x)$.
Secondly, the compound LINEX loss function (LNXL), see [33].
$L(\Delta) = L_\delta(\Delta) + L_{-\delta}(\Delta) = e^{\delta\Delta} + e^{-\delta\Delta} - 2,\ \delta > 0.$

It is named LINEX-based loss function, where

$L_\delta(\Delta) = exp[\,\delta\Delta] - \delta\Delta - 1,\ \Delta = \hat{g}(\theta) - g(\theta),$
$\delta \neq 0,$

is the LINEX loss function (LLF) that has been introduced in [25] and [31]. The Bayes estimator of the parameter θ that minimizes the risk function can be derived as follows:

$$\hat{g}_L(\theta) = \frac{1}{2\delta} \log\left(\frac{E\left(e^{\delta g(\theta)}\middle|X\right)}{E\left(e^{-\delta g(\theta)}\middle|X\right)}\right).$$

3.1 Informative prior

We consider the unknown parameters α, β and γ have independent gamma prior distributions with the joint probability density function, which is given by:

$$h(\alpha, \beta, \gamma) \propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha - d\beta - f\gamma}, \qquad (14)$$

where the hyper-parameter a, b, c, d, e and f are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

3.2 Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function $g(\alpha, \beta, \gamma)$ with support on ($\mathbf{0}, \infty$), which is defined as

$$\hat{g}(\alpha, \beta, \gamma) = \frac{1}{Nh_1h_2h_3} \sum_{i=1}^{N} K\left(\frac{\alpha - \alpha_i}{h_1}, \frac{\beta - \beta_i}{h_2}, \frac{\gamma - \gamma_i}{h_3}\right), \qquad (15)$$

$h_i, i = 1,2,3$ are called the bandwidths or smoothing parameters, which chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$. The optimal choices for $\square_i$ for minimizing mean squared errors are $h_i = 1.06 S_i n^{-0.2}$, where S_i is the sample standard deviations of the random variables α_i , β_i and γ_i respectively**.** The optimal choice for the kernel function $\boldsymbol{K}(.,.,.)$ can be used as the **tr**ivariate standard normal distribution. The basic elements associated with the kernel density estimation function have been studied extensively in [10, 11]. Based on the properties of the maximum likelihood estimates (MLEs) of the parameters, which converge in probability to the original parameters, the kernel prior estimate can be derived. The kernel prior has been used in [1] and [18, 19, 20]. Thus, using the joint priors (14) and (15) with the likelihood function of the GPHCS (4), the posterior density for the parameters **α**, **β** and **γ** can be written in a unified form as follows:

$f(\alpha, \beta, \gamma|\underline{X}) = Kq(\alpha, \beta, \gamma)L(\underline{X}\,; \alpha, \beta, \gamma)$, where

$$q(\alpha,\beta,\gamma) = \hat{g}(\alpha,\beta,\gamma)h(\alpha,\beta,\gamma)$$

$$= \hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\beta)\,\hat{g}_3^{p_3}(\gamma)\,\alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma},$$

is the general prior distribution function with $p_1 = p_2 = p_3 = 0$ for the informative prior (14), and $p_1 = p_2 = p_3 = 1,\ a = c = e = 1$, and $b = d = f = 0$ for the kernel prior (15).
Thus, the log posterior density can be written as

$$H(\alpha,\beta,\gamma) = [p_1 \ln(\hat{g}_1(\alpha)) + p_2 \ln(\hat{g}_2(\beta))$$
$$+p_3 \ln(\hat{g}_3(\gamma)) + (N+a-1)\ln\alpha - b\alpha - d\beta$$
$$+(N+c-1)\ln\beta + (N+e-1)\ln\gamma - f\gamma$$
$$+(\alpha-1)\sum_{i=1}^{N}\ln x_i - \sum_{i=1}^{N}\ln(1+\beta x_i{}^{\alpha})$$
$$-\gamma\left[\sum_{i=1}^{N}(1+R_i^*)\ln(1+\beta x_i{}^{\alpha}) + \delta R_T^*\ln(1+\beta T^{\alpha})\right]. \quad (16)$$

Thus, based on (16), we can use the Tierney-Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [30] introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero in the interior of the parameter space. For details, let $w(\alpha,\beta,\gamma)$ be a smooth, positive function on the parameter space. The posterior expectation of $w(\alpha,\beta,\gamma)$ can be obtained as

$$w^* = E\left(w(\alpha,\beta,\gamma)\middle|\underline{X}\right) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{NH^*(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}{\int_0^\infty\int_0^\infty\int_0^\infty e^{NH(\alpha,\beta,\gamma)}d\alpha d\beta d,} \quad (17)$$

where $H = \ln f(\alpha,\beta,\gamma|\underline{X})/N$, and

$$H^* = H + \ln w(\alpha,\beta,\gamma)/N.$$

For (α,β,γ) the Bayes estimator using Tierney-Kadane approximation for $w(\alpha,\beta,\gamma)$ can be obtained as

$$w^* = \sqrt{|\Sigma^*|/|\Sigma|}\exp[N[H^*(\alpha,\beta,\gamma) - H(\alpha,\beta,\gamma)]'$$

where $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ maximize the $H(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $H^*(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$, respectively.
Let

$$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1} \quad \text{and}$$

$$|\Sigma^*| = \begin{vmatrix} H_{11}^* & H_{12}^* & H_{13}^* \\ H_{21}^* & H_{22}^* & H_{23}^* \\ H_{31}^* & H_{32}^* & H_{33}^* \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ at $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ respectively. The derivatives of $H(\alpha,\beta,\gamma)$ and $H^*(\alpha,\beta,\gamma)$ have been derived in the Appendix A.

4 Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the three-parameter Burr type XII model in practice and to illustrate that this distribution can be considered a good lifetime model for some new areas of applications, compared with many other known distributions such as the Weibull distribution. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Chi-Square (CH2) tests for a significance level equal to 0.05.

4.1 The Reactor Pumps Data

In this section, a real data set for secondary nuclear pumps has been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feedwater pump that takes from the desecrator storage tank feed water pressure up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high temperature, high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the times between the failures of the secondary reactor pumps. [26] and [27] have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:
2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060,

0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the Burr type-XII model to be a good fit for this dataset as shown in Figure (1 a). For studying the reliability of these reactor pumps based on this dataset,
we obtain estimates for parameters representing the shape of the failures between pumps using our model to determine the behavior of the failed pumps. We noticed that the conditional and Bayes estimates for α lies in the interval [0.5,0.6], for γ lies in the interval [0.9, 1.0] and for β lies in the interval [0.01,0.02] indicate that the above dataset is heavily right-skewed, which means the failure rate decreases with increasing time, see Figure (1 b), and that means decreasing the reliability of safety mechanisms with increasing time.

4.2 The Vinyl Chloride Data

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. where it is known that a concentration of vinyl chloride in drinking water of 0.5 mg/liter has been calculated to be associated with an increased risk of liver and brain tumors for exposure beginning in adulthood, and that it would double cancer risk for continuous exposure beginning at birth. Therefore, we consider the dataset used by [4], which represents 34 data points in mg/L from the vinyl chloride obtained from clean upgrade monitoring wells, as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3,3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.
We found the Burr type-XII model to be a very good fit for this dataset, as shown in Figure (2 a). For studying the concentration of vinyl chloride in the water of these wells based on this dataset, we find estimates for the parameters that represent the shape of the concentration using our model to determine the average concentration in the water. We discovered that the conditional and Bayesian estimates for α lie in interval [0.7, 0.8] and for γ lies in the interval [1.1, 1.3] indicating that the above dataset is moderately right skewed and that means the concentration decreases with increasing time, see Figures (2 b). Also, the conditional and Bayes estimates for β lies in the interval [0.07, 0.09], which ensures the dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time, therefore, monitoring these wells is very significant. From the results in Table 2, the three-parameter Burr type-XII model is a very good fit for these data sets since the calculated values for the goodness of fit tests are less than the critical values. Moreover, the power of the tests is greater than the significance level of 0.05 for the tests. From the results in Table 3, the estimated values of MSEs based on the conditional inference are smaller than those for the Bayesian inference for these data sets, considering the MLEs are the true values of the parameters.

5 Simulation Study

The purpose of the simulation study is to compare the performance of the estimates using the conditional and Bayes methods based on the informative gamma and kernel priors with two different loss functions using two criteria: the average bias (AVB) and the mean squared error (MSE) as given by:

$$\mathrm{AVB} = \frac{1}{L}\sum_{i=1}^{L} |\hat{\theta}_i - \theta\,|, \qquad \mathrm{MSE} = \sum_{i=1}^{L}\left(\widehat{\theta_i} - \theta\right)^2 / L$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the hyperparameters of α, β and γ as follows:
$a = c = e = 2, b = d = f = 5$ and the values for the parameters are $\alpha = (1,2)$, $\beta = (0.75,\ 1.75)$ and $\gamma = (2,\ 3)$ respectively. Using the parameter values for generating different samples from the three-parameter Burr type-XII distribution with sizes $N = 20$, 40 and 60 to represent small, moderate and large sizes. To assess the performance of these estimates, the average Bias (AVB) and the MSEs for each were calculated using 1000 replicates. An algorithm for generating the generalized progressive hybrid censoring scheme has been written, see [21, 22].

From the simulation results in Tables 3-8, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

i. It is obvious that, in general, for the parameters α, β and γ the average Bias and MSEs values based on the conditional inference outperform the corresponding values based on the Bayes inference for the different loss functions.
ii. The estimated AVB and MSE values increase with increasing hyperparameters and decrease with increasing parameter values.

iii. The estimated AVB and MSEs values decrease with increasing the termination time of the experiment T, and the sample sizes, as expected for all methods.

iv. In general, the estimated MSE values for the Bayes method based on the LINEX-based loss function are less than those based on the squared error loss functions and are close to those based on the conditional estimations.

In conclusion, the conditional inference competes and outperforms the Bayes inference based on the informative and kernel priors.

Conclusions

In this study, it is found that, for the generalized progressive hybrid censored data, the conditional inference is strongly unbiased and much more efficient than the Bayesian inference based on the informative and kernel priors for two different loss functions. The Bayes estimates based on the kernel prior are more efficient than those based on the informative gamma prior, and they are very close to those based on the conditional inference. Thus, the statistical significance of the conditional inference is its efficiency compared to most estimation methods and the fact that it does not depend on subjective information. It is also reliable for social sciences and psychology researchers.

Declaration of conflicting interests

The author declares no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Appendix A:

Based on the loglikelihood function (16), we derive the following derivatives:

$$\frac{\partial H}{\partial \alpha} = [p_1\frac{\hat{g}_1^{/}(\alpha)}{\hat{g}_1(\alpha)} + (N+a-1)/\alpha - b + \sum_{i=1}^{N} \ln x_i$$

$$-\sum_{i=1}^{N}\frac{\beta {x_i}^{\alpha}\ln x_i}{1+\beta {x_i}^{\alpha}} \; - \gamma(\sum_{i=1}^{N}(1+R_i^*)\frac{\beta {x_i}^{\alpha}\ln x_i}{1+\beta {x_i}^{\alpha}}$$

$$+\frac{\beta T^{\alpha}\ln T}{1+\beta T^{\alpha}})]/N$$

$$\frac{\partial^2 H}{\partial \alpha^2} = [p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{//}(\alpha)-{\hat{g}^{/}_1}^2(\alpha)}{\hat{g}_1^2(\alpha)} - (N+a-1)/\alpha^2$$

$$-\sum_{i=1}^{N}\frac{(1+\beta {x_i}^{\alpha})\beta {x_i}^{\alpha}(\ln x_i)^2-(\beta {x_i}^{\alpha}\ln x_i)^2}{(1+\beta {x_i}^{\alpha})^2}$$

$$-\gamma[\textstyle\sum_{i=1}^{N}(1+R_i^*)\frac{(1+\beta {x_i}^{\alpha})\beta {x_i}^{\alpha}(\ln x_i)^2-(\beta {x_i}^{\alpha}\ln x_i)^2}{(1+\beta {x_i}^{\alpha})^2}$$

$$+\delta R_T^*\frac{(1+\beta T^{\alpha})\beta T^{\alpha}(\ln T)^2-(\beta T^{\alpha}\ln T)^2}{(1+\beta T^{\alpha})^2}]/N$$

$$\frac{\partial^2 H}{\partial \alpha\,\partial \beta} = [-\sum_{i=1}^{N}\frac{(1+\beta {x_i}^{\alpha}){x_i}^{\alpha}\ln x_i-\beta {x_i}^{2\alpha}\ln x_i}{(1+\beta {x_i}^{\alpha})^2}$$

$$-\gamma[\sum_{i=1}^{N}(1+R_i^*)\frac{(1+\beta {x_i}^{\alpha}){x_i}^{\alpha}\ln x_i-\beta {x_i}^{2\alpha}\ln x_i}{(1+\beta {x_i}^{\alpha})^2}$$

$$+\delta R_T^*\frac{(1+\beta T^{\alpha})T^{\alpha}\ln T-\beta T^{2\alpha}\ln T}{(1+\beta T^{\alpha})^2}]/N$$

$$\frac{\partial H}{\partial \beta} = [p_2\frac{\hat{g}_2^{'}(\beta)}{\hat{g}_2(\beta)} + (N+c-1)/\beta - d - \sum_{i-1}^{N}\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}}$$

$$-\gamma[\textstyle\sum_{i-1}^{N}(1+R_i)\frac{x_i^{\alpha}}{1+\beta x_i^{\alpha}} + \delta R_T^*\frac{T^{\alpha}}{1+\beta T^{\alpha}}]]/N$$

$$\frac{\partial^2 H}{\partial \beta^2} = [p_2\frac{\hat{g}_2(\beta)g_2^{//}(\beta)-{\hat{g}_2^{/}}^2(\beta)}{{\hat{g}_2}^2(\beta)} - (N+c-1)/\beta^2$$

$$+\sum_{i-1}^{N}\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2} + \gamma[\sum_{i-1}^{N}(1+R_i)\frac{x_i^{2\alpha}}{(1+\beta x_i^{\alpha})^2} + \delta R_T^*\frac{T^{2\alpha}}{(1+\beta T^{\alpha})^2}]]/N.$$

$$\frac{\partial H}{\partial \gamma} = [p_3\frac{\hat{g}_3^{'}(\gamma)}{\hat{g}_3(\gamma)} + (N+e-1)/\gamma - f$$

$$-[\sum_{i=1}^{N}(1+R_i^*)\ln(1+\beta x_i^{\alpha}) + \delta R^*\ln(1+\beta T^{\alpha})]]/N$$

$$\frac{\partial^2 H}{\partial \gamma^2} = [p_3\frac{\hat{g}_3(\gamma)g_3^{//}(\gamma)-{\hat{g}_3^{/}}^2(\gamma)}{{\hat{g}_3}^2(\gamma)} - (N+e-1)/\gamma^2]/N$$

$$\frac{\partial H}{\partial \gamma\,\partial \alpha} = -[\sum_{i=1}^{N}(1+R_i^*)\frac{\beta {x_i}^{\alpha}\ln x_i}{1+\beta {x_i}^{\alpha}} + \delta R_T^*\frac{\beta T^{\alpha}\ln T}{1+\beta T^{\alpha}}]/N$$

$$\frac{\partial H}{\partial \gamma\,\partial \beta} = -[\sum_{i=1}^{N}(1+R_i^*)\frac{{x_i}^{\alpha}}{1+\beta {x_i}^{\alpha}} + \delta R_T^*\frac{T^{\alpha}}{1+\beta T^{\alpha}}]/N$$

where the r^{th} derivative of the kernel density estimation can be defined as

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r} = {\hat{g}_1}^r(\alpha) = \frac{1}{Nh_1^{r+1}}\textstyle\sum_{i=1}^{N}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right),$$

where r=0,1,2,3,-----. (*)

Using the standard normal kernel function in (*), we have

$$\hat{g}_1(\alpha) = \frac{1}{Nh_1\sqrt{2\pi}} \sum_{i=1}^{N} e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha) = -\frac{1}{Nh_1^2\sqrt{2\pi}} \sum_{i=1}^{N} (\frac{\alpha-\hat{\alpha}_i}{h_1})\, e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2}$$

$$\hat{g}_1''(\alpha) = \frac{1}{Nh_1^3\sqrt{2\pi}} \sum_{i=1}^{N} [(\frac{\alpha-\alpha_i}{h_1})^2 - 1]\, e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

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Table 1: The critical and calculated values for the K-S, A-D and CH2 tests and their powers(p-values) for the Burr type-XII model.

Data	The Tests	Calculated value	Critical value	p-values	MLES		
					α	β	γ
Reactor pumps data N=23	K-S	0.4719	0.9001	0.8163	0.6944	0.0202	1.2035
	A-D	0.2466	0.7714	0.7563			
	CH2	9.5741	12.234	0.1355			
The vinyl Chloride data N=34	K-S	0.6064	0.9074	0.4872	0.9234	0.1238	1.4186
	A-D	0.3010	0.7711	0.6134			
	CH2	5.6472	15.086	0.3854			

Table 2: The estimate and the average bias (AVB) and mean squared errors (MSEs) for the parameter α, β and γ based on the conditional and Bayes methods under squared error loss function based on the GPHCS: for $m = N/2,\ k = m/2$.

Samples	T	Param	Conditional Estamite		Gamma Prior		Kernel prior	
			Estimate	AVB(MSE)	Estimate	AVB(MSE)	Estimate	AVB(MSE)
The reactor Pumps data n=23	0.5	α	0.6349	0.0594(0.00353)	0.5732	0.121(0.0147)	0.5747	0.119(0.0143)
		β	0.00816	0.0121(1.45E-04)	0.0164	0.004(1.5E05)	0.0164	0.004(1.5E-05)
		γ	1.0829	0.1205(0.0145)	0.9741	0.229(0.0526)	0.9836	0.219(0.0483)
	3.5	α	0.6337	0.0607(0.00368)	0.5763	0.118(0.0139)	0.5773	0.117(0.0137)
		B	0.008162	0.01206(1.5E-04)	0.0165	0.004(1.4E05)	0.0165	0.004(1.4E-05)
		γ	1.08292	0.12049(0.0145)	0.9819	0.223(0.0491)	0.9892	0.214(0.0459)
The vinyl Chloride data n=34	0.25	α	0.8374	0.0860(0.00739)	0.7524	0.171(0.0293)	0.7567	0.167(0.0278)
		β	0.07168	0.05215(0.002719)	0.0979	0.026(6.7E04)	0.0985	0.025(6.4E-04)
		γ	1.2752	0.1434(0.02507)	1.1262	0.292(0.0855)	1.1501	0.268(0.0721)
	4.5	α	0.8348	0.0887(0.007865)	0.7641	0.159(0.0254)	0.7668	0.157(0.0245)
		β	0.07175	0.05209(0.00271)	0.0982	0.026(6.6E04)	0.0992	0.026(6.1E-04)
		γ	1.2758	0.1428(0.02038)	1.13264	0.286(0.0818)	1.1572	0.261(0.0683)

Table 3: The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the Burr-XII parameter α using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior SQEL	Gamma Prior LNXL	Kernel Prior SQEL	Kernel Prior LNXL
20	10	5	0.75	1	2	0.0905(0.0087)	0.1744(0.0314)	0.1179(0.0144)	0.1658(0.0288)	0.1205(0.0150)
				2	3	0.0779(0.0066)	0.3965(0.1600)	0.2722(0.0742)	0.3481(0.1395)	0.2779(0.0773)
			1.5	1	2	0.2185(0.0480)	0.1684(0.0302)	0.1104(0.0127)	0.1466(0.0224)	0.1126(0.0132)
				2	3	0.2163(0.0469)	0.4643(0.3120)	0.2706(0.0733)	0.3301(0.1440)	0.2746(0.0755)
		8	0.75	1	2	0.0934(0.0092)	0.1577(0.0274)	0.1183(0.0146)	0.1534(0.0241)	0.1211(0.0152)
				2	3	0.0816(0.0071)	0.3473(0.1214)	0.2734(0.0749)	0.3273(0.1078)	0.2793(0.0781)
			1.5	1	2	0.2198(0.0485)	0.1521(0.0244)	0.1154(0.0137)	0.1446(0.0220)	0.1176(0.0142)
				2	3	0.2155(0.0465)	0.3471(0.1291)	0.2718(0.0739)	0.3138(0.0990)	0.2760(0.0762)
	15	8	0.75	1	2	0.0927(0.0091)	0.1596(0.0263)	0.1208(0.0151)	0.1559(0.0248)	0.1234(0.0156)
				2	3	0.0818(0.0071)	0.3474(0.1219)	0.2738(0.0751)	0.3309(0.1179)	0.2797(0.0783)
			1.5	1	2	0.2202(0.0486)	0.1486(0.0233)	0.1130(0.0132)	0.1410(0.0204)	0.1152(0.0137)
				2	3	0.2158(0.0467)	0.3568(0.2960)	0.2721(0.0741)	0.3126(0.0979)	0.2762(0.0764)
		11	0.75	1	2	0.0970(0.0098)	0.1542(0.0262)	0.1239(0.0157)	0.1782(0.7657)	0.1263(0.0163)
				2	3	0.0828(0.0072)	0.3284(0.1088)	0.2746(0.0755)	0.3183(0.1019)	0.2805(0.0788)
			1.5	1	2	0.2229(0.0498)	0.1414(0.0210)	0.1157(0.0138)	0.1385(0.0196)	0.1178(0.0142)
				2	3	0.2170(0.0472)	0.3234(0.1082)	0.2720(0.0741)	0.3066(0.0941)	0.2765(0.0765)
40	20	10	0.75	1	2	0.0892(0.0083)	0.1454(0.0215)	0.1180(0.0143)	0.1459(0.0261)	0.1199(0.0147)
				2	3	0.0787(0.0065)	0.3298(0.1099)	0.2725(0.0743)	0.3146(0.1003)	0.2766(0.0765)
			1.5	1	2	0.2180(0.0476)	0.1390(0.0200)	0.1120(0.0128)	0.1322(0.0178)	0.1135(0.0131)
				2	3	0.2156(0.0466)	0.3454(0.1272)	0.2717(0.0738)	0.3045(0.0954)	0.2743(0.0753)
		15	0.75	1	2	0.0909(0.0086)	0.1413(0.0203)	0.1202(0.0147)	0.1411(0.0202)	0.1221(0.0151)
				2	3	0.0803(0.0067)	0.3137(0.0987)	0.2724(0.0743)	0.3047(0.0931)	0.2768(0.0767)
			1.5	1	2	0.2190(0.0481)	0.1320(0.0177)	0.1128(0.0130)	0.1295(0.0170)	0.1143(0.0133)
				2	3	0.2164(0.0469)	0.3126(0.0986)	0.2718(0.0739)	0.2978(0.0888)	0.2746(0.0755)
	30	15	0.75	1	2	0.0918(0.0087)	0.1403(0.0201)	0.1187(0.0144)	0.1392(0.0197)	0.1206(0.0148)
				2	3	0.0805(0.0067)	0.3137(0.0989)	0.2726(0.0744)	0.3054(0.0935)	0.2769(0.0767)
			1.5	1	2	0.2196(0.0483)	0.1336(0.0181)	0.1141(0.0132)	0.1306(0.0173)	0.1156(0.0135)
				2	3	0.2161(0.0468)	0.3117(0.0975)	0.2717(0.0739)	0.2979(0.0891)	0.2746(0.0754)
		23	0.75	1	2	0.0971(0.0096)	0.1372(0.0190)	0.1237(0.0155)	0.1379(0.0192)	0.1253(0.0159)
				2	3	0.0832(0.0071)	0.3001(0.0902)	0.2745(0.0754)	0.2988(0.0893)	0.2788(0.0778)
			1.5	1	2	0.2238(0.0501)	0.1299(0.0186)	0.1159(0.0136)	0.1282(0.0166)	0.1174(0.0139)
				2	3	0.2172(0.0472)	0.2961(0.0880)	0.2721(0.0741)	0.2924(0.0856)	0.2753(0.0758)
60	30	15	0.75	1	2	0.0894(0.0082)	0.1342(0.0182)	0.1193(0.0144)	0.1343(0.0182)	0.1208(0.0147)
				2	3	0.0787(0.0064)	0.3040(0.0925)	0.2728(0.0745)	0.2995(0.0900)	0.2762(0.0763)
			1.5	1	2	0.2177(0.0475)	0.1278(0.0165)	0.1133(0.0130)	0.1260(0.0160)	0.1144(0.0132)
				2	3	0.2157(0.0466)	0.3031(0.0921)	0.2716(0.0738)	0.2930(0.0862)	0.2739(0.0750)
		23	0.75	1	2	0.0906(0.0084)	0.1327(0.0178)	0.1188(0.0143)	0.1337(0.0183)	0.1203(0.0146)
				2	3	0.0802(0.0066)	0.2999(0.0900)	0.2729(0.0745)	0.2971(0.0884)	0.2764(0.0764)
			1.5	1	2	0.2188(0.0479)	0.1273(0.0169)	0.1139(0.0131)	0.1255(0.0159)	0.1151(0.0134)
				2	3	0.2164(0.0468)	0.2976(0.0887)	0.2717(0.0738)	0.2903(0.0844)	0.2740(0.0751)
	45	23	0.75	1	2	0.0910(0.0085)	0.1330(0.0179)	0.1187(0.0143)	0.1330(0.0179)	0.1202(0.0146)
				2	3	0.0803(0.0066)	0.2994(0.0897)	0.2727(0.0744)	0.2973(0.0893)	0.2762(0.0763)
			1.5	1	2	0.2186(0.0478)	0.1263(0.0163)	0.1133(0.0130)	0.1249(0.0158)	0.1145(0.0133)
				2	3	0.2163(0.0468)	0.2975(0.0888)	0.2721(0.0740)	0.2912(0.0849)	0.2744(0.0753)
		34	0.75	1	2	0.0973(0.0096)	0.1323(0.0177)	0.1229(0.0153)	0.1330(0.0178)	0.1243(0.0156)
				2	3	0.0827(0.0070)	0.2917(0.0851)	0.2741(0.0752)	0.2919(0.0852)	0.2777(0.0771)
			1.5	1	2	0.2231(0.0498)	0.1235(0.0154)	0.1148(0.0133)	0.1235(0.0154)	0.1159(0.0136)
				2	3	0.2171(0.0472)	0.2888(0.0840)	0.2720(0.0740)	0.2865(0.0821)	0.2745(0.0754)

Table 4: The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the Burr-XII parameter α using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=2 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.0928(0.0091)	0.1561(0.0250)	0.1220(0.0154)	0.1547(0.0246)	0.1246(0.0159)
				2	3	0.0803(0.0069)	0.3373(0.1142)	0.2730(0.0746)	0.3238(0.1068)	0.2792(0.0780)
			1.5	1	2	0.2193(0.0483)	0.1458(0.0220)	0.1140(0.0134)	0.1411(0.0207)	0.1162(0.0139)
				2	3	0.2165(0.0469)	0.3346(0.1130)	0.2719(0.0740)	0.3109(0.0969)	0.2762(0.0764)
		8	0.75	1	2	0.0916(0.0089)	0.1548(0.0245)	0.1203(0.0149)	0.1532(0.0239)	0.1229(0.0155)
				2	3	0.0797(0.0068)	0.3463(0.1203)	0.2730(0.0747)	0.3263(0.1083)	0.2790(0.0779)
			1.5	1	2	0.2194(0.0483)	0.1481(0.0226)	0.1127(0.0131)	0.1410(0.0204)	0.1149(0.0136)
				2	3	0.2161(0.0468)	0.3460(0.1219)	0.2720(0.0740)	0.3133(0.0986)	0.2762(0.0763)
	15	8	0.75	1	2	0.0940(0.0093)	0.1555(0.0246)	0.1218(0.0153)	0.1537(0.0241)	0.1243(0.0158)
				2	3	0.0797(0.0068)	0.3380(0.1145)	0.2736(0.0749)	0.3222(0.1040)	0.2796(0.0782)
			1.5	1	2	0.2197(0.0484)	0.1461(0.0222)	0.1146(0.0135)	0.1406(0.0202)	0.1167(0.0140)
				2	3	0.2161(0.0468)	0.3351(0.1133)	0.2720(0.0741)	0.3126(0.1004)	0.2763(0.0764)
		11	0.75	1	2	0.0964(0.0097)	0.1527(0.0241)	0.1245(0.0159)	0.1514(0.0232)	0.1268(0.0164)
				2	3	0.0828(0.0072)	0.3269(0.1077)	0.2741(0.0753)	0.3165(0.1003)	0.2802(0.0786)
			1.5	1	2	0.2233(0.0500)	0.1411(0.0208)	0.1151(0.0137)	0.1381(0.0195)	0.1173(0.0141)
				2	3	0.2167(0.0471)	0.3211(0.1046)	0.2719(0.0740)	0.3062(0.0938)	0.2764(0.0764)
40	20	10	0.75	1	2	0.0933(0.0090)	0.1370(0.0191)	0.1211(0.0150)	0.1377(0.0192)	0.1229(0.0154)
				2	3	0.0808(0.0068)	0.3133(0.1823)	0.2734(0.0748)	0.3012(0.0912)	0.2778(0.0772)
			1.5	1	2	0.2203(0.0486)	0.1291(0.0170)	0.1144(0.0133)	0.1282(0.0167)	0.1159(0.0136)
				2	3	0.2162(0.0468)	0.3015(0.0949)	0.2718(0.0739)	0.2937(0.0863)	0.2749(0.0756)
		15	0.75	1	2	0.0941(0.0091)	0.1381(0.0195)	0.1217(0.0150)	0.1381(0.0192)	0.1235(0.0154)
				2	3	0.0811(0.0068)	0.3045(0.0934)	0.2736(0.0749)	0.3006(0.0904)	0.2780(0.0773)
			1.5	1	2	0.2206(0.0487)	0.1283(0.0168)	0.1142(0.0132)	0.1279(0.0166)	0.1157(0.0136)
				2	3	0.2160(0.0467)	0.2984(0.0894)	0.2717(0.0739)	0.2932(0.0861)	0.2748(0.0755)
	30	15	0.75	1	2	0.0929(0.0089)	0.1377(0.0194)	0.1225(0.0152)	0.1373(0.0190)	0.1242(0.0156)
				2	3	0.0807(0.0067)	0.3020(0.0913)	0.2736(0.0749)	0.2997(0.0899)	0.2780(0.0773)
			1.5	1	2	0.2201(0.0485)	0.1308(0.0220)	0.1147(0.0134)	0.1279(0.0166)	0.1162(0.0137)
				2	3	0.2163(0.0468)	0.2989(0.0899)	0.2719(0.0739)	0.2932(0.0860)	0.2750(0.0757)
		23	0.75	1	2	0.0981(0.0098)	0.1370(0.0190)	0.1234(0.0154)	0.1374(0.0191)	0.1250(0.0158)
				2	3	0.0835(0.0072)	0.3008(0.0907)	0.2744(0.0753)	0.2987(0.0892)	0.2787(0.0777)
			1.5	1	2	0.2235(0.0500)	0.1281(0.0167)	0.1159(0.0136)	0.1282(0.0166)	0.1173(0.0139)
				2	3	0.2170(0.0471)	0.2955(0.0875)	0.2719(0.0740)	0.2919(0.0852)	0.2751(0.0757)
60	30	15	0.75	1	2	0.0941(0.0090)	0.1334(0.0194)	0.1214(0.0149)	0.1328(0.0178)	0.1229(0.0152)
				2	3	0.0816(0.0068)	0.2938(0.0864)	0.2735(0.0748)	0.2932(0.0860)	0.2771(0.0768)
			1.5	1	2	0.2212(0.0490)	0.1248(0.0158)	0.1149(0.0133)	0.1245(0.0156)	0.1161(0.0136)
				2	3	0.2160(0.0467)	0.2904(0.0844)	0.2719(0.0740)	0.2878(0.0828)	0.2744(0.0753)
		23	0.75	1	2	0.0934(0.0089)	0.1323(0.0177)	0.1216(0.0149)	0.1328(0.0178)	0.1230(0.0153)
				2	3	0.0807(0.0067)	0.2948(0.0889)	0.2734(0.0748)	0.2930(0.0859)	0.2770(0.0767)
			1.5	1	2	0.2202(0.0485)	0.1236(0.0154)	0.1140(0.0131)	0.1236(0.0154)	0.1152(0.0134)
				2	3	0.2163(0.0468)	0.2908(0.0847)	0.2719(0.0740)	0.2877(0.0828)	0.2744(0.0753)
	45	23	0.75	1	2	0.0943(0.0091)	0.1320(0.0176)	0.1226(0.0152)	0.1325(0.0177)	0.1239(0.0155)
				2	3	0.0813(0.0067)	0.2928(0.0860)	0.2740(0.0751)	0.2922(0.0854)	0.2776(0.0771)
			1.5	1	2	0.2209(0.0488)	0.1240(0.0156)	0.1150(0.0134)	0.1239(0.0155)	0.1162(0.0136)
				2	3	0.2163(0.0468)	0.2880(0.0831)	0.2716(0.0738)	0.2864(0.0821)	0.2742(0.0752)
		34	0.75	1	2	0.0973(0.0096)	0.1335(0.0183)	0.1235(0.0154)	0.1334(0.0179)	0.1248(0.0157)
				2	3	0.0827(0.0070)	0.2918(0.0853)	0.2739(0.0751)	0.2917(0.0851)	0.2775(0.0770)
			1.5	1	2	0.2232(0.0499)	0.1238(0.0156)	0.1150(0.0134)	0.1237(0.0154)	0.1162(0.0136)
				2	3	0.2169(0.0471)	0.2881(0.0831)	0.2720(0.0740)	0.2863(0.0820)	0.2745(0.0754)

Table 5: The Average bias (ABS) and Mean Square Errors (MSEs) in parentheses for the Burr-XII parameter β using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1085(0.0118)	0.2922(0.3084)	0.1307(0.0171)	0.2430(0.0936)	0.1301(0.0169)
				2	3	0.1944(0.0380)	0.2806(0.4337)	0.1224(0.0150)	0.2492(0.0945)	0.1221(0.0149)
			1.5	1	2	0.0956(0.0091)	0.6098(0.7360)	0.3083(0.0951)	0.4466(0.2806)	0.2813(0.0791)
				2	3	0.0536(0.0042)	0.5632(0.7684)	0.2562(0.0657)	0.4198(0.2365)	0.2461(0.0606)
		8	0.75	1	2	0.1004(0.0101)	0.2494(0.1390)	0.1242(0.0154)	0.2369(0.5647)	0.1242(0.0154)
				2	3	0.1211(0.0148)	0.2479(0.1718)	0.1200(0.0144)	0.1948(0.0419)	0.1199(0.0144)
			1.5	1	2	0.0932(0.0087)	0.5756(0.9279)	0.2766(0.0765)	0.3779(0.1521)	0.2624(0.0688)
				2	3	0.0389(0.0015)	0.5761(0.6784)	0.2487(0.0618)	0.3594(0.1366)	0.2423(0.0587)
	15	8	0.75	1	2	0.1005(0.0101)	0.2459(0.1141)	0.1242(0.0154)	0.2174(0.3523)	0.1242(0.0154)
				2	3	0.1217(0.0149)	0.2529(0.1368)	0.1200(0.0144)	0.1933(0.0403)	0.1199(0.0144)
			1.5	1	2	0.0932(0.0087)	0.5030(0.4391)	0.2767(0.0766)	0.3750(0.1582)	0.2626(0.0690)
				2	3	0.0390(0.0015)	0.5212(0.7760)	0.2486(0.0618)	0.3577(0.1519)	0.2423(0.0587)
		11	0.75	1	2	0.0971(0.0094)	0.2226(0.0724)	0.1213(0.0147)	0.1880(0.0553)	0.1214(0.0147)
				2	3	0.0928(0.0087)	0.2257(0.0875)	0.1187(0.0141)	0.1732(0.0368)	0.1187(0.0141)
			1.5	1	2	0.0920(0.0085)	0.4872(0.5025)	0.2643(0.0699)	0.3508(0.2113)	0.2545(0.0648)
				2	3	0.0330(0.0011)	0.4950(0.5886)	0.2446(0.0598)	0.3233(0.1064)	0.2400(0.0576)
40	20	10	0.75	1	2	0.1066(0.0114)	0.2412(0.2080)	0.1287(0.0166)	0.1980(0.0562)	0.1291(0.0167)
				2	3	0.1744(0.0305)	0.2270(0.1134)	0.1217(0.0148)	0.1960(0.0487)	0.1219(0.0149)
			1.5	1	2	0.0954(0.0091)	0.5113(0.4431)	0.2990(0.0894)	0.3792(0.1530)	0.2839(0.0806)
				2	3	0.0516(0.0027)	0.4902(0.5161)	0.2545(0.0648)	0.3914(0.7522)	0.2485(0.0618)
		15	0.75	1	2	0.1012(0.0102)	0.2153(0.0957)	0.1247(0.0156)	0.1765(0.0353)	0.1251(0.0157)
				2	3	0.1282(0.0165)	0.2228(0.1759)	0.1202(0.0144)	0.1687(0.0307)	0.1204(0.0145)
			1.5	1	2	0.0934(0.0087)	0.4669(0.6635)	0.2793(0.0780)	0.3449(0.1339)	0.2700(0.0729)
				2	3	0.0404(0.0016)	0.4704(0.6949)	0.2492(0.0621)	0.3303(0.1226)	0.2451(0.0601)
	30	15	0.75	1	2	0.1013(0.0103)	0.2141(0.0908)	0.1248(0.0156)	0.1754(0.0352)	0.1252(0.0157)
				2	3	0.1280(0.0165)	0.2113(0.0712)	0.1201(0.0144)	0.1702(0.0432)	0.1204(0.0145)
			1.5	1	2	0.0934(0.0087)	0.4367(0.2973)	0.2791(0.0779)	0.3471(0.1353)	0.2696(0.0727)
				2	3	0.0402(0.0016)	0.4727(0.7433)	0.2492(0.0621)	0.3560(0.6997)	0.2451(0.0601)
		23	0.75	1	2	0.0966(0.0093)	0.1846(0.0607)	0.1209(0.0146)	0.1521(0.0247)	0.1213(0.0147)
				2	3	0.0880(0.0078)	0.1769(0.0391)	0.1185(0.0140)	0.1469(0.0219)	0.1187(0.0141)
			1.5	1	2	0.0919(0.0084)	0.4016(0.2801)	0.2620(0.0687)	0.3094(0.1072)	0.2568(0.0659)
				2	3	0.0322(0.0010)	0.4337(0.7658)	0.2440(0.0595)	0.2967(0.0900)	0.2414(0.0583)
60	30	15	0.75	1	2	0.1044(0.0109)	0.1970(0.1502)	0.1253(0.0157)	0.1671(0.0350)	0.1260(0.0159)
				2	3	0.1682(0.0283)	0.1964(0.0802)	0.1207(0.0146)	0.1633(0.0285)	0.1211(0.0147)
			1.5	1	2	0.0953(0.0091)	0.4489(0.5535)	0.2857(0.0816)	0.3400(0.1229)	0.2775(0.0770)
				2	3	0.0513(0.0026)	0.4562(0.7476)	0.2509(0.0630)	0.3309(0.1242)	0.2476(0.0613)
		23	0.75	1	2	0.1009(0.0102)	0.1892(0.0575)	0.1245(0.0155)	0.1619(0.0287)	0.1251(0.0156)
				2	3	0.1249(0.0156)	0.1971(0.0909)	0.1200(0.0144)	0.1546(0.0247)	0.1204(0.0145)
			1.5	1	2	0.0933(0.0087)	0.4330(0.4771)	0.2779(0.0772)	0.3286(0.1140)	0.2714(0.0736)
				2	3	0.0398(0.0016)	0.4452(0.7879)	0.2488(0.0619)	0.3168(0.1412)	0.2460(0.0605)
	45	23	0.75	1	2	0.1009(0.0102)	0.1871(0.0655)	0.1245(0.0155)	0.1599(0.0272)	0.1251(0.0157)
				2	3	0.1248(0.0156)	0.1927(0.0744)	0.1200(0.0144)	0.1580(0.0439)	0.1204(0.0145)
			1.5	1	2	0.0933(0.0087)	0.4184(0.3256)	0.2779(0.0772)	0.3273(0.1133)	0.2714(0.0737)
				2	3	0.0399(0.0016)	0.4309(0.4882)	0.2489(0.0619)	0.3143(0.1132)	0.2460(0.0605)
		34	0.75	1	2	0.0967(0.0093)	0.1576(0.0270)	0.1210(0.0146)	0.1429(0.0207)	0.1215(0.0148)
				2	3	0.0887(0.0079)	0.1666(0.0508)	0.1185(0.0140)	0.1385(0.0193)	0.1188(0.0141)
			1.5	1	2	0.0919(0.0085)	0.3589(0.1781)	0.2625(0.0689)	0.2964(0.0888)	0.2589(0.0670)
				2	3	0.0323(0.0010)	0.3805(0.3230)	0.2440(0.0595)	0.2815(0.0795)	0.2422(0.0587)

Table 6: The Average bias (AVB) and Mean Square Errors (MSEs) in parentheses for the Burr-XII parameter β using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=2 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1016(0.0103)	0.2514(0.1665)	0.1228(0.0151)	0.1916(0.0405)	0.1228(0.0151)
				2	3	0.1380(0.0192)	0.2549(0.2866)	0.1194(0.0143)	0.1825(0.0356)	0.1194(0.0143)
			1.5	1	2	0.0938(0.0088)	0.5944(0.7864)	0.2714(0.0737)	0.3584(0.1350)	0.2592(0.0672)
				2	3	0.0426(0.0018)	0.5043(0.5938)	0.2470(0.0610)	0.3456(0.1328)	0.2414(0.0583)
		8	0.75	1	2	0.1004(0.0101)	0.2724(0.6434)	0.1234(0.0152)	0.1957(0.0458)	0.1233(0.0152)
				2	3	0.1219(0.0150)	0.2631(0.5834)	0.1199(0.0144)	0.1922(0.0429)	0.1198(0.0144)
			1.5	1	2	0.0932(0.0087)	0.5751(0.7845)	0.2766(0.0765)	0.3704(0.1453)	0.2626(0.0689)
				2	3	0.0392(0.0016)	0.6048(0.8794)	0.2487(0.0619)	0.3637(0.1896)	0.2424(0.0587)
	15	8	0.75	1	2	0.1000(0.0100)	0.2542(0.2185)	0.1229(0.0151)	0.1921(0.0462)	0.1229(0.0151)
				2	3	0.1218(0.0149)	0.2411(0.1562)	0.1194(0.0143)	0.1839(0.0388)	0.1194(0.0143)
			1.5	1	2	0.0932(0.0087)	0.5572(0.7684)	0.2710(0.0735)	0.3556(0.1318)	0.2591(0.0671)
				2	3	0.0391(0.0015)	0.5330(0.7856)	0.2471(0.0611)	0.3527(0.1541)	0.2415(0.0583)
		11	0.75	1	2	0.0972(0.0094)	0.2368(0.3377)	0.1212(0.0147)	0.1785(0.0402)	0.1213(0.0147)
				2	3	0.0923(0.0086)	0.2360(0.1389)	0.1187(0.0141)	0.1683(0.0290)	0.1187(0.0141)
			1.5	1	2	0.0920(0.0085)	0.5269(0.8254)	0.2642(0.0698)	0.3595(0.2988)	0.2545(0.0648)
				2	3	0.0331(0.0011)	0.4811(0.7121)	0.2443(0.0597)	0.3210(0.1052)	0.2398(0.0575)
40	20	10	0.75	1	2	0.0989(0.0098)	0.1944(0.0874)	0.1219(0.0149)	0.1655(0.0432)	0.1223(0.0150)
				2	3	0.1138(0.0130)	0.2041(0.0955)	0.1190(0.0142)	0.1536(0.0248)	0.1192(0.0142)
			1.5	1	2	0.0929(0.0086)	0.4829(0.7683)	0.2667(0.0711)	0.3180(0.1042)	0.2603(0.0678)
				2	3	0.0375(0.0014)	0.4684(0.7689)	0.2454(0.0602)	0.3031(0.0944)	0.2424(0.0588)
		15	0.75	1	2	0.0988(0.0098)	0.2018(0.1094)	0.1219(0.0149)	0.1567(0.0251)	0.1223(0.0150)
				2	3	0.1084(0.0118)	0.2130(0.2316)	0.1190(0.0142)	0.1546(0.0349)	0.1192(0.0142)
			1.5	1	2	0.0927(0.0086)	0.4375(0.6661)	0.2667(0.0711)	0.3186(0.1049)	0.2603(0.0678)
				2	3	0.0364(0.0013)	0.4413(0.9599)	0.2454(0.0602)	0.3041(0.0948)	0.2424(0.0588)
	30	15	0.75	1	2	0.0992(0.0098)	0.1837(0.0541)	0.1212(0.0147)	0.1570(0.0297)	0.1216(0.0148)
				2	3	0.1139(0.0130)	0.1997(0.1671)	0.1188(0.0141)	0.1499(0.0228)	0.1190(0.0142)
			1.5	1	2	0.0929(0.0086)	0.4059(0.2807)	0.2650(0.0702)	0.3171(0.1044)	0.2590(0.0671)
				2	3	0.0377(0.0014)	0.4509(0.6014)	0.2449(0.0600)	0.3021(0.0964)	0.2421(0.0586)
		23	0.75	1	2	0.0966(0.0093)	0.1970(0.4392)	0.1209(0.0146)	0.1580(0.0416)	0.1213(0.0147)
				2	3	0.0879(0.0078)	0.1897(0.0834)	0.1185(0.0140)	0.1464(0.0217)	0.1187(0.0141)
			1.5	1	2	0.0919(0.0084)	0.4115(0.3514)	0.2622(0.0688)	0.3132(0.1119)	0.2569(0.0660)
				2	3	0.0321(0.0010)	0.4232(0.8564)	0.2439(0.0595)	0.2928(0.0862)	0.2413(0.0582)
60	30	15	0.75	1	2	0.0981(0.0096)	0.1680(0.0391)	0.1219(0.0149)	0.1474(0.0221)	0.1225(0.0150)
				2	3	0.1014(0.0103)	0.1780(0.0545)	0.1190(0.0142)	0.1430(0.0207)	0.1193(0.0142)
			1.5	1	2	0.0924(0.0085)	0.3842(0.2573)	0.2665(0.0710)	0.3063(0.0952)	0.2622(0.0687)
				2	3	0.0351(0.0012)	0.3983(0.5649)	0.2454(0.0602)	0.2908(0.0860)	0.2434(0.0592)
		23	0.75	1	2	0.0984(0.0097)	0.1684(0.0537)	0.1219(0.0149)	0.1477(0.0228)	0.1225(0.0150)
				2	3	0.1046(0.0110)	0.2117(0.6757)	0.1190(0.0142)	0.1433(0.0208)	0.1193(0.0142)
			1.5	1	2	0.0925(0.0086)	0.3653(0.1962)	0.2665(0.0710)	0.3032(0.0928)	0.2622(0.0688)
				2	3	0.0357(0.0013)	0.4153(0.5456)	0.2454(0.0602)	0.2915(0.0885)	0.2434(0.0592)
	45	23	0.75	1	2	0.0981(0.0096)	0.1597(0.0394)	0.1209(0.0146)	0.1425(0.0205)	0.1214(0.0147)
				2	3	0.1046(0.0110)	0.1753(0.0921)	0.1186(0.0141)	0.1399(0.0197)	0.1190(0.0141)
			1.5	1	2	0.0926(0.0086)	0.3802(0.3286)	0.2633(0.0694)	0.2991(0.0912)	0.2596(0.0674)
				2	3	0.0357(0.0013)	0.3633(0.1913)	0.2442(0.0597)	0.2830(0.0806)	0.2424(0.0588)
		34	0.75	1	2	0.0967(0.0093)	0.1615(0.0406)	0.1208(0.0146)	0.1455(0.0330)	0.1214(0.0147)
				2	3	0.0888(0.0079)	0.1932(0.5028)	0.1185(0.0140)	0.1393(0.0198)	0.1188(0.0141)
			1.5	1	2	0.0919(0.0085)	0.3809(0.3139)	0.2624(0.0688)	0.2982(0.0981)	0.2588(0.0670)
				2	3	0.0324(0.0011)	0.3683(0.2451)	0.2440(0.0595)	0.2838(0.0818)	0.2423(0.0587)

Table 7: The Average Bias (AVB) and Mean Square Errors (MSEs) in parentheses for the Burr-XII parameter γ using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1584(0.0251)	0.6267(0.7150)	0.3348(0.1121)	0.4683(0.3860)	0.3179(0.1011)
				2	3	0.3859(0.1496)	0.9979(0.7865)	0.5069(0.2570)	0.5779(0.3644)	0.4679(0.2189)
			1.5	1	2	0.1503(0.0226)	0.6654(0.5933)	0.3787(0.1435)	0.4758(0.2502)	0.3402(0.1157)
				2	3	0.2749(0.0756)	0.5648(0.8952)	0.5378(0.2894)	0.5885(0.3555)	0.4777(0.2282)
		8	0.75	1	2	0.1443(0.0208)	0.5679(0.6112)	0.3206(0.1028)	0.4248(0.2144)	0.3119(0.0973)
				2	3	0.2995(0.0899)	0.9591(0.8765)	0.4896(0.2398)	0.5578(0.3215)	0.4649(0.2162)
			1.5	1	2	0.1414(0.0200)	0.6237(0.7684)	0.3438(0.1182)	0.4414(0.3054)	0.3255(0.1060)
				2	3	0.2445(0.0598)	0.5683(0.7854)	0.5111(0.2613)	0.5618(0.3241)	0.4731(0.2238)
	15	8	0.75	1	2	0.1445(0.0209)	0.5864(0.7684)	0.3205(0.1027)	0.4272(0.2406)	0.3118(0.0972)
				2	3	0.2995(0.0899)	0.9201(0.8462)	0.4897(0.2398)	0.5609(0.3316)	0.4650(0.2162)
			1.5	1	2	0.1414(0.0200)	0.6304(0.6784)	0.3444(0.1187)	0.4188(0.1848)	0.3257(0.1061)
				2	3	0.2446(0.0598)	0.8454(0.9820)	0.5110(0.2612)	0.5538(0.3100)	0.4730(0.2238)
		11	0.75	1	2	0.1385(0.0192)	0.5115(0.3888)	0.3139(0.0985)	0.4305(0.8622)	0.3087(0.0953)
				2	3	0.2652(0.0704)	0.8296(0.7684)	0.4802(0.2306)	0.5387(0.3027)	0.4627(0.2141)
			1.5	1	2	0.1370(0.0188)	0.5220(0.5228)	0.3299(0.1088)	0.3921(0.1597)	0.3186(0.1015)
				2	3	0.2296(0.0527)	0.8460(0.7864)	0.4963(0.2463)	0.5485(0.4373)	0.4696(0.2205)
40	20	10	0.75	1	2	0.1547(0.0239)	0.5981(0.7683)	0.3309(0.1095)	0.4075(0.1787)	0.3194(0.1020)
				2	3	0.3618(0.1311)	0.5943(0.8659)	0.5032(0.2532)	0.5583(0.3353)	0.4725(0.2233)
			1.5	1	2	0.1494(0.0223)	0.6190(0.8794)	0.3694(0.1365)	0.4346(0.2035)	0.3434(0.1180)
				2	3	0.2752(0.0758)	0.9799(0.6758)	0.5332(0.2844)	0.5688(0.3474)	0.4853(0.2355)
		15	0.75	1	2	0.1459(0.0213)	0.5415(0.8602)	0.3219(0.1036)	0.4002(0.2239)	0.3146(0.0990)
				2	3	0.3085(0.0952)	0.7710(0.8954)	0.4919(0.2420)	0.5469(0.4601)	0.4696(0.2206)
			1.5	1	2	0.1426(0.0203)	0.5400(0.7879)	0.3476(0.1208)	0.3990(0.1642)	0.3317(0.1101)
				2	3	0.2486(0.0618)	0.8188(0.9675)	0.5147(0.2650)	0.5444(0.2986)	0.4804(0.2308)
	30	15	0.75	1	2	0.1460(0.0213)	0.5236(0.7953)	0.3221(0.1037)	0.3982(0.2628)	0.3147(0.0990)
				2	3	0.3079(0.0949)	0.9048(0.7953)	0.4918(0.2419)	0.5342(0.2883)	0.4696(0.2205)
			1.5	1	2	0.1426(0.0203)	0.5612(0.7684)	0.3475(0.1208)	0.4003(0.1641)	0.3317(0.1100)
				2	3	0.2486(0.0618)	0.8355(0.4532)	0.5147(0.2649)	0.5444(0.3000)	0.4804(0.2308)
		23	0.75	1	2	0.1377(0.0190)	0.4397(0.3431)	0.3132(0.0981)	0.3541(0.1263)	0.3095(0.0958)
				2	3	0.2607(0.0680)	0.6651(0.6952)	0.4791(0.2295)	0.5131(0.2642)	0.4655(0.2167)
			1.5	1	2	0.1365(0.0186)	0.4780(0.6783)	0.3283(0.1078)	0.3644(0.1340)	0.3204(0.1027)
				2	3	0.2280(0.0520)	0.7405(0.7658)	0.4947(0.2448)	0.5260(0.2886)	0.4738(0.2245)
60	30	15	0.75	1	2	0.1526(0.0233)	0.4727(0.6408)	0.3233(0.1046)	0.3805(0.1750)	0.3166(0.1002)
				2	3	0.3681(0.1357)	0.7988(0.6758)	0.4963(0.2463)	0.5418(0.3817)	0.4737(0.2244)
			1.5	1	2	0.1497(0.0224)	0.5066(0.4513)	0.3548(0.1259)	0.4012(0.1833)	0.3386(0.1147)
				2	3	0.2754(0.0759)	0.7934(0.6759)	0.5214(0.2719)	0.5486(0.3060)	0.4867(0.2368)
		23	0.75	1	2	0.1453(0.0211)	0.4490(0.4001)	0.3214(0.1033)	0.3761(0.1624)	0.3154(0.0995)
				2	3	0.3048(0.0930)	0.6914(0.6744)	0.4913(0.2414)	0.5448(0.6305)	0.4720(0.2228)
			1.5	1	2	0.1422(0.0202)	0.5434(0.6738)	0.3462(0.1198)	0.3890(0.1717)	0.3333(0.1111)
				2	3	0.2475(0.0613)	0.8067(0.6758)	0.5138(0.2640)	0.5367(0.2892)	0.4840(0.2343)
	45	23	0.75	1	2	0.1453(0.0211)	0.4511(0.2868)	0.3215(0.1034)	0.3706(0.1440)	0.3155(0.0995)
				2	3	0.3045(0.0928)	0.7520(0.6574)	0.4912(0.2413)	0.5252(0.2827)	0.4719(0.2227)
			1.5	1	2	0.1422(0.0202)	0.5098(0.6658)	0.3462(0.1199)	0.3844(0.1505)	0.3333(0.1111)
				2	3	0.2475(0.0613)	0.8107(0.7843)	0.5138(0.2640)	0.5384(0.2930)	0.4840(0.2343)
		34	0.75	1	2	0.1379(0.0190)	0.3849(0.1631)	0.3134(0.0982)	0.3463(0.1221)	0.3102(0.0963)
				2	3	0.2621(0.0687)	0.6079(0.4234)	0.4795(0.2299)	0.5037(0.2540)	0.4674(0.2185)
			1.5	1	2	0.1367(0.0187)	0.4146(0.2300)	0.3287(0.1081)	0.3562(0.1275)	0.3220(0.1037)
				2	3	0.2287(0.0523)	0.6582(0.7453)	0.4953(0.2453)	0.5139(0.2642)	0.4768(0.2273)

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ble 8: The Average bias (AVB) and Mean Square Errors (MSEs) in parentheses for the Burr–XII parameter γ using the Kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=2 and $\delta = 2$ for LINEX loss function.

N	m	k	β	α	γ	Conditional Method	Gamma Prior		Kernel Prior	
							SQEL	LNXL	SQEL	LNXL
20	10	5	0.75	1	2	0.1464(0.0214)	0.5700(0.8947)	0.3175(0.1008)	0.4179(0.2873)	0.3104(0.0963)
				2	3	0.3191(0.1021)	0.9562(0.8957)	0.4858(0.2360)	0.5536(0.3294)	0.4641(0.2154)
			1.5	1	2	0.1436(0.0206)	0.5758(0.8120)	0.3383(0.1144)	0.4120(0.1783)	0.3228(0.1042)
				2	3	0.2525(0.0638)	0.5673(0.8964)	0.5053(0.2554)	0.5507(0.3058)	0.4718(0.2226)
		8	0.75	1	2	0.1443(0.0208)	0.5622(0.5758)	0.3186(0.1015)	0.4167(0.2013)	0.3109(0.0967)
				2	3	0.3009(0.0908)	0.9350(0.7982)	0.4894(0.2395)	0.5731(0.7594)	0.4648(0.2161)
			1.5	1	2	0.1414(0.0200)	0.6025(0.6327)	0.3443(0.1186)	0.4214(0.1833)	0.3256(0.1060)
				2	3	0.2448(0.0599)	0.9609(0.6584)	0.5112(0.2614)	0.5589(0.3164)	0.4731(0.2238)
	15	8	0.75	1	2	0.1437(0.0207)	0.5814(0.9855)	0.3177(0.1009)	0.4091(0.2052)	0.3105(0.0964)
				2	3	0.3005(0.0905)	0.9646(0.8875)	0.4859(0.2361)	0.5596(0.4604)	0.4641(0.2154)
			1.5	1	2	0.1414(0.0200)	0.5638(0.5611)	0.3385(0.1146)	0.4081(0.1717)	0.3229(0.1043)
				2	3	0.2448(0.0599)	0.9009(0.9835)	0.5055(0.2555)	0.5626(0.3906)	0.4719(0.2227)
		11	0.75	1	2	0.1385(0.0192)	0.5115(0.4815)	0.3139(0.0985)	0.3971(0.2664)	0.3086(0.0953)
				2	3	0.2652(0.0704)	0.8540(0.8956)	0.4801(0.2305)	0.5350(0.2979)	0.4627(0.2141)
			1.5	1	2	0.1370(0.0188)	0.5463(0.8786)	0.3299(0.1088)	0.3903(0.1544)	0.3186(0.1015)
				2	3	0.2296(0.0527)	0.9039(0.8952)	0.4960(0.2460)	0.5389(0.3022)	0.4695(0.2204)
40	20	10	0.75	1	2	0.1417(0.0201)	0.4599(0.4114)	0.3156(0.0996)	0.3711(0.1464)	0.3110(0.0967)
				2	3	0.2913(0.0850)	0.7558(0.8953)	0.4829(0.2332)	0.5247(0.2840)	0.4668(0.2179)
			1.5	1	2	0.1406(0.0198)	0.5082(0.6584)	0.3335(0.1112)	0.3763(0.1442)	0.3235(0.1047)
				2	3	0.2418(0.0585)	0.7532(0.7682)	0.5007(0.2507)	0.5326(0.3067)	0.4760(0.2266)
		15	0.75	1	2	0.1415(0.0200)	0.4893(0.6820)	0.3157(0.0996)	0.3672(0.1410)	0.3110(0.0967)
				2	3	0.2847(0.0812)	0.7655(0.8867)	0.4829(0.2332)	0.5172(0.2682)	0.4668(0.2179)
			1.5	1	2	0.1397(0.0195)	0.4825(0.4708)	0.3335(0.1112)	0.3741(0.1417)	0.3235(0.1047)
				2	3	0.2389(0.0571)	0.7629(0.7936)	0.5006(0.2506)	0.5306(0.2936)	0.4759(0.2265)
	30	15	0.75	1	2	0.1423(0.0203)	0.4507(0.3722)	0.3139(0.0985)	0.3612(0.1462)	0.3099(0.0961)
				2	3	0.2917(0.0852)	0.7328(0.7953)	0.4815(0.2318)	0.5173(0.2685)	0.4663(0.2175)
			1.5	1	2	0.1406(0.0198)	0.4562(0.3332)	0.3315(0.1099)	0.3732(0.1426)	0.3223(0.1039)
				2	3	0.2419(0.0585)	0.7693(0.8935)	0.4986(0.2486)	0.5257(0.2772)	0.4752(0.2259)
		23	0.75	1	2	0.1376(0.0189)	0.4148(0.2058)	0.3131(0.0980)	0.3556(0.1291)	0.3094(0.0958)
				2	3	0.2606(0.0680)	0.6968(0.6583)	0.4790(0.2295)	0.5123(0.2632)	0.4655(0.2167)
			1.5	1	2	0.1365(0.0186)	0.4514(0.5376)	0.3282(0.1077)	0.3644(0.1335)	0.3203(0.1026)
				2	3	0.2280(0.0520)	0.7749(0.6572)	0.4946(0.2446)	0.5197(0.2704)	0.4738(0.2245)
60	30	15	0.75	1	2	0.1404(0.0197)	0.4091(0.2225)	0.3156(0.0996)	0.3512(0.1242)	0.3117(0.0972)
				2	3	0.2768(0.0767)	0.7104(0.6573)	0.4830(0.2333)	0.5110(0.2620)	0.4689(0.2198)
			1.5	1	2	0.1387(0.0193)	0.4384(0.2999)	0.3333(0.1111)	0.3645(0.1335)	0.3251(0.1057)
				2	3	0.2355(0.0555)	0.6844(0.7683)	0.5007(0.2507)	0.5215(0.2729)	0.4791(0.2295)
		23	0.75	1	2	0.1409(0.0198)	0.3946(0.1744)	0.3156(0.0996)	0.3536(0.1288)	0.3117(0.0972)
				2	3	0.2810(0.0790)	0.7240(0.7943)	0.4830(0.2332)	0.5113(0.2628)	0.4688(0.2198)
			1.5	1	2	0.1392(0.0194)	0.4471(0.4246)	0.3335(0.1112)	0.3674(0.1529)	0.3251(0.1057)
				2	3	0.2373(0.0563)	0.7426(0.8956)	0.5008(0.2508)	0.5212(0.2721)	0.4791(0.2295)
	45	23	0.75	1	2	0.1405(0.0197)	0.3883(0.1990)	0.3132(0.0981)	0.3436(0.1184)	0.3101(0.0962)
				2	3	0.2808(0.0789)	0.6586(0.6793)	0.4803(0.2307)	0.5059(0.2562)	0.4678(0.2188)
			1.5	1	2	0.1392(0.0194)	0.4280(0.2858)	0.3298(0.1087)	0.3583(0.1292)	0.3227(0.1041)
				2	3	0.2373(0.0563)	0.6892(0.6743)	0.4965(0.2465)	0.5157(0.2662)	0.4773(0.2278)
		34	0.75	1	2	0.1378(0.0190)	0.3856(0.1694)	0.3132(0.0981)	0.3438(0.1186)	0.3101(0.0961)
				2	3	0.2621(0.0687)	0.6249(0.7173)	0.4795(0.2299)	0.5028(0.2534)	0.4674(0.2185)
			1.5	1	2	0.1367(0.0187)	0.4442(0.7842)	0.3287(0.1081)	0.3551(0.1263)	0.3220(0.1037)
				2	3	0.2287(0.0523)	0.6528(0.6167)	0.4953(0.2454)	0.5167(0.2689)	0.4768(0.2273)

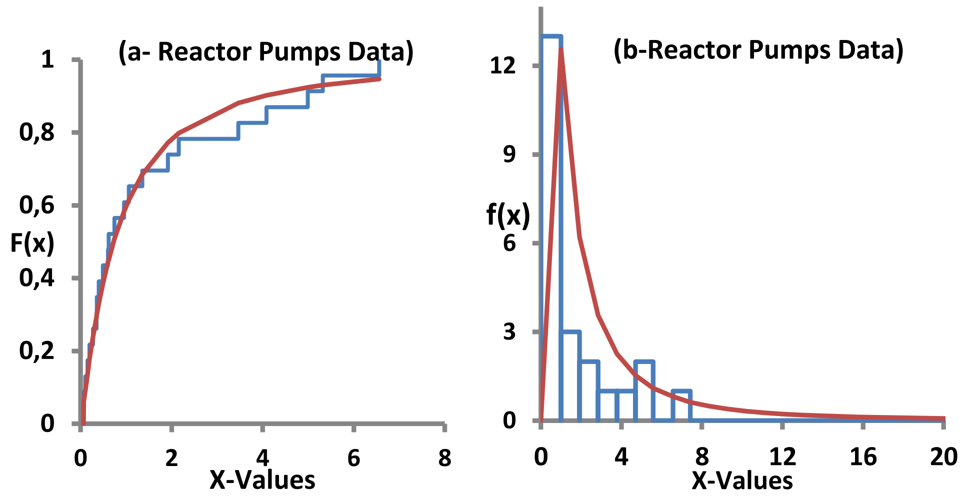

Figure 1: a) The Empirical CDF and the fitted CDF for the Reactor Pumps Data
b) The Histogram and the fitted PDF for the Reactor Pumps Data

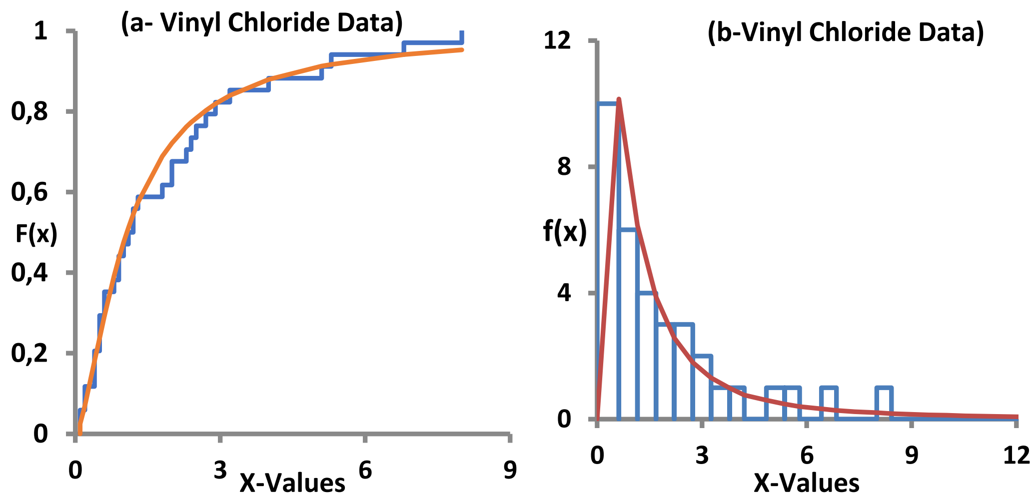

Figure 2: a) The Empirical CDF and the fitted CDF for the Vinyl Chloride Data
b) The Histogram and the fitted PDF for the Vinyl Chloride Data