From Zadeh's Fuzziness to Smarandache's Neutrosophy: A Review

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Abstract: —The present paper reviews the process that led from Zadeh's fuzziness to Smarandache's neutrosophy. Starting from fuzzy sets and logic and Atanassov's intuitionistic fuzzy sets, it proceeds to a detailed study of the concept of neutrosophic set, which takes in account the existing in real life indeterminacy. The basic operations on neutrosophic sets are defined and the classical notion of topological space is extended to the notion of neutrosophic topological space, where fundamental properties and concepts like convergence, continuity, compactness and Hausdorff topological spaces are naturally extended.

Keywords: —bivalent logic (BL), probability, fuzzy set (FS), fuzzy logic (FL), intuitionistic FS (IFS), neutrosophic set (NS), neutrosophic topological space (NTS).

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1. Introduction

The frequent appearance of *uncertainty* in everyday life and science is due to a shortage of knowledge. Roughly speaking, the amount of the existing uncertainty is equal to the difference of the amount of the necessary knowledge needed for interpreting or determining the evolution of the corresponding situation, minus the already existing knowledge about this situation. In other words, uncertainty represents the total amount of potential information in the situation, which implies that a reduction of uncertainty due to new evidence (e.g. receipt of a message) indicates a gain of an equal amount of information. This is why the classical measures of uncertainty under crisp or fuzzy conditions (Hartley's formula, Shannon's entropy, etc. [1, Chapter 5]) have been also adopted as measures of information.

Different types of uncertainties arise in real situations including *randomness*, *imprecision*, *vagueness*, *ambiguity, inconsistency,* etc. [1]. The uncertainty due to randomness is related to well-defined events whose outcomes cannot be predicted in advance, like the turning of a coin, the throwing of a die, etc. Imprecision, on the other hand, occurs when the events are well defined, but the possible outcomes cannot be expressed with an exact numerical value; e.g. "the temperature tomorrow will be between 27^0 and 32^0 C". Vagueness is created when one is unable to clearly differentiate between two classes, such as "a person of average height" and "a tall person". In case of ambiguity the existing information leads to several interpretations by different observers. For example, the statement "Boy no girl" written as "Boy, no girl" means boy, but written as "Boy no, girl" means girl! Finally, inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, "the probability for raining tomorrow is 80%, but this does not mean that the probability for not raining is 20%, because they might be hidden weather factors".

Note that several other taxonomies of uncertainty have been proposed. One such taxonomy, for example, includes the *epistemic (or subjective)* uncertainty and the *linguistic* uncertainty. The former is due to lack of knowledge, whether the latter is produced by statements expressed in natural language.

Probability theory was proved to be sufficient for managing the uncertainty due to randomness [2]. Starting from Zadeh's *fuzzy set (FS)* and the connected to it *fuzzy logic (FL)*, however, several other theories developed during the last 60 years for managing more effectively all the types of uncertainty (e.g. see [3]). FSs, for example, tackle successfully the uncertainty due to vagueness.

Atanassov introduced the concept of *intuitionistic FS (IFS)* by adding to Zadeh's *membership degree* the degree of *non-membership* of each element of the universal set. Smarandanche, motivated by the various neutral situations appearing in real life - like <friend, neutral, enemy>, <positive, zero, negative>, <small, medium, high>, <male, transgender, female>, <win, draw, defeat>, etc. - introduced the concept of *neutrosophic set (NS)* by adding a third degree of *indeterminacy* between the degrees of membership and non-membership.

The present paper reviews the process that led from fuzziness to neutrosophy. More explicitly, the next section discusses the differences and similarities between probability and fuzzy logic. The third section contains basic definitions and examples of IFSs and NSs, while in fourth section basic operations on NSs are defined and the classical notion of *topological space* is extended to *neutrosophic topological spaces (NTSs),* where fundamental properties and concepts like convergence, continuity, compactness and Hausdorff topological spaces are naturally extended. The article closes with the final conclusions and some hints for future research presented in the fifth section.

2. Fuziness vs Probability

Logic is the study of the correct reasoning, involving the drawing of inferences. There is no doubt that the enormous progress of science and technology owes a lot to the Aristotle's (384-322 BC) *bivalent logic (BL)*, which dominated for centuries the human way of thinking. BL is based on the law of the *excluded middle* - according to which, for all propositions p, either p or not p must be true and there is no middle (third) true proposition between them - all its other principles being mere elaborations of this law [4].

From the time of Buddha Siddhartha Gautama, however, who lived in India around 500 BC, of Heraclitus (535-475 BC) and of Plato (427-377 BC) views appeared too discussing the existence of a third area between "true" and "false", where those two opposites can exist together. The first integrated propositions of multivalued logics, however, appeared only during the 20th century by Lukasiewicz (1858-1956) and Tarski (1901-1983) [5, Section 2].

Max Black [6] introduced in 1937 the concept of *vague set* being a premonition of the FS, defined by Zadeh [7] in 1965 as follows:

Definition 1: Let U be the universal set of the discourse, then a FS A in U is defined with the help of its *membership function* m: $U \rightarrow [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the *membership degree* of x in A**.** The greater m(x), the more x satisfies the characteristic property of A. Many authors, for reasons of simplicity, identify a fuzzy set with its membership function.

A crisp subset A of U is a FS in U with membership function taking the values m(x) = 1, if x belongs to A, and 0 otherwise. The classical operations on crisp sets (intersection, union, complement, etc.) are generalized in a natural way to FSs [1, Chapter 2].

. The infinite-valued on the interval [0, 1] FL [8] is defined with the help of the concept of FS. Through FL, the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner [9]. An important advantage of FL is that its rules are set in natural language with the help of linguistic, and therefore fuzzy, variables [10].

Many of the traditional supporters of BL, based on a culture of centuries, argued that, since this logic works effectively in science, functions the computers and explains satisfactorily the phenomena of the real world, except perhaps those that happen in the boundaries, there is no reason to make things more complicated by introducing the unstable principles of a multi-valued logic

FL, however, aims exactly at smoothing the situation in the boundaries! Look, for example, at the graph of Fig. 1 representing the FS T of "tall people". People with heights less than 1.50 m are considered of having membership degree 0 in T. The membership degree is continuously increasing for heights greater than 1.50m, taking its maximal value 1 for heights equal or greater than 1.80 m. Therefore, the "fuzzy part" of the graph - which is conventionally represented in Fig. 1 by the straight line segment AC, but its exact form depends upon the way in which the membership function has been defined - lies in the area of the rectangle ABCD defined by the OX axis, its parallel through the point E and the two perpendicular to it lines at the points A and B.

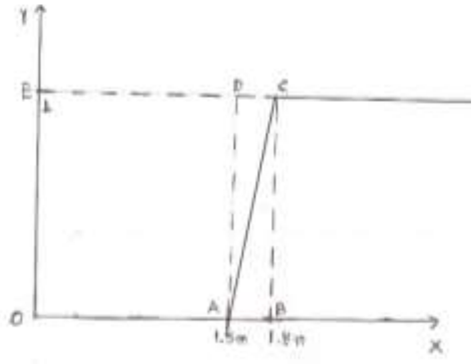

Fig. 1. The fuzzy set of "tall people"

On the contrary, BL defines a bound, e.g. 1.80 m, above which people are considered to be tall and under which are considered to be short. Consequently, one with height 1.79 m is considered to be short, whether another with height 1.81 m is tall!

The way of perceiving a concept (e.g. "tall") is different from person to person, depending on the "signals" that each one receives from the real world about it. Mathematically speaking, this means that the definition of the membership function of a FS is not unique, depending on the observer's personal criteria. The only restriction is that its definition must be compatible with common logic, because otherwise the corresponding FS does not give a reliable description of the corresponding real situation. This could happen, for example, in the previously mentioned FS T of "tall people", if persons with heights less than 1.50 m possessed membership degrees ≥ 0.5.

BL is able to verify the validity/consistency of an argument only, but not its truth. A deductive argument is always valid, even if its inference is false. A characteristic example can be found in the function of computers. A computer is unable to judge, if the input data inserted to it are correct, and therefore, if the result obtained by elaborating these data is correct and useful for the user. The only thing that it guarantees is that, if the input is correct, then the output will be correct too. On the contrary, always under the BL's approach, an inductive argument is never valid, even if its inference is true. To put it in a different way, if a property p is true for a sufficiently large number of cases, the expression "the property p is possibly true in general" is not acceptable, since it does not satisfy the principle of the excluded middle.

People, however, always want to know the truth in order to organize better, or even to protect, their lives. Consequently, under this option, the significance of an argument has greater importance than its validity/precision. In Fig. 2 retrieved from [11], for example, the extra precision on the left makes things worse for the poor man in danger, who has to spend too much time trying to understand the data and misses the opportunity to take the much needed action of getting out of the way. On the contrary, the rough / fuzzy warning on the right could save his life.

Fig. 2 illustrates very successfully the importance of FL for real life situations. Real-world knowledge has generally a different structure and requires different formalization than the existing formal systems. FL, which according to Zadeh is "a precise logic of imprecision and approximate reasoning" [10], serves as a link between classical logic and human reasoning/experience, which are two incommensurable approaches. Having a much higher generality than bivalent logic, FL is capable of generalizing any bivalent logic-based theory.

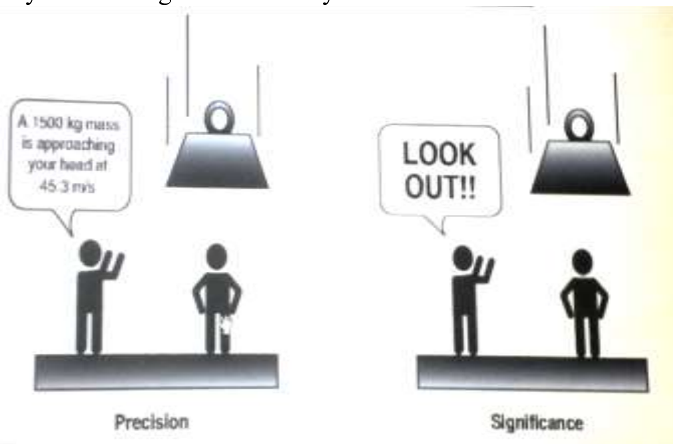

Fig. 2. Validity/precision vs significance

Probability and *Statistics* are related mathematical topics having, however, fundamental differences. In fact, Probability is a branch of theoretical mathematics dealing with the estimation of the likelihood of future events, whereas Statistics is a branch of applied mathematics, which tries to make sense by analyzing the frequencies of past events.

When FL was introduced, a great part of probability theorists claimed that it cannot do any more than probability does. Membership degrees, taking values in the same probabilities interval [0, 1], are actually hidden probabilities, fuzziness is a kind of disguised randomness, and the multi-valued logic is not a new idea. It took a long time to become universally understood that fuzziness does not oppose probability, but actually supports and completes it by treating successfully the cases of the existing in the real world uncertainty which is caused by reasons different from randomness.

The expressions "John's membership degree in the FS of clever people is 0.7" and "the probability of John to be clever is 0.7", although they look similar, they actually have essentially different meanings. The former means that John is a rather clever person, whereas the latter means that John, according to the principle of the excluded middle, is either clever or not, but his outlines (heredity, academic studies, etc.) suggests that the probability to be clever is high (70%).

There are also other differences between the two theories mainly arising from the way of defining the corresponding notions and operations. For instance, whereas the sum of the probabilities of all the single events of the universal set U (singleton subsets of U) is always equal to 1 (probability of the certain event), this is not necessarily true for the membership degrees. Consequently a probability distribution could be used to define membership degrees, but the converse does not hold in general.

Note that E. Jaynes, Professor of Physics at the University of Washington, argued that Probability theory can be considered as a generalization of the BL reducing to it in the special cases where our hypothesis is either absolutely true or absolutely false [12]. Many eminent scientists have been inspired by the ideas of Jaynes', like the expert in Algebraic Geometry D. Mumford, who believes that Probability and Statistics are emerging as a better way for building scientific models [13]. Nevertheless, both Probability and Statistics, developed on the basis of the principles of BL, are able to tackle effectively only the cases of uncertainty which are due to randomness [2]. As a result, Jaynes' probabilistic logic "covers" only the cases of uncertainty which are due to randomness, thus being subordinate to FL.

Zadeh realized that FSs are connected to words (adjectives and adverbs) of the natural language [14]; e.g. the adjective "tall" indicates the FS of the tall people, since "how tall is everyone" is a matter of degree. A grammatical sentence may contain many adjectives and/or adverbs, therefore it correlates a number of FSs. A synthesis of grammatical sentences, i.e. a group of FSs related to each other, forms what we call a *fuzzy system*. A fuzzy system provides empirical advice, mnemonic rules and common logic in general. It is not only able to use its own knowledge to represent and explain the real world, but can also increase it with the help of the new data; in other words, it learns from the experience. This is actually the way in which humans think. Nowadays, for example, a fuzzy system can control the function of an electric washing-machine or send signals for purchasing shares from the stock exchange, etc. [15].

Fuzzy systems are considered to be a part of the wider class of *Soft Computing*, which also includes probabilistic reasoning and *neural networks* (see Fig. 3) [16].

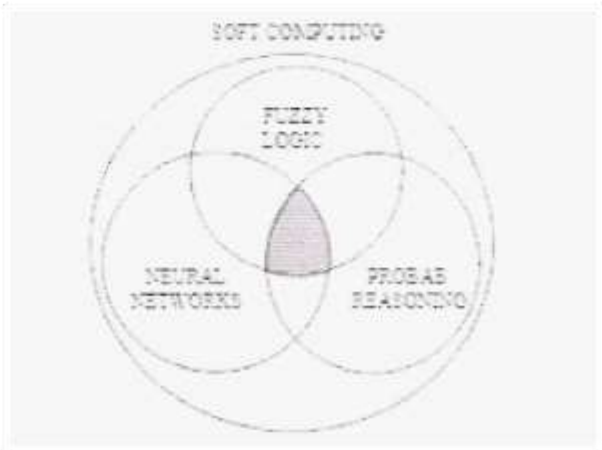

Fig. 3. A graphical approach of the contents of Soft Computing

One may argue that fuzzy systems and neural networks try to emulate the operation of the human brain. Neural networks have the ability to learn and also have a parallel structure which can rapidly process the information. In other words they concentrate on the structure of human brain, i.e. on the "hardware", emulating its basic functions. On the other hand, fuzzy systems concentrate on the "software", emulating fuzzy and symbolic reasoning. Fuzzy systems make decisions based on the raw and ambiguous data given to them, whereas neural networks try to learn from the data, incorporating the same way involved in the biological neural networks.

Intersections in Fig. 3 include *neuro-fuzzy systems* and techniques, probabilistic approaches to neural networks and *Bayesian Reasoning* [17]. A neuro-fuzzy system is a fuzzy system that uses a learning algorithm derived from or inspired by neural network theory to determine its parameters (FSs and fuzzy rules) by processing data samples. Characteristic examples of such systems are the *Adaptive Neuro-Fuzzy Inference Systems (ANFIS)* [18].

3. Intuitionistic Fuzzy Sets , Neutrosophic Sets and Other Generalizations of Fuzzy Sets

For a more accurate quantification of the uncertainty K. Atanassov, Professor of Mathematics at the Bulgarian Academy of Sciences, introduced in 1986 the concept of IFS [19] as follows:

Definition 2: An IFS A in the universe U is defined with the help of its membership function m: $U \rightarrow [0,1]$ and its *non-membership function* n: $U \rightarrow [0,1]$ as the set of the ordered triples

$$A = \{(x, m(x), n(x)): x \in U, 0 \leq m(x) + n(x) \leq 1\} \quad (2)$$

One can write m(x) + n(x) + h(x) = 1, where h(x) is called the *hesitation* or *uncertainty degree* of x. When h(x) = 0, then the corresponding IFS is reduced to an ordinary FS. The characterization intuitionistic is due to the fact that an IFS contains the intuitionistic idea, as it incorporates the degree of hesitation.

For example, if A is the IFS of the clever students of a class and (x, 0.6, 0.2) ∈ A, then there is a 60% probability for the student x to be characterized as clever, a 20% probability to be characterized as not clever, and a 20% hesitation to be characterized as either clever or not. Most notions and operations concerning FS can be extended to IFS, which successfully simulate the existing imprecision in human thinking [20].

The Romanian writer and mathematician F. Smarandache, Professor at the branch of Gallup of the New Mexico University, introduced in 1995 the degree of *indeterminacy/neutrality* of the elements of the universal set U in a subset of U and defined the concept of NS [21] as follows:

Definition 3: A *single valued NS (SVNS)* A in U is of the form

$$A = \{(x,T(x),I(x),F(x)): x \in U, T(x),I(x),F(x) \in [0,1], 0 \leq T(x)+I(x)+F(x) \leq 3\} \quad (3)$$

In (3) T(x), I(x), F(x) are the degrees of *truth, indeterminacy and falsity* of x in A respectively, called the *neutrosophic components* of x. For simplicity, we write A<T, I, F>. The etymology of the term "neutrosophy" comes from the adjective "neutral΄ and the Greek word "sophia" (wisdom) and, according to Smarandanche who introduced it, means "the knowledge of neutral thought".

For example, let U be the set of the players of a football team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in [0, 1]. For instance, x(0.6, 0.2, 0.4) ∈ A means that there is a 60% probability for x to be a good player, a 20% probability to be not certain if x is a good player or not and a 40% probability for x to be not a good player. In particular, x(0,1,0)∈ A means that we do not know absolutely nothing about x's affiliation with A.
Indeterminacy is understood to be in general everything which is between the opposites of truth and falsity [22].

In an IFS the indeterminacy is equal by default with the hesitancy, i.e. we have I(x)=1- T(x) – F(x). Also, in a FS is I(x)=0 and F(x) = 1 – T(x), whereas in a crisp set is T(x)=1 (or 0) and F(x)= 0 (or 1). In other words, crisp sets, FSs and IFSs are special cases of SVNSs.

When the sum T(x) + I(x) + F(x) of the neutrosophic components of x ∈ U in a SVNS A on U is <1, then it leaves room for incomplete information about x, when is equal to 1 for complete information and when is greater than 1 for *paraconsistent* (i.e. contradiction tolerant) information about x. A SVNS may contain simultaneously elements leaving room for all the previous types of information.

When T(x) + I(x) + F(x)<1, ∀ x ∈ U, then the corresponding SVNS is usually referred as *picture FS (PiFS)* [23]. In this case 1- T(x)-I(x)-F(x) is called the degree of *refusal membership* of x in A. The PiFSs based models are adequate in situations where we face human opinions involving answers of types yes, abstain, no and refusal. Voting is a representative example of such a situation.

The difference between the *general definition of a NS* and the previously given definition of a SVNS is that in the general definition T(x), I(x) and F(x) may take values in the non-standard unit interval]−0, 1+[(including values <0 or >1) [6]. This could happen in real world situations. For example, in a company with full-time work for its employees 40 hours per week an employee, upon his work, could belong by $\frac{40}{40}$=1 to the company (full-time job) or by $\frac{30}{40}$ <1 (part-time job) or by $\frac{45}{40}$ >1 (over-time job). Assume further that a full-time employee caused damage to his job's equipment, the cost of which must be taken from his salary. Then, if the cost is equal to $\frac{50}{40}$ of his weekly salary, the employee belongs this week to the company by -$\frac{10}{40}$ <0.

NSs, apart from vagueness, manage as well the cases of uncertainty due to ambiguity and inconsistency (see our Introduction).

The difficulty, however, of defining properly the neutrosophic components of an object still exists, for the same reason as for the membership function of a FS described in the previous section. The same also happens with IFSs, and generally for any generalization of FSs involving membership functions. This led in 1975 to the introduction of the concept of the *interval-valued FS (IVFS)* defined by a mapping F from the universe U to the set of closed intervals in [0, 1] [24]. Other related to FSs theories were also developed, in which the definition of a membership function is either not necessary (*grey systems/numbers* [25]) or it is overpassed by using either a pair of sets which give the lower and the upper approximation of the original crisp set (*rough sets* [26]) or a suitable set of linguistic parameters (*soft sets* [27]). Note also that proper combinations of the previous theories frequently give better results. The present author, for example, has recently proposed hybrid models for assessment and decision making using soft sets and grey numbers as tools [28, 29].

4. Fuzzy Topological Spaces

4.1 Operations on Neutrosophic Sets

The classical operations on crisp sets are generalized to NSs [30]. Here, for simplicity, we consider SVNSs and we define the subset and the complement of a SVNS, as well as the union and intersection of two SVNSs.

Definition 4: Let A$<T_A, I_A, F_A>$ and B$<T_B, I_B, F_B>$ be two SVNSs in the universe U. Then A is called a *subset* of B ($A \subseteq B$), if, and only if, $T_A(x) \le T_B(x)$, $I_A(x) \le I_B(x)$ and $F_A(x) \ge F_B(x)$, $\forall$ x $\in$ U. If we have simultaneously $A \subseteq B$ and $B \subseteq A$, then A and B are called *equal* SVNSs (A=B).

Definition 5: The *complement* of a SVNS A$<T_A, I_A, F_A>$ in U is the SVNS c(A)$<F_A, 1-I_A, T_A>$ in U.

Definition 6: Let A$<T_A, I_A, F_A>$ and B$<T_B, I_B, F_B>$ be two SVNSs in the universe U. Then the *union* A∪B is the SVNS C$<T_C, I_C, F_C>$ in U, with $T_C = \max\{T_A, T_B\}$, $I_C = \max\{I_A, I_B\}$ and $F_C = \min\{F_A, F_B\}$.

Definition 7: Let A$<T_A, I_A, F_A>$ and B$<T_B, I_B, F_B>$ be two SVNSs in the universe U. Then the *intersection* A∩B is the SVNS C$<T_C, I_C, F_C>$ in U, with $T_C = \min\{T_A, T_B\}$, $I_C = \min\{I_A, I_B\}$ and $F_C = \max\{F_A, F_B\}$.

It is straightforward to check that if A, B are crisp sets (FSs, IFSs) then the previous definitions are reduced to the corresponding definitions for crisp sets (FSs, IFSs).

Example 1: Let U= $\{x_1, x_2, x_3\}$ be the universal set and letA=$\{(0.3,0.3,0.6,x_1),(0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)\}$and B=$\{(0.6,0.1,0.2,x_1), (0.3,0.2,0.5, x_2), (0.3,0.1,0.6,x_3)\}$be two SVNSs in U. Then:

i) Neither $A \subseteq B$, nor $B \subseteq A$ (definition 4)

ii) c(A)=$\{(0.6,0.7,0.3,x_1),(0.4,0.7,0.5,x_2), (0.5,0.8, 0.7,x_3)\}$ andc(B)=$\{(0.2,0.9,0.6,x_1),0.5,0.8,0.3,x_2) (0.6,0.9,0.3..,x_3)\}$ (definition 5)

iii)A∪B=$\{(0.6,0.3,0.6,x_1),(0.5,0.3,0.5,x_2), (0.3,0.1,0.5,x_3)\}$ (definition 6)

iv)A∩B=$\{(0.3,0.1,0.2,x_1),(0.3,0.2,0.4,x_2), (0.7,0.2,0.6,x_3)\}$ (definition 7)

4.2 Neutrosophic Topological Spaces

FSs, FL and the related theories for managing the uncertainty have found many and important applications to almost all sectors of human activity (e.g. see [1], Chapter 6, [15], [31, 32], etc.). Fuzzy mathematics, however, has also significantly developed at a theoretical level giving important insights even to traditional branches of pure mathematics, like Algebra, Geometry, Analysis, Topology, etc.

Topological spaces is the most general category of mathematical spaces, in which fundamental mathematical concepts like convergence, continuity, compactness, etc. are defined (e.g. see [33]). Metric spaces and manifolds are special forms of topological spaces satisfying some extra conditions. It is recalled that the concept of a topological space is defined as follows:

Definition 8: A *topology* T on a non-empty set U is defined as a collection of subsets of U such that:
1. U and the empty set belong to T, and
2. The intersection of any two elements of T and the union of any number (finite or infinite) of elements of T belong also to T.

Trivial examples are the *discrete topology* of all subsets of U and the *non-discrete topology* T= {U, ∅}. The *usual* topology on the set ***R*** of real numbers is defined as the set of all subsets A of ***R*** with the property that, for each a in A, there exists $\varepsilon>0$, such that $(a-\varepsilon, a+\varepsilon) \subseteq A$.

The elements of a topology T on U are called *open subsets* of U and their complements are called *closed subsets* of U. The pair (U, T) defines a *topological space (TS)* on U.

The concept of TS has been extended to *fuzzy TS* [34], to *intuitionistic fuzzy TS* [35], to *soft TC* [36], etc. Here we describe how one can extend the concept of TS to *neutrosophic TS* [37].

Definition 9: i) The *empty NS* $\emptyset_N$ on the universe U is defined to be $\emptyset_N = \{(x,0,0,1): x \in U\}$.
ii) The *universal NS* U_N on U is defined to be
$U_N = \{(x,1,1,0): x \in U\}$.
It is straightforward to check that for each NS A in U is $A \cup U_N = U_N$, $A \cap U_N = A$, $A \cup \emptyset_N = A$ and $A \cap \emptyset_N = \emptyset_N$.

Definition 10: A *neutrosophic topology* T on a non-empty set U is defined as a collection of NSs on U such that:
1. U_N and $\emptyset_N$ belong to T, and
2. The intersection of any two elements of T and the union of any number (finite or infinite) of elements of T belong also to T.

Trivial examples are the *discrete neutrosophic topology* of all NSs in U and the *non-discrete neutrosophic topology* T= $\{U_N, \emptyset_N\}$.

The elements of a neutrosophic topology T on U are called *open NSs* in U and their complements are called *closed NSs* in U. The pair (U, T) defines a *neutrosophic topological space (NTS)* on U.

Example 2: Let U = {u} and let A = {(u ,0.5,0.5,0.4)}, B = {(u ,0.4,0.6,0.8)}, C = {(u ,0.5,0.6,0.4)}, D ={(u ,0.4,0.5,0.8)} be NSs in U. Then it is straightforward to check that the collection T = $\{\emptyset_N, U_N$, A, B, C, D} is a neutrosophic topology on U.

We close by extending the concepts of *convergence*, *continuity*, *compact* TS and Hausdorf TS to NTSs.

Definition 11: Given two NSs A and B of the NTS (U, T), B is said to be a *neighborhood* of A, if there exists an open NS O such that $A \subseteq O \subset B$. Further, we say that a sequence $\{A_n\}$ of NSs of (U, T) *converges* to the NS A of (U, T), if there exists a positive integer m such that for each integer $n \ge m$ and each neighborhood B of A we have that $A_n \subset B$.

The following lemma generalizes *Zadeh's extension principle* for FSs [10] to NSs:

Lemma 1: Let U and V be two non-empty crisp sets and let g: U $\rightarrow$ V be a function. Then g can be extended to a function G mapping NSs in U to NSs sets in V.

Proof: Let A<T_A, I_A, F_A> be a NS in U. Then its image G(A) is a NS B in V, whose neutrosophic components are defined as follows: Given v in V, consider the set $g^{-1}(v)=\{u \in U: g(u)=v\}$. If $g^{-1}(v)=\emptyset$, then $T_B(v)=0$, and if $g^{-1}(v)\neq\emptyset$, then $T_B(v)$ is equal to the maximal value of all $T_A(u)$ such that $u \in g^{-1}(v)$. Conversely, the inverse image $G^{-1}(B)$ is the NS A in U with truth membership function $T_A(u)=T_B(g(u))$, for each $u \in U$. In an analogous way one can determine the neutrosophic components I_B and F_B of B.

Definition 12: Let (U,T) and (V,S) be two NTSs and let g be a function g: U $\rightarrow$ V. Then g can be extended to a function G which maps NSs of U to NSs of V. We say then that g is a *neutrosophicaly-continuous* function, if, and only if, the inverse image of each open NS of V through G is an open NS of U.

Definition 13: A family A = $\{A_i, i\in I\}$ of NSs of a NTS (U, T) is called a *cover* of U, if U =$\bigcup_{i\in I} A_i$ If the elements of A are open NSs, then A is called an *open cover* of U. Also, each NS subset of A which is also a cover of U is called a *sub-cover* of A. The NTS (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.

Definition 14: A NTS (U, T) is called a *T_1-NTS*, if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open NSs O_1 and O_2 such that $u_1\in O_1$, $u_2 \notin O_1$ and $u_2\in O_2$, $u_1 \notin O_2$.

Definition 15: A NTS (U, T) is called a *T_2-NTS*, if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open NSs O_1 and O_2 such that $u_1\in O_1$, $u_2\in O_2$ and $O_1\cap O_2 = \emptyset_N$.

A T_2-NTS is also called a *Hausdorff* or a *separable* NTS. Obviously a T_2-NTS is always a T_1-NTS.

5. Discussion and Conclusions

In this work the concept of NS in the universe U, introduced by Smarandanche in 1995 which describes the existing in real life indeterminacy was studied. The basic operations between NSs were presented and the classical notion of TS was extended to NTS. It was further shown that convergence, continuity, compact TS and Hausdorff TS can be naturally extended to NTSs.

It looks in general that proper combinations of the theories developed for tackling the existing in real life uncertainty is a promising tool for obtaining better results in a variety of human activities characterized by uncertainty (e.g. see [28, 29]). This is, therefore, a fruitful area for future research.

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