Invariant Characterization of Liouville's System with Two Degrees of Freedom

ALEXANDER S. SUMBATOV
Federal Research Center "Computer Science and Control"
of the Russian Academy of Sciences
Vavilov str., 44 bl. 2, 119333 Moscow
RUSSIA

Abstract: - The local problem of finding separated generalized coordinates in a holonomic natural system with two degrees of freedom (if the solution exists) is reduced to the problem of integrability of the Pfaffian systems of differential equations.

Key-Words: - Separation of variables, Liouville's system, differential invariants and parameters, overdetermined PDE system, Pfaffian equations

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1 Problem Statement

Let

$$L = T + U = \frac{1}{2} a_{ij}(q^1,\, q^2)\dot{q}^i\dot{q}^j + U(q^1,\, q^2) \quad (1)$$

be the Lagrangian of a holonomic system with two degrees of freedom ($i, j = 1, 2$; the Ricci summation convention is applied throughout the paper). The system refers to local coordinates q^1, q^2 chosen on its configuration 2-D manifold M. All occurring functions of coordinates are supposed to be smooth locally up to desired order. Dot denotes the derivative with respect to time.

As known, [1], the Lagrange equations are integrable in certain generalized coordinates

$$u = u(q^1,\, q^2),\; v = v(q^1,\, q^2) \quad (2)$$

with the help of the method of separation of variables iff the Lagrangian takes the Liouville form

$$L = \frac{1}{2}[h(u) + j(v)][(\dot{u})^2 + (\dot{v})^2] + \frac{m(u)+n(v)}{h(u)+j(v)} \quad (3)$$

after transformation to these coordinates (h, j, m, n are arbitrary functions).

In the case $U \equiv 0$ the problem of recognizing the existence of the Liouville form for a metric

$$ds^2 = 2T\,dt^2 = a_{ij}(q^1,\, q^2)dq^i dq^j \quad (4)$$

was stated in the Theory of Surfaces and has some history, [2], [3], [4], [5], [6], [7]. Nevertheless, it was solved in the explicit form quite recently, [7]. The obtained conditions are very cumbersome but it is the consequence of nature of this problem not of the method used in [7].

The way to specify the separated variables (2) for the metric (4) is not discussed in literature except for [3]. Below we do this for (1) provided that

$$(\partial_1 U)^2 + (\partial_2 U)^2 \neq 0 \quad (\partial_j = \partial/\partial q^j)$$

2 Reducing to the Pfaffian Systems

2.1 Basic Differential Parameters

For any functions φ, ψ of $q^1,\, q^2$ let us introduce

$$\Delta_1\varphi = a^{ij}\partial_i\varphi\,\partial_j\varphi\,, \quad \Delta(\varphi,\psi) = a^{ij}\partial_i\varphi\,\partial_j\psi\,,$$

$$\Theta(\varphi,\psi) = \frac{1}{\delta}\left(\partial_1\varphi\,\partial_2\psi - \partial_2\varphi\,\partial_1\psi\right),$$

$$\Delta_2\varphi = \frac{1}{\delta}\partial_i(\delta\, a^{ij}\partial_j\varphi),$$

where

$$\delta^2 = a_{11}a_{22} - a_{12}a_{21}\,, \quad ||a^{ij}|| = ||a_{ij}||^{-1}$$

These formulae give the basic differential parameters of functions φ and ψ, [2]. If by any reversible point transformation (2) the old metric (4) becomes the new one, then each of these parameters $I = I'$, where I' is I in the accented variables.

Any functions of a_{ij} and their derivatives satisfying this condition is called a differential invariant of the form (4). The classic example is the Gaussian curvature K of the configuration manifold M endowed with a metric (4).

2.2 One Hidden Cyclic Coordinate Existence

When $j = n = 0$ in (3) the coordinate v is cyclic. The problem to recognize, if a metric (4) given in arbitrary generalized coordinates has a hidden (in Hertz's sense) generalized coordinate, was first solved in [2]. Namely, when $K \not\equiv const$ the hidden cyclic coordinate exists iff the differential invariants $\Delta_1 K$ and $\Delta_2 K$ are functions of the Gaussian curvature K. When $K \equiv const$, the geodesic flow for (4) has 3 linear first integrals.

A hidden cyclic coordinate in (1) exists iff the differential parameters $\Delta_1 U$ and $\Delta_2 U$ depend on the force function U only, [8]. Hence, the particular case of the considered problem is exhausted.

2.3 Generic Case

Let the Lagrangian (1) have the Liouville form (3). Then the metric (4) is

$$a_{ij}(q^1, q^2)dq^i dq^j = [h(u)+j(v)][(du)^2+(dv)^2]$$

and the unknown function $u = u(q^1, q^2)$ (or v) satisfies the equations

$$\Delta_2 u = 0, \qquad (5)$$

$$\Delta\left(\frac{\Theta(u,\ \Delta_1 u)}{(\Delta_1 u)^3}\ , u\right) = 0 \qquad (6)$$

written in this metric, and vice versa, [3].

But according to (3) the conformal metric

$$U(q^1, q^2)[a_{ij}(q^1, q^2)dq^i dq^j] = [m(u)+n(v)][(du)^2+(dv)^2]$$

also takes the Liouville form in the same coordinates u, v. Hence, we have 2 supplemental equations such as (5) and (6) but referred to this conformal metric.

It can be easily proved, [9], that after transforming these supplemental equations to the original metric (4), we obtain (5) once more and (6) in the form

$$\Theta(u, \Delta_1 u)\Delta(u, U) - - (\Delta_1 u)^3 \Delta\left(\frac{\Theta(u,U)}{(\Delta_1 u)^2}\ , u\right) = 0 \qquad (7)$$

Thus, if the invariant equations (5), (6) and (7) have a non-trivial solution $u = u(q^1, q^2)$ the Lagrangian (1) can be transformed to the Liouville form, and vice versa.

The system (5-7) of PDE is overdetermined, there are three PDE for one function. Two equations are of the second order and one of the third order.

Now we construct the first continuation of this PDE system. By differentiation of (5) and (7) with respect to q^1, q^2 we obtain four equations which are algebraically linear relative to the derivatives of $u(q^1, q^2)$ of the third order

$$r_1, r_2, s_1, s_2, t_1, t_2 \qquad (8)$$

(the indices specify the numbers of the independent variables q^1, q^2 with respect to which the differentiations were fulfilled). Here, as usual,

$$r = \partial_1 p,\ \ s = \partial_1 q = \partial_2 p,\ \ t = \partial_2 q,$$
$$p = \partial_1 u,\ \ q = \partial_2 u$$

Since $s_1 = r_2$, $s_2 = t_1$, the list (8) of the independent derivatives is shorter

$$r_1, r_2, t_1, t_2 \qquad (9)$$

The determinant of the linear system mentioned above with quantities (9) is not zero: when (4) is written in isothermal (isometric) coordinates, i.e.

$$ds^2 = \lambda(q^1, q^2)[(dq^1)^2 + (dq^2)^2],$$

the determinant equals

$$9\, \lambda(q^1, q^2)(p^2+q^2)[(\partial_1 U)^2 + (\partial_2 U)^2]$$

Hence, the unknowns (9) can be specified by solving the linear system as functions

$$r_1 = f_1\left(p, q, r, s, t, \partial_i U, \partial_{ij}^2 U, \partial_{ijk}^3 U\right), \qquad (10)$$
$$r_2 = f_2(\bullet), \quad t_1 = g_1(\bullet), \quad t_2 = g_2(\bullet)$$

$(i, j, k = 1, 2)$.

The obtained formulae are cumbersome and we do not give them here. The formulae can be derived easily with the help of symbolic computations, e.g. of the system Mathematica. Copy of the notebook Mathematica with detailed explanations is available from the Author.

With the help of (10) the third derivatives of $u(q^1, q^2)$ can be eliminated in (6). As a result, we obtain the quadratic equation with respect to s. When its discriminant is negative the Lagrangian (1) has not the Liouville form.

Denote by s_+ and s_- the real roots as the explicit functions of the unknowns p, q, r, t. Substituting each of them successively in the right-hand sides of (10) we can write two Pfaffian systems

$$du = p\, dq^1 + q\, dq^2,$$
$$dp = r\, dq^1 + s_\pm dq^2,\ dq = s_\pm\, dq^1 + t\, dq^2, \qquad (11)$$
$$dr = f_1\, dq^1 + f_2\, dq^2,\ dt = g_1 dq^1 + g_2\, dq^2$$

Each of these systems is closed with 5 unknown functions u, p, q, r, t.

3 Main Result

Theorem. The Lagrangian (1) has the Liouville form (3) iff one of the Pfaffian systems (11) is completely integrable, i.e. the exterior derivatives of its right-hand sides vanish by virtue of (11).

The converse theorem is evident.

The validity of the direct theorem follows from the structure of the system (11) and from the invariance property of the function *first* differential $dz(x) = \nabla z\, dx$ (∇z is covector, dx is vector). Thus, $(dz) = d(z)$ where the brackets denote any replacement $x = w(y)$.

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4 Conclusion

The formulae of this paper were verified for the simplest cases (Euclidean metric, linear and quadratic force function, elliptic and parabolic coordinates, etc.). Symbolic computations have shown the effectiveness of the proposed approach.

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