Systems of Discrete Walsh-Like Sequential Functions

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Abstract: - In this paper the algorithm for constructing the discrete (0,1)-sequent functions constituting the whole symmetric systems of orthogonal equidistant functions on the example of the eighth-order systems developed. Discrete sequential functions form by replacing their piecewise constant values +1 or -1 in the time domain (from the original space) with numerical values 0 and 1 in the image space. We refer to Walsh-like functions as (0,1)-sequent functions in which the number of zeros and ones in each half of the definition interval is not necessarily the same as in classical Walsh functions. By the directed search method, each of the 30 formed whole groups of equidistant sequent functions unfolds, like the group of classical Walsh functions of the eighth order, into 28 symmetric systems of sequent functions. The main result achieved in this work should consider an expansion of the set of Walsh-like systems of the eighth order by more than an order of magnitude. The algorithm's simplicity for synthesizing such systems of sequential functions and the high speed of spectral processing of discrete signals provided by the proposed bases open the Walsh-like systems for broad prospects of application in various fields of science and technology.

Key-Words: - The sequential functions and systems, the method of a directed enumeration.

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1 Introduction

The theory and technique of spectral analysis of signals mainly focus on signals of sinusoidal forms. Also, non-sinusoidal signals (functions) use in information transmission systems, radiolocation, and other applications [1-3]. A typical example of non-sinusoidal functions is the Walsh function [4]. Their distinctive feature is that the Walsh functions take piecewise constant values to equal $+1$ or -1 in the original space.

Spectral analysis of discrete signals is usually performed based on discrete exponential functions formed by temporal discretization of complex-valued harmonic signals. It knew [5] that discrete Fourier transform (DFT) bases have some requirements, the most important of which are the following. First, it is desirable in the form of basic transform functions to be as close as possible to the state of the analyzed signal. And secondly, the basis function systems must support such a speed of the DFT processors that enables real-time signal processing.

Thus, the choice of a system of basic functions determines by the requirements of convenience calculations. And finally, by the labor intensity of algorithms of realization of the sought transformation. Based on these considerations, the use of fundamental bases of Walsh systems and their extension, Walsh-like sequential systems (the definition of such systems is given further in the text), seems relevant and promising for the digital processing of broadband signals.

Put together and numbered orthogonal Walsh functions of different orders form a system. Let us introduce the notation $\boldsymbol{W}(k,t)$ for discrete Walsh systems, where k is the function's order and $t-$ is the normalized time (argument), whereby $k, t=\overline{0,\ N-1}$. An example of the Walsh functions $\boldsymbol{h}(k,t)$ ordered by Adamar depict in Fig. 1.

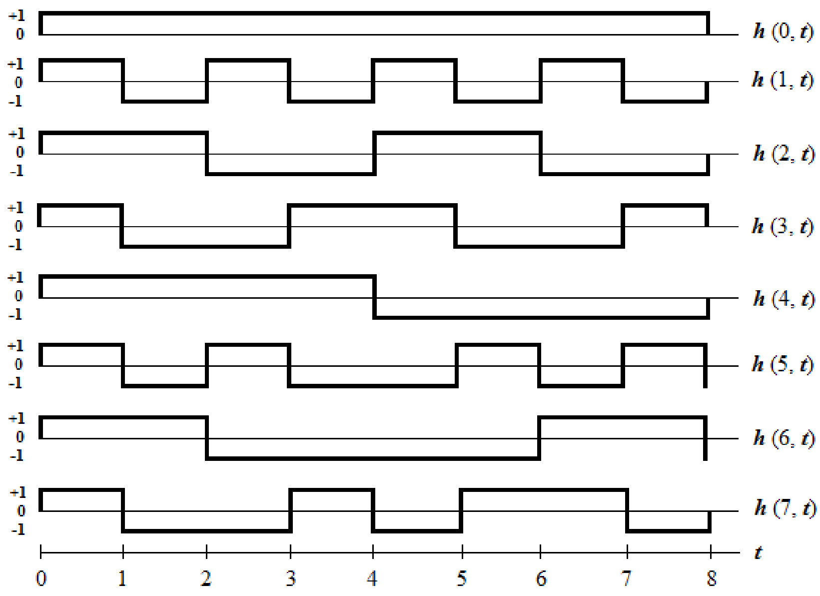

Figure 1. Walsh-Adamar function systems

Replacing the piecewise constant functions $\boldsymbol{w}(k,t)$ with their discrete values $+1$ and -1, we arrive at the matrix forms of Walsh systems in the original space. Below is a sequence of matrices $\boldsymbol{P}_N$ of Paley's first (degenerate), second, and fourth orders of Walsh systems.

$$\boldsymbol{P}_1 = [+1]; \quad \boldsymbol{P}_2 = \{p(k,t)\} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \; t \\ \begin{array}{c} 0 \\ 1 \\ k \end{array} & \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix} \end{array}; \quad \boldsymbol{P}_4 = \{p(k,t)\} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \; t \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ k \end{array} & \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix} \end{array}. \tag{1}$$

A more convenient way to represent systems of Walsh functions is to represent them as square matrices in which each row is a Walsh function. For simplicity, instead of element values $+1$ and -1, write only their signs $+$ and $-$. So, for example, a system of Walsh-Paley functions of the eighth order looks like this:

$$\boldsymbol{P}_8 = \{p(k,t)\} = \begin{array}{c c} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \; t \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ k \end{array} & \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & + & - & + & - \\ + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \end{bmatrix} \end{array}. \tag{2}$$

The Paley matrices $\boldsymbol{P}_N$ (1) or (2) at an arbitrary but binary-degree ordering $N = 2^n$, $n = 1, 2, \ldots$, can construct directly using a simple mnemonic rule [6], the essence of which reduce to the following transformations. At the initial formation stage $\boldsymbol{P}_N$, each row of the previous Paley matrix $\boldsymbol{P}_{N/2}$ writes twice. Then to the first of them (row), the same elements are attributed to the right, i.e., the details of the right half of the row repeat the elements of the left half of the row, and of the second one, the opposite (*complementary*) elements attributed. The above method of forming Walsh-Paley systems is implemented in the image space using the code tree shown in Fig. 2.

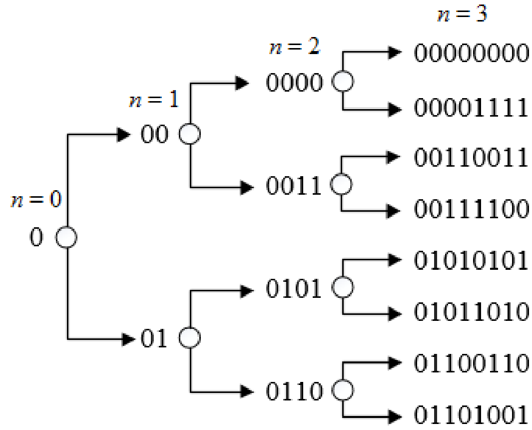

Figure 2. Repeatedly-complementary algorithm synthesis of Walsh-Paley function system

We come to the image space by replacing the discrete values of the Walsh functions $+1$ and -1 in matrices (1) or the $+$ and $-$ signs in matrices (2) with numbers 0 and 1. The matrix Walsh-Paley system of the eighth order in the space of images is represented below by the following relation

$$\boldsymbol{P}_8 = \{ p(k,t) \} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \quad t \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ k \end{matrix} & \begin{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \end{matrix} \end{array} . \qquad (3)$$

The translation of Walsh matrices from the original area, for example, matrix (2), into the space of images, matrix (3), is accompanied by a change in the operation of element-by-element multiplication of the two discrete Walsh functions $\boldsymbol{u} = \{u_i\}$ and $\boldsymbol{v} = \{v_j\}$, $i, j = \overline{0, N-1}$, to the operation of their element-by-element addition modulo 2. Such operations perform, in particular, when calculating the scalar product of these functions $(\boldsymbol{u}, \boldsymbol{v})$ to confirm their orthogonality, given by condition $(\boldsymbol{u}, \boldsymbol{v}) = 0$.

Sequent analysis, a generalization and alternative to spectral harmonic analysis, was formed as an independent discipline at the turn of the 1970s-80s, primarily due to the actual results obtained in the works of H. Hartmut [7, 8]. The success of sequential analysis basis on the fact that instead of sinusoidal signals, Walsh functions and other non-sinusoidal waves use. To date, a sufficiently large number of publications devoted to the theory and application of sequential analysis in various fields of science and technology have appeared, among which we will distinguish a textbook [9], dissertations [10, 11], journal articles [12, 13], etc.

This paper aims to develop algorithms for synthesizing Walsh-like discrete sequential functions that form complete symmetric systems of orthogonal equidistant functions $\boldsymbol{s}(k,t)$,

$k,t=\overline{0,N-1}$, on the example of the eighth-order systems, i.e., for $N=2^3$.

The completeness of a system of discrete sequential functions means that it cannot augment with any new function that would be orthogonal to all other system functions simultaneously. An *equidistance* of $N-$bit sequential functions means that any pair of functions of the system, such as the functions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, is at a Hamming distance d equal to $N/2$, i.e., $d(\boldsymbol{s}_1,\boldsymbol{s}_2)=N/2$.

2 General ratios

We will refer to Walsh-like functions such that the number of zeros and ones in each half of the definition interval is not necessarily the same as, for example, in representations of classical Walsh functions. In the future, for brevity, we will also call sequent function sequences. Thus, apart from zero bytes, the only type of binary (binary) code combinations (codes) considered in this paper are uniform (codes of the same length) eight-bit sequent functions (sequents) with a weight (number of units in a code) equal to four.

Let us form a complete set of sequent functions of the eighth order, including into the set only those functions which begin with zero. It means that the number 0 is placed in the senior (left) bit of each sequent, and three zeros and four ones place in the remaining minor seven bits. Hence, the complete set of such non-zero functions L_8 contains 35 sequences of the eighth order. All these functions are summarized (together with the zero sequent) in Table 1.

Table 1. The set of sequent functions of the eighth order

Number of sequent	Number of the digit									Number of sequent	Number of the digit							
	0	1	2	3	4	5	6	7			0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0		18	0	1	0	1	1	0	0	1
1	0	1	1	1	1	0	0	0		19	0	0	1	1	1	0	0	1
2	0	1	1	1	0	1	0	0		20	0	1	1	0	0	1	0	1
3	0	1	1	0	1	1	0	0		21	0	1	0	1	0	1	0	1
4	0	1	0	1	1	1	0	0		22	0	0	1	1	0	1	0	1
5	0	0	1	1	1	1	0	0		23	0	1	0	0	1	1	0	1
6	0	1	1	1	0	0	1	0		24	0	0	1	0	1	1	0	1
7	0	1	1	0	1	0	1	0		25	0	0	0	1	1	1	0	1
8	0	1	0	1	1	0	1	0		26	0	1	1	0	0	0	1	1
9	0	0	1	1	1	0	1	0		27	0	1	0	1	0	0	1	1
10	0	1	1	0	0	1	1	0		28	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	1	0		29	0	1	0	0	1	0	1	1
12	0	0	1	1	0	1	1	0		30	0	0	1	0	1	0	1	1
13	0	1	0	0	1	1	1	0		31	0	0	0	1	1	0	1	1
14	0	0	1	0	1	1	1	0		32	0	1	0	0	0	1	1	1
15	0	0	0	1	1	1	1	0		33	0	0	1	0	0	1	1	1
16	0	1	1	1	0	0	0	1		34	0	0	0	1	0	1	1	1
17	0	1	1	0	1	0	0	1		35	0	0	0	0	1	1	1	1

Let us compare each non-zero sequent function $\hat{\boldsymbol{s}}_i$, $i=\overline{1,L_N}$, from Table 1 with a set of sequents $\boldsymbol{s}_j\in\boldsymbol{S}_{i,j}$, distant from $\hat{\boldsymbol{s}}_i$ the Hamming distance d, equal to $N/2$, i.e., in our case $d(\boldsymbol{s}_i,\boldsymbol{s}_j)=4$. Let us summarize the functions $\hat{\boldsymbol{s}}_i$ (called *forming sequents*) and the sets $\boldsymbol{S}_{i,j}$ in Table 2. The left column of Table 2 shows the numbers of functions $\hat{\boldsymbol{s}}_i$, and the top row shows the numbers of the sequents that form the sets $\boldsymbol{S}_{i,j}$. Each set $\boldsymbol{S}_{i,j}$ also includes the zero sequent $\boldsymbol{s}_0=\{0,0,0,0,0,0,0,0\}$, not shown in Table 2.

Let us pay attention to such features in Table 2. First, Table 2 is symmetric to the main diagonal. Second, each row of Table 2 includes, in addition to the forming sequent (the light diagonal element highlighted by the bold frame), 18 sequents $\boldsymbol{s}_j$ distant from the forming element at the Hamming distance $d(\hat{\boldsymbol{s}}_k,\boldsymbol{s}_j)=4$. Finally, thirdly, the whole set $\boldsymbol{\Omega}$ of Table rows $\boldsymbol{S}_i$ can be divided into 10 non-intersecting subsets $\boldsymbol{\Omega}_l$, $l=\overline{1,10}$. At the same time, the $l-$th subset includes consecutive rows $\boldsymbol{S}_i$, containing the same number n_l of sequents arranged on the left side of the $\hat{s}_k$ forming sequents. For example, subset $\mathbf{\Omega}_1$ generated by the sequents $\hat{s}_j$, $j = \overline{1, 5}$, with $n_1 = 0$. The second subset $\mathbf{\Omega}_2$ is formed by the sequents $\hat{s}_j$, $j = \overline{6, 9}$, for which $n_2 = 4$, etc. Information about the numerical characteristics of the subsets gives in Table 3.

Table 2. The set of sequent functions distant from the forming sequents at the Hamming distance

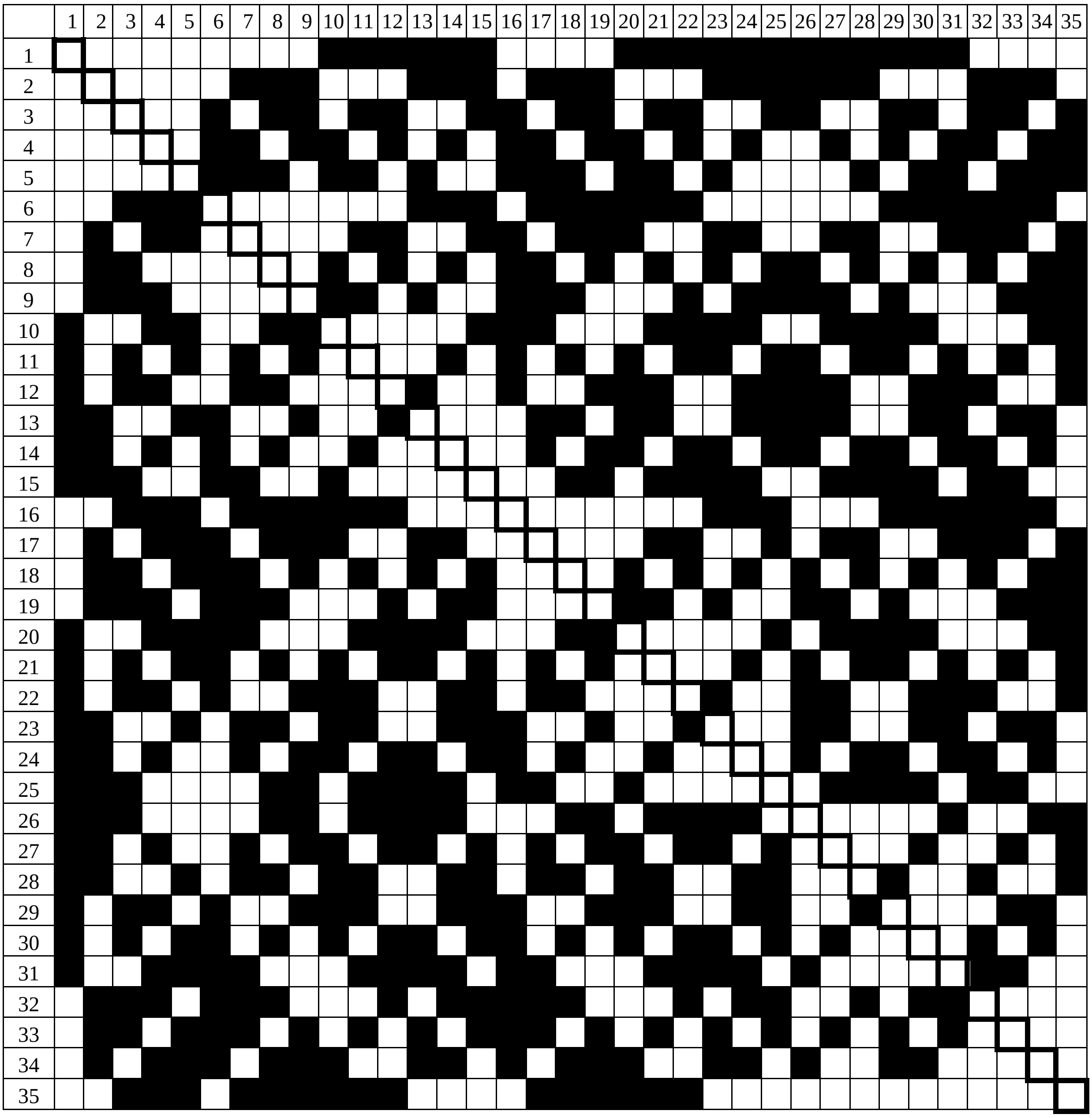

Table 3. Composition of Subsets $\mathbf{\Omega}_l$ of Sequential Functions

	Sequent subset number (l)									
	1	2	3	4	5	6	7	8	9	10
Sequents $\hat{s}_k$	1-5	6-9	10-12	13-15	16-19	20-22	23-25	26-28	29-31	31-35
n_l	0	3	5	6	9	11	12	15	16	18

As the results of elementary calculations have shown, the forming sequents $\hat{\boldsymbol{s}}_k$, together with the zero-sequent function $\boldsymbol{s}_0$ and 18 sequents, which are in the rows of Table 2, form six complete equidistant code combinations (we will call them groups for brevity). Each row in Table 1 corresponds to six groups with eight equidistant sequents. The groups of Sequent equidistant functions $\boldsymbol{SF}_{i,j}$ formed by the forming, for example, sequents $\hat{\boldsymbol{s}}_i$ of the subset $\boldsymbol{\Omega}_1$, are given in Table 4.

Table 4. Composition of groups formed by the forming sequents of a subset $\boldsymbol{\Omega}_1$

Group	Sequents of groups																			
$\boldsymbol{SF}_{1,j}$	0	1	10	11	12	13	14	15	20	21	22	23	24	25	26	27	28	29	30	31
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{2,j}$	0	2	7	8	9	13	14	15	17	18	19	23	24	25	26	27	28	32	33	34
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{3,j}$	0	3	6	8	9	11	12	15	16	18	19	21	22	25	26	29	30	32	33	35
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{4,j}$	0	4	6	7	9	10	12	14	16	17	19	20	22	24	27	29	31	32	34	35
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{5,j}$	0	5	6	7	8	10	11	13	16	17	18	20	21	23	28	30	31	33	34	35
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■

Sequents $\boldsymbol{s}_j$ included groups $\boldsymbol{SF}_{i,j}$ marked by gray cells in the rows of Table 4. And their corresponding numbers of sequential functions are in the black rows of Table 4 located directly above the sequents, and the row element containing the number i of the forming sequents $\hat{\boldsymbol{s}}_i$ lightened. The left column of Table 4 shows the numbers $j=\overline{1,6}$ of groups $\boldsymbol{SF}_{i,j}$ formed by the sequents of $\hat{\boldsymbol{s}}_i$, $i=\overline{1,5}$, that make up the subset of $\boldsymbol{\Omega}_1$.

Table 4 contains all groups $\boldsymbol{SF}$ of equidistant functions generated by formative elements $\hat{\boldsymbol{s}}_i$ of the first subset of sequents $\boldsymbol{\Omega}_1$, which characterizes by the peculiarity that in the rows of Table 2 to the left of the sequents $\hat{\boldsymbol{s}}_1 - \hat{\boldsymbol{s}}_5$, there are no other sequents

$\boldsymbol{s}$. The 30 groups, summarized in Table 3 and corresponding to a subset of sequential functions $\boldsymbol{\Omega}_1$, constitute a complete set of sequential equidistant byte functions. That means — the group of functions formed by any $\hat{\boldsymbol{s}}_j$, $6 \le j \le 35$, absorbed by one of the groups $\boldsymbol{SF}_{i,j}$ subset $\boldsymbol{\Omega}_1$.

Let us confirm this statement with concrete examples. For this purpose, let us choose, for example, the sequents forming $\hat{\boldsymbol{s}}_{17}$ and $\hat{\boldsymbol{s}}_{33}$, and their corresponding complete groups of equidistant functions presented in Table 5.

Table 5. Composition of groups of sequential functions formed by the elements $\hat{\boldsymbol{s}}_{17}$ and $\hat{\boldsymbol{s}}_{33}$

Groups	The sequential of groups																			
$\boldsymbol{SF}_{17,j}$	0	2	4	5	6	8	9	10	13	14	**17**	21	22	25	27	28	31	32	33	35
1																				
2																				
3																				
4																				
5																				
6																				
$\boldsymbol{SF}_{33,j}$	0	2	3	5	6	7	9	11	13	15	16	17	19	21	23	25	27	29	31	**33**
1																				
2																				
3																				
4																				
5																				
6																				

From the data comparison, we can easily see that any group in Table 5 is in one of the rows of Table 4. The correspondence between groups $\boldsymbol{SF}_{17,j}$, $\boldsymbol{SF}_{33,j}$, $j = \overline{1,6}$, and group (i, j) subset $\boldsymbol{\Omega}_1$ showed in Table 6.

Table 6. Composition of sequent function groups, formed by the elements $\hat{\boldsymbol{s}}_{17}$ and $\hat{\boldsymbol{s}}_{33}$

$\boldsymbol{SF}_{17,j}$	1	2	3	4	5	6
(i, j)	2,3	2,5	4,1	4,6	5,1	5,6
$\boldsymbol{SF}_{33,j}$	1	2	3	4	5	6
(i, j)	2,2	2,5	3,2	3,5	5,1	5,3

In the same way, the redundancy of groups generated by sequences $\hat{\boldsymbol{s}}_k$ for all $6 \le k \le 35$ establish.

Let us pay attention to the mosaic of rectangular squares in Table 3. The coloring of all the sites turns out to be the same. And this provides an opportunity to significantly reduce the labor intensity of calculating the composition of groups A of subset B. Suppose a site mosaic maiden for group $\boldsymbol{SF}_{1,j}$, in which the sequent is $\hat{\boldsymbol{s}}_1$. To calculate the sequences of any group $\boldsymbol{SF}_{i,j}$ generated by $\hat{\boldsymbol{s}}_i$, $i \ne 1$, replace the top string $\boldsymbol{S}_1$ in group $\boldsymbol{SF}_{i,j}$ with the string $\boldsymbol{S}_i$.

3 Synthesis of symmetric sequential systems

In applications, it often may be interesting not the complete systems of sequent functions themselves, but some orderings of them, such as, for example, systems of functions forming symmetric bases. Such bases are particularly interesting for spectral analysis of signals or solving other problems of discrete signal processing. In this section of the work, we consider the problem of construction (synthesis) of symmetric sequential bases from the complete set of equidistant sequential groups $\boldsymbol{SF}_{i,j}$.

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Different approaches to the solution of the problem are possible. The synthesis of a symmetric system based on sequent functions basis on the method of direct enumeration [14, 15]. This method allows you to discard unacceptable options in advance. Let us choose from Table 3 as the initial set of sequences for the complete eighth-order group $\boldsymbol{SF}_{1,1} = \{\boldsymbol{s}_0, \hat{\boldsymbol{s}}_1, \boldsymbol{s}_{10}, \boldsymbol{s}_{15}, \boldsymbol{s}_{21}, \boldsymbol{s}_{24}, \boldsymbol{s}_{28}, \boldsymbol{s}_{29}\}$.

Using the data in Table 1, we compose a matrix of group elements $\boldsymbol{SF}_{1,1}$, denoting it by $\boldsymbol{S}(k, t)$, which is not symmetric.

$$\boldsymbol{S}(k,t) = \begin{array}{cc} & \\ 0 & 0 \\ 1 & 1 \\ 2 & 10 \\ 3 & 15 \\ 4 & 21 \\ 5 & 24 \\ 6 & 28 \\ 7 & 29 \\ \downarrow k & \downarrow n \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \to t \\ \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&1&1&1&1&0&0&0 \\ 0&1&1&0&0&1&1&0 \\ 0&0&0&1&1&1&1&0 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&1&0&0&1&0&1&1 \end{bmatrix} \\ \end{array} . \tag{4}$$

In matrix (4), the parameter k is the basis of function order, coinciding with the order number of the functions in the system; t – the function's argument (discrete normalized time); n – is the number of the sequent in Table 1.

In any sequential system in image space, the basis of the zero-order function cannot be rearranged on any other line. The reason is that such a permutation leads to the loss of the symmetry of matrix $\boldsymbol{S}_i(k, t)$. Since all sequents begin with zero, the left column of the matrix is zero by definition, i.e., it consists of only zeros. For this reason, the zero rows of a matrix are "doomed" to occupy its top row. Otherwise, the symmetry condition is violated: each column must coincide with the corresponding matrix row (by the number) in any symmetric matrix.

The following (first) row of matrix $\boldsymbol{S}_i(k, t)$ can contain any of the remaining rows (basis functions) of the matrix (4). Let us choose such a basic function of the first order, i.e., a sequent $\boldsymbol{s}_1$, which results in the first two rows and two columns of the matrix to formed $\boldsymbol{S}_1(k, t)$, namely

$$\boldsymbol{S}_1(k,t) = \begin{array}{cc} & \\ 0 & 0 \\ 1 & 1 \\ 2 & \mathbf{10}, 21, 29 \\ 3 & \\ 4 & \\ 5 & \\ 6 & \\ 7 & \\ \downarrow k & \downarrow n \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \to t \\ \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&1&1&1&1&0&0&0 \\ (0&1) & & & & & & \\ 0&1 & & & & & & \\ 0&1 & & & & & & \\ 0&0 & & & & & & \\ 0&0 & & & & & & \\ 0&0 & & & & & & \end{bmatrix} \\ \end{array} . \tag{5}$$

The possibility of choosing the next (second) row is limited by the condition of maintaining the symmetry of the matrix. To observe this condition, from the remaining rows of the matrix (4), we need to choose only those whose initial elements coincide with the initial elements of the second row of the matrix (5), enclosed in parentheses. The bracketed elements correspond to sequents $\boldsymbol{s}_{10}$, $\boldsymbol{s}_{21}$, $\boldsymbol{s}_{29}$ and matrices (4), whose numbers (10, 21, and 29) write in (5) to the left of the parentheses. Placing in the second row of the matrix $\boldsymbol{S}_1(k, t)$ the basis function (sequent) $\boldsymbol{s}_{10}$ (the number of this sequent marked in bold in matrix $\boldsymbol{S}_1(k, t)$) and continuing the synthesis procedure similarly, we come to a symmetric basis

$$\boldsymbol{S}_1(k,t)=\begin{array}{cc} & \\ 0 & 0 \\ 1 & 1 \\ 2 & \mathbf{10},21,29 \\ 3 & \mathbf{21},29 \\ 4 & 29 \\ 5 & 28 \\ 6 & 24 \\ 7 & 15 \\ \downarrow k & \downarrow n \end{array}\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \rightarrow t \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \\ \end{array}. \qquad (6)$$

Let us turn to the matrix (6). In this matrix, in place of the third row, we can use not only the sequent $\boldsymbol{s}_{21}$ but also the sequent $\boldsymbol{s}_{29}$ and, as a result of further substitutions, we obtain

$$\boldsymbol{S}_2(k,t)=\begin{array}{cc} & \\ 0 & 0 \\ 1 & 1 \\ 2 & \mathbf{10},21,29 \\ 3 & 29 \\ 4 & 21 \\ 5 & 24 \\ 6 & 28 \\ 7 & 15 \\ \downarrow k & \downarrow n \end{array}\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \rightarrow t \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \\ \end{array}. \qquad (7)$$

By the example of matrices (6) and (7), we convince that in the binary image space, the set of sequent functions of the basis closed under the operation of digit addition modulo 2. In contrast, in the original area, the sequent functions of basis matrices are closed under the operation of the element-by-element multiplication of functions.

If a deadlock occurs at any stage of synthesis, proceed as follows. In the row of the synthesized matrix containing at least two alternative sequent numbers $\boldsymbol{s}_i$, the nearest to the "deadlock" row, the left sequent replaces by its neighbor to the right, which may (or may not) resolve the emerging deadlock. If the "deadlock" problem persists with the proposed substitution, select another possible sequent substitution, or go to another first-order sequent function.

$$
\begin{array}{cc|cccccccc}
 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \;\; \to t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 10 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
2 & \mathbf{21},29 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
3 & 28 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
4 & & (0 & 0 & 0 & 0) & & & & \\
5 & & 0 & 1 & 1 & 0 & & & & \\
6 & & 0 & 1 & 0 & 1 & & & & \\
7 & & 0 & 0 & 1 & 1 & & & & \\
\downarrow k & \downarrow n & & & & & & & &
\end{array}
.
$$

In the example under consideration, the deadlock successfully resolves, which leads to a symmetric basis

$$
\boldsymbol{S}_8(k,t) =
\begin{array}{cc}
 & \\
0 & 0 \\
1 & 10 \\
2 & 29 \\
3 & 15 \\
4 & 28 \\
5 & 21 \\
6 & 1 \\
7 & 24 \\
\downarrow k & \downarrow n
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \;\; \to t \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1
\end{bmatrix} \\

\end{array}
.
$$

Based on the considered algorithm of the directed search of basic functions, we come to the complete set consisting of 28 permutations of the sequent $\boldsymbol{s}_i$ group $\boldsymbol{SF}_{1,1}$ (Table 7), each of which generates a symmetric system of sequent functions (an orthogonal basis that has the property of completeness).

Table 7. Transpositions of equidistant sequents of a group $\boldsymbol{SF}_{1,1}$, generating a symmetric basis

Number of basis	Sequent number								Number of basis	Sequent number							
1	0	1	10	21	29	28	24	15	15	0	21	24	29	28	10	15	1
2	0	1	10	29	21	24	28	15	16	0	21	28	1	15	29	24	10
3	0	1	21	10	29	28	15	24	17	0	24	1	28	10	29	15	21
4	0	1	29	21	10	15	24	28	18	0	24	10	15	21	1	28	29
5	0	10	1	24	28	21	29	15	19	0	24	21	28	29	10	15	1
6	0	10	1	28	24	29	21	15	20	0	24	29	15	1	21	28	10
7	0	10	21	24	15	1	29	28	21	0	28	1	10	24	15	21	29
8	0	10	29	15	28	21	1	24	22	0	28	10	29	15	24	1	21
9	0	15	24	21	10	1	29	28	23	0	28	21	1	15	24	29	10
10	0	15	24	29	1	10	21	28	24	0	28	29	21	24	15	10	1
11	0	15	28	1	21	29	10	24	25	0	29	15	24	1	28	10	21
12	0	15	28	10	29	21	1	24	26	0	29	15	28	10	24	1	21
13	0	21	15	1	28	10	24	29	27	0	29	24	15	1	28	21	10
14	0	21	15	10	24	1	28	29	28	0	29	28	24	21	15	10	1

The classical Walsh functions occupy the last row in Table 4, forming the 30-th group of sequential functions $\boldsymbol{SF}_{5,6}$. Group $\boldsymbol{SF}_{5,6}$, as well as all other groups belonging to the subset $\boldsymbol{\Omega}_1$, has its own 28 symmetric bases. Hence, there exist a total of bases of Walsh-like sequent byte functions.

4 Spectral applications

Let's call the input frequency scale of the DFT processor the abscissa axis X of the Cartesian coordinate system on which the normalized frequencies m of the input complex-exponential signal locate

$$\dot{x}_m(t) = \exp\left(j\,\frac{2\pi}{N}\,m\,t \right);\; m, t = \overline{0, N-1};\; N = 2^k . \quad (8)$$

Let's call the output frequency scale the Y ordinate axis intended for placing the numbers of k – th output channels of the processor, from which the k – th complex harmonic takes

$$\dot{\boldsymbol{X}}_m(k) = \sum_{t=0}^{N-1} \dot{x}_m(t) \cdot \varphi(k, t) . \quad (9)$$

We will say that some basis provides the frequency scales of the DFT processor with linear connectivity if the harmonics (9) of the discrete signal (8) with maximum amplitude located on the bisector of the right angle formed by the coordinates m and k. As an example of a basis delivering linear connectivity to the frequency scales of a DFT processor, we can cite the basis of discrete exponential functions (DEF). Similar bases also exist in Walsh systems [16]. In particular, in the set of classical Walsh systems, they are called Walsh-Cooley bases ($\boldsymbol{C}$), and in the set of sequent Walsh-like systems, they are the Walsh-Tukey bases ($\boldsymbol{T}$). Below are the 16-order matrices corresponding to the systems C and T, respectively.

$$\boldsymbol{C}_{16} = c(k,t) = \begin{array}{c|c} k \backslash t & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{matrix} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\ 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0\\ 0&0&1&1&0&0&1&1&1&1&0&0&1&1&0&0\\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0\\ 0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0\\ 0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0\\ 0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1\\ 0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1\\ 0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&0&1&1&1&1&0&0&1&1&0&0&0&0&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix} \end{array} , \quad (10)$$

$$\boldsymbol{T}_{16} = \tau(k,t) = \begin{array}{cc} & \begin{array}{cccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \quad t \\ \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\ 0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0\\ 0&1&1&0&0&0&0&1&1&0&0&1&1&1&1&0\\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0\\ 0&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1\\ 0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1\\ 0&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1\\ 0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1\\ 0&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1\\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\ 0&0&0&1&1&0&0&1&1&1&1&0&0&1&1&0\\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0 \end{pmatrix} \\ k & \end{array} . \qquad (11)$$

The amplitude and phase characteristics of the 16-point DFT processor in the Walsh-Cooley and Walsh-Tukey function bases calculated by formulas (8)-(11) are shown in Fig. 3 and in Table 8, respectively.

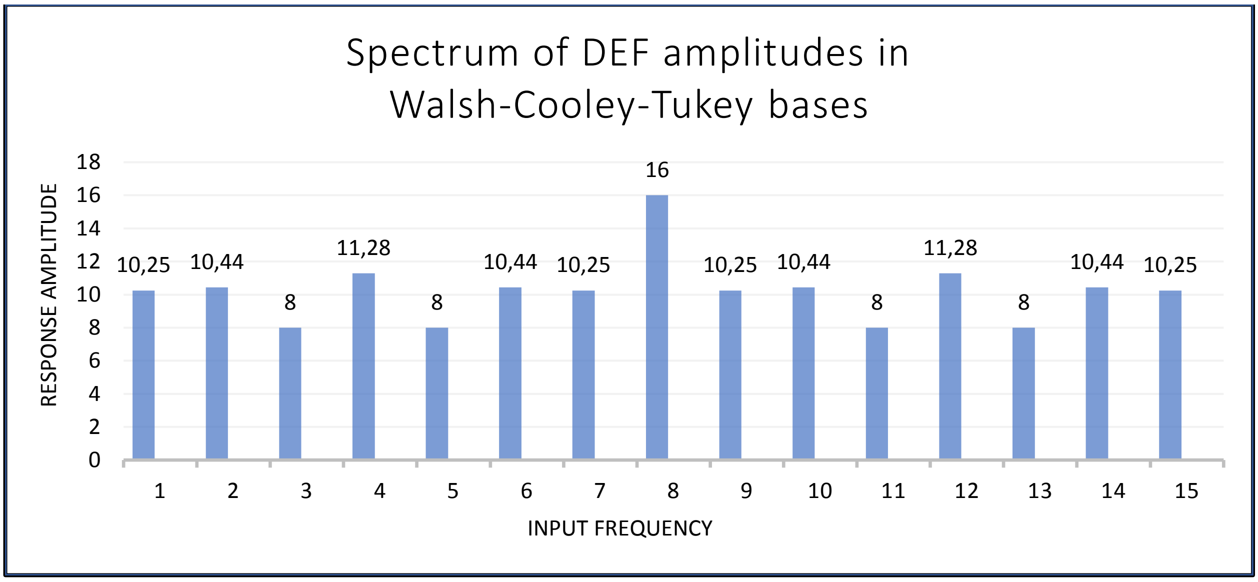

Figure 3. Amplitude-frequency characteristics of complex 16-point DEFs in Cooley and Tukey bases

Table 8. Phase-frequency characteristics of complex 16-point DFTs in Cooley and Tukey bases

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\to k$
Cooley	– 0.20	– 0.39	– 0.65	– 0.79	– 0.92	– 1.18	– 1.37	0	– 0.20	– 0.39	– 0.65	– 0.79	– 0.92	– 1.18	– 1.37	
Tukey	– 1.37	– 1.18	– 0.92	– 0.79	– 0.65	– 0.39	– 0.20	0	– 1.37	– 1.18	– 0.92	– 0.79	– 0.65	– 0.39	– 0.20	

The above data shows that the amplitude spectra of signals in the Walsh-Cooley and Walsh-Tukey bases coincide, while the phase spectra are opposite. If in some *m*-output channel of the DFT point processor, the response phase in the Walsh-Cooley basis is equal to $\varphi_c(k)$, then in the Walsh-Tukey bases $\varphi_\tau(k) = \varphi_c(N-k)$.

5 Results and discussion

The main results achieved by this study are as follows. First, the set of groups $\boldsymbol{SF}_{i,j}$, $j=\overline{1,6}$, of arbitrary degree order N, is replenished only by those sequents $\hat{\boldsymbol{s}}_i$, to the left of which there are no other sequents (except for $\boldsymbol{s}_0$), and this rule does not depend on N. Second, for a subset of sequent $\boldsymbol{\Omega}_1$ of the eighth order, the young sequent $\hat{\boldsymbol{s}}_1$ is eight digits apart from the sequent closest $\boldsymbol{s}$ to its right; the next sequent $\hat{\boldsymbol{s}}_2$ is apart by the sequent nearest to its right at four digits, and so on. And finally, thirdly, the following feature of Walsh-like systems of sequential functions is noticed. As it turned out, each of 29 equidistant sequent groups, not taking into account the 30-th group, which unites the classical Walsh functions, corresponds to 28 symmetric systems, i.e., to the same number as the set of classical Walsh functions of length $N=8$. It knows [14, 15] that Walsh systems of order $N=2^n$, where $n-$is a natural number, are uniquely defined by the so-called indicator matrices (IM) of $n-$order. IM is right-sided symmetric binary matrices in the ring of subtractions modulo 2 (i.e., symmetrical to the auxiliary diagonal). But if one-to-one mappings exist between IMs and their corresponding systems for classical Walsh systems (of arbitrary order), then such correspondence for sequent systems should specify.

6 Future research

Briefly formulated above, the main results of the work predetermine, at least, such directions for further research:

1. Generalize the results for sequent systems of arbitrary binary degree order exceeding eight.
2. Confirm (or disprove) the hypothesis about the existence of a relationship between indicator matrices and their corresponding symmetric Walsh-like systems of sequential functions.
3. Evaluate the feasibility of using one-dimensional (as well as two-dimensional) FFTs on the bases of sequential functions for various applications.

7 Conclusions

The main result achieved by this paper should be considered an expansion by more than an order of magnitude (more precisely, by a factor of 30) of the set of Walshe-like systems of the eighth order. The algorithm's simplicity for synthesizing Walshe-like systems of sequential functions and the high speed of spectral processing of discrete signals provided by the proposed bases open up to such systems (bases) a broad prospect of application in various fields of science and technology.

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