Computation of Troesch Problem by a Modified Newton's Approach

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Abstract: We introduce an improved Newton scheme to solve the nonlinear Troesch problem; which is a nonlinear parameter-sensitive differential equation applied to model plasma confinement in a column.. A predictor-corrector scheme based on a modified backward Euler method was adopted for the Newton's update. The hyperbolic sine component of the governing equation was converted to its logarithmic analog in order to handle high gradients and discontinuities. This was also found to be useful when the initial position values are not very close to the projected equilibrium. The proposed algorithm is straightforward and offers a relatively high degree of accuracy for a remarkably wide range of values of the sensitivity parameter. The trade-off is a slightly more computation time than the classical Newton's approach.

Keywords: Troesch problem, Newton's method, backward Euler, predictor-corrector, sensitivity parameter, nonlinear, hyperbolic sine, logarithmic analog.

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1. Introduction

Consider the following nonlinear boundary value problem (BVP):

$$u''(x)+\lambda\sinh\left(\lambda u(x)\right)=0,\quad 0\le x\le 1,\qquad (1)$$

$$u(0)=0,\qquad u(1)=1\qquad (2)$$

Equations (1) subjected to Equation (2) is known as the Troesch problem. It arises from the confinement of plasma column by radiation pressure as well as in the theory of gas porous electrodes. The numerical rigor that accompanies the solution is connected with the fact that the larger the sensitivity parameter λ, the more difficult it is to determine the solution. In such a case, the dependent variable remains almost constant $\left(u(x)\approx 0\right)$ for $x\ge 0$, but rapidly fulfils the boundary condition at $x=1$. Hence we have a boundary layer close to the right end of the problem domain characterized by a high gradient of the dependent variable.

The closed form solution of this problem in terms of the Jacobi elliptic function is given as:

$$u(x)=$$

$$2\Big/\lambda\sinh^{-1}\left[\frac{u''(0)}{2}sc\left(\lambda x\middle|1-0.25u'(0)^2\right)\right]\qquad (3)$$

where $u'(0)$ is the derivative at $u=0$ and is expressed as: $u'(0)=2\sqrt{1-m}$ where m is the solution of an implicit transcendental equation:

$$\left(\sinh(\lambda/2)\right)\Big/\sqrt{(1-m)}=sc(\lambda/m)\qquad (4)$$

The elliptic Jacob function is defined as: $sc(\lambda/m)=\tan\alpha$ Both α *and* λ are related by the integral equation: $\lambda=\int_0^{\alpha}1/\left(\cosh\theta-m\right)d\theta$. For more details of these derivations see [1 ,2]. From equation (1), it can be noticed that $u(x)$ has a singularity located at $sc(\lambda x/m)$ *or at* $x=\ln\left(8/u'(0)\right)$. It can therefore be deduced that $u'(0)>8e^{-1}$ lies within the integration range. Consequently, the Troesch problem constitutes a considerable challenge to many numerical techniques especially as the value of λ gets bigger [3,4,5]. A closed form solution was obtained by Robert and Shipman[6]. This was followed by several attempts [7,8,9,10]. More recent application includes the so called homotopy methods. They belong to a family of continuation methods and essentially involves a way of finding a solution to a problem by constructing another simpler problem which is developed stepwise into the original one until the final stage of the deformation process leads to the desired solution. [11-15].

In the work reported herein we adopt a finite difference discretization of the governing differential equation after converting the nonlinear hyperbolic sine term to its logarithmic equivalent. We then deploy a predictor-corrector implicit approach involving the first two terms of the Taylor's expansion to facilitate convergence of the dependent variable update. The ease with which the high gradient of the scalar profile resulting from large values of the sensitivity parameter λ was efficiently handled is a notable attractive feature of this numerical technique. The resulting system of nonlinear algebraic equations is then solved iteratively. Finally the accuracy of the proposed technique is demonstrated by comparing the numerical results with those available in literature.

2. Mathematical Formulation

The classical Newton's method for solving nonlinear operator equations can be viewed as a discrete dynamical system that seeks a convergence which is as close as possible to a root. For example if we consider an open subset $\mathbf{\Omega} \subset \mathbf{X}$ and a continuos operator represented as $\mathbf{F}:\mathbf{\Omega} \to \mathbf{Y}$; we seek zeros $\mathbf{x} \in \Omega$ *of* $\mathbf{F}$ of a matrix equation given by :

$$\mathbf{x} \in \Omega: \quad \mathbf{F}(\mathbf{x}) = 0 \tag{5}$$

In order to initiate an iterative procedure, Equation (5) requires a guessed solution or a starting vector that is as close as possible to the solution.. A sequence of iterates $\{x\}_{n=0}^{\infty}$ is then computed by the formula:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{\Delta x} \tag{6}$$

where $\mathbf{\Delta x}_n \in \mathbf{X}$ the increment or correction vector is obtained by solving the following matrix equation.

$$\mathbf{x}_{n+1} = \mathbf{x}_n - (\mathbf{J}_n)^{(-1)} \mathbf{F}(x_n) \tag{7}$$

where $\mathbf{J}_n = \partial \mathbf{F}(x_n) / \partial x$ is the Jacob matrix and $\mathbf{F}(\mathbf{x}_n)$ is obtained by substituting the computed values into the function $\mathbf{F}$. Once the initial guess is properly specified, then the solution tends towards convergence in such a way that $\|\mathbf{F}(\mathbf{x_0})\|$ will be smallut Most attempts to improve on the method have centered on ways of better handling the Jacob matrix as well as guaranteeing that the final determination of the solution vector as suggeted by equation (6) not only avoids a chaotic behavior, but at the same time quarantees guarantees quadratic convergence and stability.

Blankeevoort *et al.* [16] implemented an explicit forward Euler scheme:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Phi \mathbf{F}(\mathbf{x}_n) \tag{8}$$

where Φ is the step size. Though equation (8) has the advantage of being computationally inexepensive, more often than not, it exhibits instability especially when the first guess is outside of the proximity of the solution vector.

As a result, a predictor-corrector approach was adopted to give a more reliable update as indicated in the following equations:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Phi \left(\mathbf{I} - \Phi \mathbf{J}(\mathbf{x}_n) \right)^{(-1)} \mathbf{F}(\mathbf{x}_n) \tag{9}$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Phi \left[\left(\mathbf{F}(\mathbf{x}_n) + \mathbf{F}(\mathbf{x}_{n+1}) \right) / 2 \right] \tag{10}$$

where $\mathbf{I}$ is the identity matrix, $0 \le \Phi \le 1$, after some numerical experiments, the value of Φ was found to be 0.65

polynomial type equations resulting from the logarithmic equivalence of the $\sinh(u)$ term have been found to cope with boundary layers efficiently [17]. Hence, equation (1) and its finite difference discrete analog can be written as:

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{(x^2 + 1)} \right) \tag{11a}$$

$$F(u) = \left[\ln\left[\left(u''/\lambda\right) + \sqrt{\left(\left(u''/\lambda\right)^2 + 1\right)} \right] \right] - \lambda u \qquad (11b)$$

Next, the Jacob matrix is easily determined and should be tridiagonal because each $F_i(u)$ depends on $u_{i-1},\, u_i$ *and* u_{i+1}

$$\mathbf{J} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & a_i & b_i & c_i \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \qquad (12)$$

where

$$\mathbf{J}(i,i{-}1) = \partial \mathbf{F}_i(u)/\partial u_{i,i-1} = a_i = \varphi/(\phi+1) \qquad (13a)$$

$$\mathbf{J}(i,i) = \partial \mathbf{F}_i(u)/\partial u_{i,i} = b_i = -\lambda - 2\varphi/(\phi+1) \qquad (13b)$$

$$\mathbf{J}(i,i{+}1) = \partial \mathbf{F}_i(u)/\partial u_{i,i+1} = c_i = \varphi/(\phi+1) \qquad (13c)$$

and

$$\phi + 1 = \left[\varphi\sqrt{\left(u_{i-1} - 2u_i + u_{i+1}\right)}\right]^2 + 1, \quad \varphi = 1/h^2$$

3. Numerical Results and Discusion

In this section, we verify the level of accuracy and convergence of the proposed algorithm. In all the examples unless otherwise stated, the grid size is $h = .025$ and the allowable convergence error is $e = 10^{-6}$ Tables 1-4 show the numerical results obtained herein compared with some existing literature results [18-23]. In Figures 1 and 2, we display the graphical results for moderate $(\lambda = 3, 5, 8, 10)$

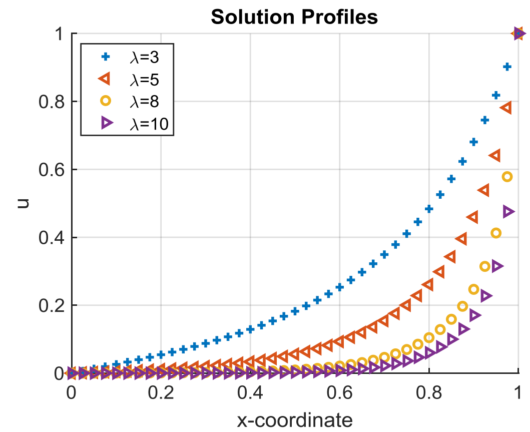

Fig.1 Numerical solution profiles

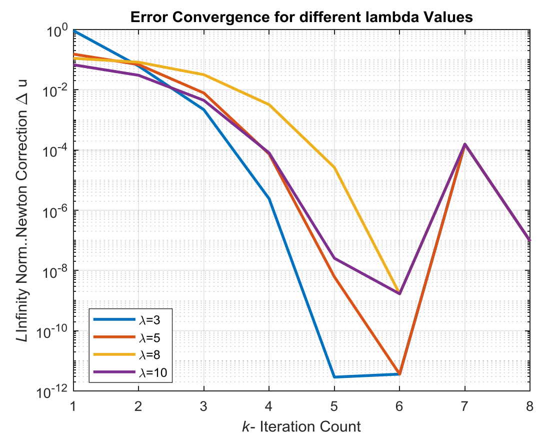

Fig. 2 Convergence trends for Newton updates as well as the converging trends of their updates.

Next, we tested for much larger values of λ. Convergence was achieved after a relatively small number of iterations. Figs. 3-4 display the numerical solutions for $\lambda = 40$ *and* 60.

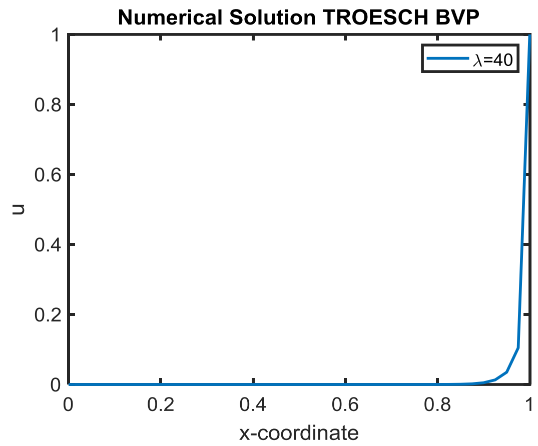

Fig. 3 Numerical solution for $\lambda = 40$

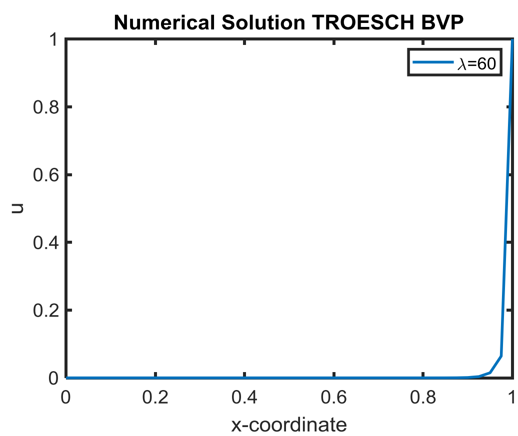

Fig. 4 Numerical solution for $\lambda = 60$

The boundary layer effect as well as accompanying high gradients can be seen as they all approach $x = 1$. The accuracy of the proposed algorithm was further confirmed by comparing the absolute errors between the numerical results and the semi-analytic results of [18]. Fig. 5 shows little or no error for a significant section of the problem domain where the numerical solution lies on top of the x-axis and has a zero gradient.

The magnitude of the errors confirms that the proposed method is highly accurate and can provide faithful results.

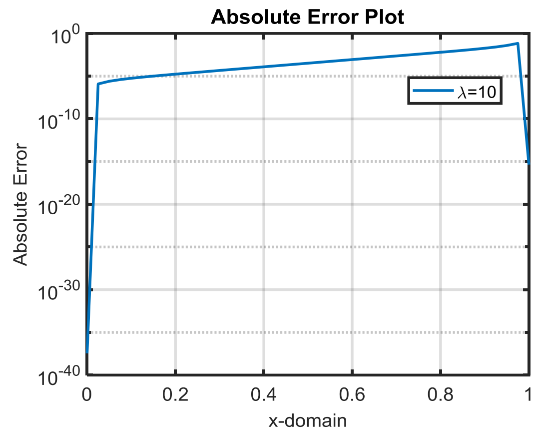

Fig. 5 : Absolute error plot for $\lambda = 10$

4. Conclusion

In this work, we have presented a modified Newton's approach for solving the Troesch's problem in plasma physics. After some modifications, the problem was finally reduced to an iterative solution of algebraic equations. Comparison of numerical results with those available in literature demostrates the utility and the applicability of the proposed algorithm. Most importantly both the formulation and the magnitude of the errors obtained when compared with literature results demonstrate that the method is simple and accurate because by selecting a relatively small number of grid points and a stringent error tolerance highly acceptable numerical results were obtained.

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Table 1. Troesch problem $\lambda = 0.5$

X	Exact	HPM [18]	ADM [21]	This work
0.1	0.0959443493	0.0959443155	0.0959383534	0.0959444533
0.2	0.1921287477	0.1921286848	0.1921180592	0.1921228901
0.3	0.2887944009	0.2887943176	0.2887803297	0.2887946900
0.4	0.3861848464	0.3861847539	0.3861687095	0.3861852045
0.5	0.4845471647	0.4845470753	0.4845302901	0.4845475568
0.6	0.5841332484	0.5841331729	0.5841169798	o.5841336653
0.7	0.6852011483	0.685201943	0.6851868451	0.6852015406
0.8	0.7880165227	0.7880164925	0.7880055691	0.7880168436
0.9	0.8928542161	0.8928542059	0.8928480234	0.8928544098

Table 2. Results of Troesch problem $\lambda = 1.0$

X	Exact	HPM [18]	ADM [21]	This work
0.1	0.0846612565	0.0846607585	0.084248760	0.0846627550
0.2	0.1701713582	0.1701704581	0.169430700	0.1701742936
0.3	0.2573939080	0.2573927827	0.256414500	0.2573981530
0.4	0.3472226551	0.3472217324	0.346085720	0.3472282606
0.5	0.4405998351	0.4405989511	0.439401885	0.4406059971
0.6	0.5385343980	0.5385339413	0.537365700	0.5385409594
0.7	0.6421286091	0.6421286573	0.641083800	0.6421350055
0.8	0.7526080939	0.7526085475	0.751788000	0.7536135539
0.9	0.8713625196	0.8713630450	0.870908700	0.8713659843

Table 3. Results of Troesch problem $\lambda = 5.0$

X	Doha [20]	Collocation [21]	B-spline [22]	This work
0.1	---------------	-----------------	-------------------	
0.2	0.01078872	0.00762552	0.01002027	0.0108758430
0.3	---------------	---------------	---------------------	
0.4	0.03338672	0.03817903	0.03099793	0.0335619889
0.5	---------------	----------------	--------------------	
0.6	---------------	-----------------	-------------------	
0.7	---------------	------------------	-------------------	
0.8	0.25956596	0.23252435	0.24170496	0.2607581400
0.9	0.45706638	0.44624551	0.42461830	0.4594727833

Table 4. Results of Troesch problem $\lambda = 10.0$

X	Temimi [23]	This work
0.1	0.0000421119	0.0000477583
0.2	0.0001299641	0.0001132381
0.3	0.0003589784	0.0003153550
0.4	0.0009779027	0.0008580085
0.5	0.0026590204	0.0029869420
0.6	0.0072289310	0.0080996144
0.7	0.0196640631	0.0171214972
0.8	0.0537303294	0.0599571693
0.9	0.1521140764	0.1705875017