On Some Identities of Bivariate Fibonacci and Bivariate Lucas Polynomials

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***Abstract*—In this paper, we present some identities of Bivariate Fibonacci polynomials and Bivariate Lucas polynomials. We have used their Binet's formula and Generating function to derive the identities.**

***Keywords*— Bivariate Fibonacci polynomials, bivariate Lucas polynomials, Binet's formula and Generating function.**

I. INTRODUCTION

Mario Catalani, [9, 10, 11], define generalized bivariate polynomials, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained and derived many interesting identities. Also derive a collection of identities for bivariate Fibonacci and Lucas polynomials using essentially a matrix approach as well as properties of such polynomials when the variables x and y are replaced by polynomials. Jacob, Reutenauer, Sakarovitch [3]; Belbachir and Bencherif, [4]; Inoue and Aki, [6]; Kaygisiz and Sahin, [7] and Tasci, [2], present theories of the bivariate Fibonacci and Lucas polynomials are developed. Panwar and Singh, [13], present generalized bivariate Fibonacci-Like polynomials sequence and its properties like Catalan's identity, Cassini's identity or Simpson's identity and d'ocagnes's identity for generalized bivariate Fibonacci-Like polynomials through Binet's formulas. Alves and Catarino, [1], present to the reader elements and information derived from a formal and computational mathematical model, in order to demonstrate a greater, complete and broad understanding of some studies on the subject that require a greater scientific discussion and dissemination, in order to provide an evolutionary understanding about the current research of some subjects in Mathematics and provide a systematic and simplified view of the various properties (related to the BFP, BLP, BCFP and BCLP) discussed throughout the paper. Cakmak and Karaduman, [12], present the new algebraic properties related to bivariate Fibonacci polynomials have been given and the partial derivatives of these polynomials in the form of convolution of bivariate Fibonacci polynomials. Also, they define a new recurrence relation for the r-th partial derivative sequence of bivariate Fibonacci polynomials. Panwar, Gupta and Bhandari, [14], derived many identities of bivariate Fibonacci and Lucas polynomials through Binet's formulas. Also present identities related to their sum and difference of squares involving them and describe some generalized identities involving product of bivariate Fibonacci and Lucas polynomials. In this study, we present and define some identities of Bivariate Fibonacci and Bivariate Lucas polynomials.

II. PRELIMINARIES

For $n \geq 2$, the bivariate Fibonacci polynomials sequence is defined by

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y) \tag{2.1}$$

So, the first bivariate Fibonacci polynomials are

$$\{F_n(x,y)\} = \{0, 1, x, x^2+y, x^3+2xy, x^4+3x^2y+y^2, \ldots\}$$

Binet's formula for the bivariate Fibonacci polynomials:

$$F_n(x,y) = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \tag{2.2}$$

Generating function for the bivariate Fibonacci polynomials:

$$F_n(x,y) = \frac{t}{(1-xt-yt^2)} \tag{2.3}$$

For $n \geq 2$, the bivariate Lucas polynomials sequence is defined by

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y) \tag{2.4}$$

So, the first bivariate Lucas polynomials are

$$\{L_n(x,y)\} = \{2, x, x^2+2y, x^3+3xy, x^4+4x^2y+2y^2, \ldots\}$$

Binet's formula for the bivariate Lucas polynomials:

$$L_n(x,y) = \Re_1^n + \Re_2^n \tag{2.5}$$

Generating function for the bivariate Lucas polynomials:

$$L_n\left(x,y\right)=\frac{2-xt}{\left(1-xt-yt^2\right)} \tag{2.6}$$

The characteristic equation of recurrence relation (2.1) and (2.4) is: $t^2-xt-y=0$ (2.7)

Where $x\neq 0,\ y\neq 0\ and\ x^2+4y\neq 0$.

This equation has two real roots:

$\Re_1=\frac{x+\sqrt{x^2+4y}}{2}$ and $\Re_2=\frac{x-\sqrt{x^2+4y}}{2}$.

Note that

$\Re_1+\Re_2=x, \Re_1\Re_2=-y, \Re_1-\Re_2=\sqrt{x^2+4y}$.

Also $F_{-n}(x,y)=\frac{-1}{(-y)^n}F_n(x,y)$ and

$L_{-n}(x,y)=\frac{1}{(-y)^n}L_n(x,y)$.

III. Expression of Sums of Bivariate Fibonacci and Bivariate Lucas Polynomials

In this section, we establish many identities and relations for Bivariate Fibonacci polynomials and Bivariate Lucas polynomials.

Proposition 1: For any integer $n\geq 0$ [Panwar et al., 2020],

$$\begin{aligned}\Re_1^{n+2}&=x\Re_1^{n+1}+y\Re_1^{n}\\ \Re_2^{n+2}&=x\Re_2^{n+1}+y\Re_2^{n}\end{aligned} \tag{3.1}$$

Proposition 2: If $F_n(x,y)$ is Bivariate Fibonacci polynomials, then for $p,q\in$ □

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}F_n(x,y)=F_{2p+2q}(x,y) \tag{3.2}$$

Proof: By the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}F_n(x,y)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}\left(\frac{\Re_1^n-\Re_2^n}{\Re_1-\Re_2}\right)$$

$$=\frac{1}{\Re_1-\Re_2}\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}\left(\Re_1^n-\Re_2^n\right)$$

$$=\frac{1}{\Re_1-\Re_2}\sum_{n=0}^{p+q}\binom{p+q}{n}\left\{(x\Re_1)^n-(x\Re_2)^n\right\}y^{p+q-n}$$

$$=\frac{1}{\Re_1-\Re_2}\left\{(x\Re_1+y)^{p+q}-(x\Re_2+y)^{p+q}\right\}$$

Since $\Re_1$ & $\Re_2$ are the roots of the characteristic equation $t^2-xt-y=0$, using Lemma:

$$=\frac{(\Re_1^2)^{p+q}-(\Re_2^2)^{p+q}}{\Re_1-\Re_2}$$

$$=F_{2p+2q}(x,y)$$

This completes the proof.

Theorem 3: If $F_n(x,y)$ is Bivariate Fibonacci polynomials, then for $p,q\in$ □

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n(-y)^{p+q-n}F_n(x,y)=\sum_{n=0}^{p+q}\binom{p+q}{n}(-2y)^n F_{2(p+q-n)}(x,y) \tag{3.3}$$

Proof: By the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n(-y)^{p+q-n}F_n(x,y)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}x^n(-y)^{p+q-n}\left(\frac{\Re_1^n-\Re_2^n}{\Re_1-\Re_2}\right)$$

$$=\frac{1}{\Re_1-\Re_2}\sum_{n=0}^{p+q}\binom{p+q}{n}x^n(-y)^{p+q-n}\left(\Re_1^n-\Re_2^n\right)$$

$$=\frac{1}{\Re_1-\Re_2}\sum_{n=0}^{p+q}\binom{p+q}{n}\left\{(x\Re_1)^n-(x\Re_2)^n\right\}(-y)^{p+q-n}$$

$$=\frac{1}{\Re_1-\Re_2}\left\{(x\Re_1-y)^{p+q}-(x\Re_2-y)^{p+q}\right\}$$

Since $\Re_1 and \Re_2$ are the roots of the characteristic equation

$t^2-xt-y=0$,

$x\Re_1-y=\Re_1^2-2y$

$x\Re_2-y=\Re_2^2-2y$

Thus,

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n(-y)^{p+q-n}F_n(x,y)=\frac{(\Re_1^2-2y)^{p+q}-(\Re_2^2-2y)^{p+q}}{\Re_1-\Re_2}$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}(-2y)^n F_{2(p+q-n)}(x,y)$$

This completes the proof.

PROPOSITION 4: If $L_n(x,y)$ is Bivariate Lucas polynomials, then for $p,q\in□$

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^{p+q-n}(-1)^n L_n(x,y)=L_{p+q}(x,y) \tag{3.4}$$

Proof: By the Binet's formula of bivariate Lucas polynomials,

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^{p+q-n}(-1)^n L_n(x,y)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}x^{p+q-n}(-1)^n\left(\Re_1^r+\Re_2^r\right)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}x^{p+q-n}(-1)^n\left(-\Re_1^n\right)+\sum_{n=0}^{p+q}\binom{p+q}{n}x^{p+q-n}(-1)^n\left(-\Re_2^n\right)$$

$$=(x-\Re_1)^{p+q}+(x-\Re_2)^{p+q}$$

$$=\left(\frac{-y}{\Re_1}\right)^{p+q}+\left(\frac{-y}{\Re_2}\right)^{p+q}$$

$$=(-y)^{p+q}\frac{\left(\Re_1^{p+q}+\Re_2^{p+q}\right)}{\left(\Re_1\Re_2\right)^{p+q}}$$

$$=L_{p+q}(x,y)$$

This completes the proof.

PROPOSITION 5: If $L_n(x,y)$ is Bivariate Lucas polynomials, then for $p,q\in□$

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}L_n(x,y)=L_{2p+2q}(x,y) \tag{3.5}$$

Proof: By the Binet's formula of bivariate Lucas polynomials,

$$\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}L_n(x,y)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}x^n y^{p+q-n}\left(\Re_1^n+\Re_2^n\right)$$

$$=\sum_{n=0}^{p+q}\binom{p+q}{n}(x\Re_1)^n y^{p+q-n}+\sum_{n=0}^{p+q}\binom{p+q}{n}(x\Re_2)^n y^{p+q-n}$$

$$=(x\Re_1+y)^{p+q}+(x\Re_2+y)^{p+q}$$

Since $\Re_1$ *and* $\Re_2$ are the roots of the characteristic equation $t^2-xt-y=0$,

$$=(\Re_1^2)^{p+q}+(\Re_2^2)^{p+q}$$

$$=L_{2p+2q}(x,y)$$

This completes the proof.

PROPOSITION 6: If $F_n(x,y)$ *and* $L_n(x,y)$ is Bivariate Fibonacci and Bivariate Lucas polynomials, then for $n\geq p+q$,

$$\begin{aligned}F_{n+p+q}(x,y)-(-y)^{p+q}F_{n-p-q}(x,y)\\=F_{p+q}(x,y)L_n(x,y)\end{aligned} \tag{3.6}$$

Proof: By the Binet's formula of bivariate Fibonacci polynomials and bivariate Lucas polynomials,

$$F_{n+p+q}(x,y)-(-y)^{p+q}F_{n-p-q}(x,y)$$

$$=\left(\frac{\Re_1^{n+p+q}-\Re_2^{n+p+q}}{\Re_1-\Re_2}\right)-(-y)^{p+q}\left(\frac{\Re_1^{n-p-q}-\Re_2^{n-p-q}}{\Re_1-\Re_2}\right)$$

$$=\frac{(\Re_1^{n+p+q}-\Re_2^{n+p+q})-(-y)^{p+q}(\Re_1^{n-p-q}-\Re_2^{n-p-q})}{\Re_1-\Re_2}$$

$$=\frac{(\Re_1^{n+p+q}-\Re_2^{n+p+q})-(\Re_1\Re_2)^{p+q}(\Re_1^{n-p-q}-\Re_2^{n-p-q})}{\Re_1-\Re_2}$$

$$=\frac{(\Re_1^{n+p+q}-\Re_2^{n+p+q})-(\Re_1^n\Re_2^{p+q}-\Re_1^{p+q}\Re_2^n)}{\Re_1-\Re_2}$$

$$=\left(\frac{\Re_1^{p+q}-\Re_2^{p+q}}{\Re_1-\Re_2}\right)(\Re_1^n+\Re_2^n)$$

$$=F_{p+q}(x,y)L_n(x,y)$$

This completes the proof.

IV. SUMS OF BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS WITH NEGATIVE INDICES

In this section, we study the sums of bivariate Fibonacci and Lucas polynomials for negative indices.

THEOREM 7: If $F_n(x,y)$ is Bivariate Fibonacci polynomials, then for $p\geq 1$ and q any integer,

$$\sum_{i=0}^{n}(-y)^i F_{-pi-q}(x,y)=\begin{cases}\dfrac{F_{pn+p+q}(x,y)-(-y)^p F_{pn+q}(x,y)-(-y)^q F_{p-q}(x,y)-F_q(x,y)}{(-1)^p-L_p(x,y)+1}\ ;\ if\ q<p\\[2ex]\dfrac{F_{pn+p+q}(x,y)-(-y)^p F_{pn+q}(x,y)+(-y)^q F_{q-p}(x,y)-F_q(x,y)}{(-1)^p-L_p(x,y)+1}\ ;\ otherwise\end{cases} \tag{4.1}$$

Proof: Since $\sum_{i=0}^{n}(-y)^i F_{-pi-q}(x,y)=-\sum_{i=0}^{n}F_{pi+q}(x,y)$ (4.2)

By the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{i=0}^{n}(-y)^{i}F_{-pi-q}(x,y)=-\sum_{i=0}^{n}\frac{\Re_1^{pi+q}-\Re_2^{pi+q}}{\Re_1-\Re_2}$$

$$=\frac{-1}{\Re_1-\Re_2}\left(\Re_1^{q}\sum_{i=0}^{n}\Re_1^{pi}-\Re_2^{q}\sum_{i=0}^{n}\Re_2^{pi}\right)$$

$$=\frac{-1}{\Re_1-\Re_2}\left[\frac{\Re_1^{pn+p+q}-\Re_1^{q}}{\Re_1^{p}-1}-\frac{\Re_2^{pn+p+q}-\Re_2^{q}}{\Re_2^{p}-1}\right]$$

$$=\frac{\left(\Re_1^{pn+p+q}-\Re_2^{pn+p+q}\right)-\left(\Re_1\Re_2\right)^{p}\left(\Re_1^{pn+q}-\Re_2^{pn+q}\right)-\left(\Re_1^{p}\Re_2^{q}-\Re_1^{q}\Re_2^{p}\right)-\left(\Re_1^{q}-\Re_2^{q}\right)}{(\Re_1-\Re_2)\left\{\left(\Re_1\Re_2\right)^{p}-\left(\Re_1^{p}+\Re_2^{p}\right)+1\right\}}$$

$$=\begin{cases}\dfrac{F_{pn+p+q}(x,y)-(-y)^{p}F_{pn+q}(x,y)-(-y)^{q}F_{p-q}(x,y)-F_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{if } q<p\\[2ex] \dfrac{F_{pn+p+q}(x,y)-(-y)^{p}F_{pn+q}(x,y)+(-y)^{q}F_{q-p}(x,y)-F_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{otherwise}\end{cases}$$

This completes the proof.

THEOREM 7: If $L_n(x,y)$ is Bivariate Lucas polynomials, then for $p\geq 1$ and q any integer,

$$\sum_{i=0}^{n}(-y)^{i}L_{-pi-q}(x,y) =\begin{cases}\dfrac{(-y)^{p}L_{pn+q}(x,y)-L_{pn+p+q}(x,y)-(-y)^{q}L_{p-q}(x,y)+L_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{if } q<p\\[2ex] \dfrac{(-y)^{p}L_{pn+q}(x,y)-L_{pn+p+q}(x,y)+(-y)^{q}L_{q-p}(x,y)+L_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{otherwise}\end{cases} \tag{4.3}$$

Proof: Since $\sum_{i=0}^{n}(-y)^{i}L_{-pi-q}(x,y)=\sum_{i=0}^{n}L_{pi+q}(x,y)$ (4.4)

By the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{i=0}^{n}(-y)^{i}L_{-pi-q}(x,y)=\sum_{i=0}^{n}\left(\Re_1^{pi+q}+\Re_2^{pi+q}\right)$$

$$=\Re_1^{q}\sum_{i=0}^{n}\Re_1^{pi}+\Re_2^{q}\sum_{i=0}^{n}\Re_2^{pi}$$

$$=\frac{\Re_1^{pn+p+q}-\Re_1^{q}}{\Re_1^{p}-1}+\frac{\Re_2^{pn+p+q}-\Re_2^{q}}{\Re_2^{p}-1}$$

$$=\frac{\left(\Re_1\Re_2\right)^{p}\left(\Re_1^{pn+q}+\Re_2^{pn+q}\right)-\left(\Re_1^{pn+p+q}+\Re_2^{pn+p+q}\right)-\left(\Re_1^{p}\Re_2^{q}+\Re_1^{q}\Re_2^{p}\right)+\left(\Re_1^{q}+\Re_2^{q}\right)}{(\Re_1-\Re_2)\left\{\left(\Re_1\Re_2\right)^{p}-\left(\Re_1^{p}+\Re_2^{p}\right)+1\right\}}$$

$$=\begin{cases}\dfrac{(-y)^{p}L_{pn+q}(x,y)-L_{pn+p+q}(x,y)-(-y)^{q}L_{p-q}(x,y)+L_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{if } q<p\\[2ex] \dfrac{(-y)^{p}L_{pn+q}(x,y)-L_{pn+p+q}(x,y)+(-y)^{q}L_{q-p}(x,y)+L_{q}(x,y)}{(-1)^{p}-L_{p}(x,y)+1}\,; & \text{otherwise}\end{cases}$$

This completes the proof.

V. GENERATING FUNCTION OF BIVARIATE FIBONACCI AND BIVARIATE LUCAS POLYNOMIALS

In this section, we establish some identities for bivariate Fibonacci and bivariate Lucas polynomials. Generating functions are very helpful in finding of relations for sequences of integers. Some authors found miscellaneous identities for bivariate Fibonacci and bivariate Lucas polynomials by manipulation with their generating functions. Our approach is rather different in this section.

COROLLARY 8:

$$\sum_{n=0}^{\infty}\left\{F_n(x,y)+F_{n+1}(x,y)\right\}t^{n}=\frac{(1+t)}{1-xt-yt^{2}} \tag{5.1}$$

COROLLARY 9:

$$\sum_{n=0}^{\infty}\left\{L_n(x,y)+L_{n+1}(x,y)\right\}t^{n}=\frac{(2y-x)t+(2-x)}{1-xt-yt^{2}} \tag{5.2}$$

THEOREM 10: For $p,q\in$ □ , we get

$$\sum_{n=0}^{p+q}F_n(x,y)t^{-n} =\frac{1}{t^{p+q}(t^{2}-xt-y)}\left\{t^{p+q+1}-tF_{p+q+1}(x,y)-yF_{p+q}(x,y)\right\} \tag{5.3}$$

Proof: By the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{n=0}^{p+q}F_n(x,y)t^{-n}=\sum_{n=0}^{p+q}\left(\frac{\Re_1^{n}-\Re_2^{n}}{\Re_1-\Re_2}\right)t^{-n}$$

$$=\frac{1}{\Re_1-\Re_2}\sum_{n=0}^{p+q}\left\{\left(\frac{\Re_1}{t}\right)^{n}-\left(\frac{\Re_2}{t}\right)^{n}\right\}$$

$$=\frac{1}{\Re_1-\Re_2}\left\{\frac{1-\left(\frac{\Re_1}{t}\right)^{p+q+1}}{1-\frac{\Re_1}{t}}-\frac{1-\left(\frac{\Re_2}{t}\right)^{p+q+1}}{1-\frac{\Re_2}{t}}\right\}$$

$$=\frac{1}{(\Re_1-\Re_2)t^{p+q}}\left(\frac{t^{p+q+1}-\Re_1^{p+q+1}}{t-\Re_1}-\frac{t^{p+q+1}-\Re_2^{p+q+1}}{t-\Re_2}\right)$$

$$=\frac{1}{(\Re_1-\Re_2)t^{p+q}}\left\{\frac{t^{p+q+1}\left(\Re_1-\Re_2\right)-t\left(\Re_1^{p+q+1}-\Re_2^{p+q+1}\right)-y\left(\Re_1^{p+q}-\Re_2^{p+q}\right)}{\left(t-\Re_1\right)\left(t-\Re_2\right)}\right\}$$

$$=\frac{1}{t^{p+q}(t^{2}-xt-y)}\left\{t^{p+q+1}-tF_{p+q+1}(x,y)-yF_{p+q}(x,y)\right\}$$

This completes the proof.

THEOREM 11: For $p, q \in \square$, we get

$$\sum_{n=0}^{p+q} L_n(x,y)t^{-n} = \frac{2t^2 - xt}{(t^2 - xt - y)} - \frac{1}{t^{p+q}(t^2 - xt - y)}\left\{tL_{p+q+1}(x,y) + yL_{p+q}(x,y)\right\} \tag{5.4}$$

Proof: By the Binet's formula of bivariate Lucas polynomials,

$$\sum_{n=0}^{p+q} L_n(x,y)t^{-n} = \sum_{n=0}^{p+q}\left(\Re_1^r + \Re_2^r\right)t^{-n}$$

$$= \sum_{n=0}^{p+q}\left\{\left(\frac{\Re_1}{t}\right)^n + \left(\frac{\Re_2}{t}\right)^n\right\}$$

$$= \frac{1-\left(\frac{\Re_1}{t}\right)^{p+q+1}}{1-\frac{\Re_1}{t}} + \frac{1-\left(\frac{\Re_2}{t}\right)^{p+q+1}}{1-\frac{\Re_2}{t}}$$

$$= \frac{1}{t^n}\left(\frac{t^{p+q+1} - \Re_1^{p+q+1}}{t-\Re_1} + \frac{t^{p+q+1} - \Re_2^{p+q+1}}{t-\Re_2}\right)$$

$$= \frac{2t^{p+q+2} - t\left(\Re_1^{p+q+1} + \Re_2^{p+q+1}\right) - t^{p+q+1}\left(\Re_1 + \Re_2\right) + \Re_1\Re_2\left(\Re_1^{p+q} + \Re_2^{p+q}\right)}{t^{p+q}\left(t-\Re_1\right)\left(t-\Re_2\right)}$$

$$= \frac{2t^2 - xt}{(t^2 - xt - y)} - \frac{1}{t^{p+q}(t^2 - xt - y)}\left\{tL_{p+q+1}(x,y) + yL_{p+q}(x,y)\right\}$$

This completes the proof.

VI. CONCLUSION

In this paper, we have derived many identities of bivariate Fibonacci and bivariate Lucas polynomials through Binet's formula and generating function. We describe sums of bivariate Fibonacci and bivariate Lucas polynomials. This enables us to give in a straightforward way several formulas for the sums of such Polynomials. These identities can be used to develop new identities of polynomials. Also we describe some sums with negative indices and give several interesting identities involving them.

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