Analysis of behavior of a simple eigenvalue of singular system

Maria Isabel García-Planas*, Sonia Tarragona†

* Matemàtica Aplicada I,
Universitat Politècnica de Catalunya, C. Minería 1, Esc. C, 1-3, 08038 Barcelona, Spain
(e-mail: maria.isabel.garcia@upc.edu)
† Departamento de Matemáticas,
Universidad de León, Campus de Vegazana s/n, 24071, León,
(e-mail: sonia.tarragona@unileon.es)

***Abstract*—Small perturbations of simple eigenvalues with a change of parameters is a problem of general interest in applied mathematics. The aim of this work is to study the behavior of a simple eigenvalue of singular linear system family**

$$\left.\begin{array}{rl} E(p)\dot{x} & = A(p)x + B(p)u, \\ y & = C(p)x \end{array}\right\}$$

smoothly dependent on real parameters $p = (p_1, \ldots, p_n)$.

***Index Terms*—Singular linear systems, Eigenvalues, Eigenvectors, Perturbation.**

1. Introduction

Let us consider a finite-dimensional singular linear time-invariant system

$$\left.\begin{array}{rl} E\dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{array}\right\} \quad x(t_0) = x_0, \quad (1)$$

where x is the state vector, u is the input (or control) vector, $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, $C \in M_{p\times n}(\mathbb{C})$ and $\dot{x} = dx/dt$. We will represent the systems as quadruples of matrices (E, A, B, C). In the case where $E = I_n$ the systems are standard and we will denote them, as triples (A, B, C).

Singular systems are found in engineering systems such as electrical, chemical processing circuit or power systems, aircraft guidance and control, mechanical industrial plants, acoustic noise control, among others, and they have attracted interest in recent years.

Sometimes it is possible to change the value of some eigenvalues introducing proportional and derivative feedback controls in the system and proportional and derivative output injection. The values of the eigenvalues that can not be modified by any feedback (proportional or derivative) and/or output injection (proportional or derivative), correspond to the eigenvalues of the singular pencil $\begin{pmatrix} sE - A & B \\ C & 0 \end{pmatrix}$, that we will simply call eigenvalues of the quadruple (E, A, B, C).

Perturbation theory of linear systems has been extensively studied over the last years starting from the works of Rayleigh and Schrodinger [6], and more recently different works as [2], [5],[4], can be found. This treatment of eigenvalues is a tool for efficiently approximating the influence of small perturbations on different properties of the unperturbed system.

Small perturbations of simple eigenvalues with a change of parameters is a problem of general interest in applied mathematics and concretely, this study for the kind of systems under consideration have some interest because in the case where $m = p < n$, the most generic types of systems have $n-m$ simple eigenvalues. The obtained results can be applied to analyze the frequency and damping perturbations in models of flexible structures for example, (see [7], [8]).

In the sequel and without lost of generality, we will consider systems such that matrices B and C have full rank and $m = p < n$.

2. Feedback and output injection equivalence relation

Definition 2.1: Two quadruples (E, A, B, C) and (E', A', B', C') are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_A^B, F_E^B \in M_{m\times n}(\mathbb{C})$ and $F_A^C, F_E^C \in M_{n\times p}(\mathbb{C})$ such that

$$\begin{array}{l} E' = QEP + F_E^C CP + QBF_E^B, \\ A' = QAP + F_A^C CP + QBF_A^B, \\ B' = QBR, \\ C' = SCP, \end{array} \tag{2}$$

or written in a matrix form

$$\begin{pmatrix} E' & B' & 0 & 0 \\ C' & 0 & 0 & 0 \\ 0 & 0 & A' & B' \\ 0 & 0 & C' & 0 \end{pmatrix} = \begin{pmatrix} Q & F_E^C & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & Q & F_A^C \\ 0 & 0 & 0 & S \end{pmatrix} \begin{pmatrix} E & B & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ F_E^B & R & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & F_A^B & R \end{pmatrix}.$$

It is easy to check that this relation is an equivalence relation.

A singular system (E, A, B, C), for which there exist matrices F_E^B and/or F_E^C such that $E + BF_E^B + F_E^C C$ is invertible is called standardizable, and in this case there exist matrices P, Q, F_E^B, F_E^C such that $QEP + QBF_E^B + F_E^C CP = I_n$. Consequently the equivalent system is standard. Notice that the standardizable character is invariant under the equivalence relation considered.

In the case where the original system is standard and if we want to preserve this condition under the equivalence relation we restrict the operation to the case where $Q = P^{-1}$, $F_E^B = 0$ and $F_E^C = 0$.

Definition 2.2: Let (E, A, B, C) be a system. λ_0 is an eigenvalue of this system if and only if

$$\operatorname{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} < \operatorname{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix}.$$

We denote by $\sigma(E, A, B, C)$ the set of eigenvalues of the quadruple (E, A, B, C) and we call it the spectrum of the system.

The continuous invariants under this equivalence are the eigenvalues of the system that they are defined as follows.

Proposition 2.1: Let (E, A, B, C) be a system. The spectrum of this system is invariant under equivalence relation considered.

Proof: It suffices to observe that

$$\operatorname{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} Q & \lambda F_E^C - F_A^C \\ 0 & S \end{pmatrix} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} P & \\ \lambda F_E^B - F_A^B & R \end{pmatrix}.$$

■

Associated to each eigenvalue there is an eigenvector defined in the following manner:

Definition 2.3: i) $v_0 \in M_{n\times 1}(\mathbb{C})$ is an eigenvector of this system corresponding to the eigenvalue λ_0 if and only if, there exist a vector $w_0 \in M_{m\times 1}(\mathbb{C})$ such that

$$\begin{pmatrix} \lambda_0(E + BF_E^B) - (A + BF_A^B) & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

for all F_E^B, F_A^B.

ii) $u_0 \in M_{1\times n}(\mathbb{C})$ is a left eigenvector of the system corresponding to the eigenvalue λ_0 if and only if, there exist a vector $\omega_0 \in M_{1\times p}(\mathbb{C})$ such that

$$\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \lambda_0(E + F_E^C C) - (A + F_A^C C) & B \\ C & 0 \end{pmatrix} = 0,$$

for all F_E^C, F_A^C.

Proposition 2.2: Let λ_0 be an eigenvalue and v_0 an associated eigenvector of the (E, A, B, C). Then λ_0 is an eigenvalue and v_0 an associated eigenvector of $(E + BF_E^C + F_E^C C, A + BF_A^B + F_A^C C, B, C)$ for all F_E^B, F_E^C, F_A^B, F_A^C.

Proof: Let $\overline{w}_0 = w_0 - (\lambda_0 F_E^B - F_A^B)v_0$.

$$\begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \lambda_0 F_E^B - F_A^B & I \end{pmatrix} \begin{pmatrix} v_0 \\ \overline{w}_0 \end{pmatrix} = \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ (\lambda_0 F_E^B - F_A^B)v_0 + \overline{w}_0 \end{pmatrix} = \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} I & \lambda_0 F_E^C - F_A^C \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

■

Proposition 2.3: Let λ_0 be an eigenvalue and u_0 an associated left eigenvector of the (E, A, B, C).

Then λ_0 is an eigenvalue and u_0 an associated left eigenvector of $(E + BF_E^C + F_E^C C, A + BF_A^B + F_A^C C, B, C)$ for all F_E^B, F_E^C, F_A^B, F_A^C.

Proof: Analogous to the proof of proposition 2.2, taking $\overline{\omega}_0 = \omega_0 - u_0(\lambda_0 F_E^C - F_A^C)$. ■

Remark 2.1: Unlike the case of triples of matrices (E, A, B) (see [4]) if λ_0 is an eigenvalue of the quadruple (E, A, B, C) it is not necessarily a generalized eigenvalue of the pair (E, A).

Example 2.1: Let (E, A, B, C) be a system with $E = I$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$,$C = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

$$\det \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = -3\lambda + 3 = 0.$$

Then, the eigenvalue of the system is $\lambda = 1$. Observe that $v_0 = (-3, 3)^t$ is an eigenvector associated to $\lambda = 1$ (there exist $w_0 = 1$).

But $\det(\lambda E - A) = \lambda(\lambda - 2)$, so the eigenvalues of the pair (E, A) are $\lambda_1 = 0$ and $\lambda_2 = 2$.

Definition 2.4: An eigenvalue λ_0 of the system (E, A, B, C) is called simple if and only if verifies the following conditions

i) $\operatorname{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} - 1,$ and

ii) $$\operatorname{rank} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix} + \operatorname{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix}.$$

Proposition 2.4: The simple character is invariant under equivalence relation considered.

Proof: Considering

$$\mathbf{Q} = \begin{pmatrix} Q & \lambda_0 F_E^C - F_A^C & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & F_E^C & Q & \lambda_0 F_E^C - F_A^C \\ 0 & 0 & 0 & S \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} P & 0 & 0 & 0 \\ \lambda_0 F_E^B - F_A^B & R & 0 & 0 \\ 0 & 0 & P & 0 \\ F_E^B & 0 & \lambda_0 F_E^B - F_A^B & r \end{pmatrix}$$

Therefore,

$$\operatorname{rank} \mathbf{Q} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \mathbf{P} = \operatorname{rank} \begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix}.$$

■

Proposition 2.5: Let λ_0 be a simple eigenvalue of the standard system (A, B, C). Then, there exist an associate eigenvector v_0 and an associate left eigenvector u_0 such that $u_0 v_0 = 1$.

Proof: If λ_0 is a simple eigenvalue, the system can be reduced to $\left(\begin{pmatrix} A_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \end{pmatrix} \right)$, with $\operatorname{rank} \begin{pmatrix} \lambda_0 I - A_1 & B_1 \\ C_1 & 0 \end{pmatrix} = n - 1$.

In this reduced form it is easy to observe that $v_0 = (0, ..., 0, 1)^t$ is an eigenvector and $v_0 = (0, ..., 0, 1)$ is a left eigenvector verifying $u_0 v_0 = 1$. Now, taking into account propositions 2.2 and 2.3, we can check easily that $P v_0$ is an eigenvector of the system (A, B, C) and $u_0 P^{-1}$ is a left eigenvector for some invertible matrix P. ■

Remark 2.2: In general, for singular systems this result fails, as we can see in the following example.

Example 2.2: Let (E, A, B, C) be a singular system with $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$. It is easy to observe that $\lambda_0 = 3$ is a simple eigenvalue of this system and all possible eigenvectors are $v_0 = (\alpha, 0)^t$ with $\alpha \neq 0$ and all possible left eigenvectors are $u_0 = (0, \beta)$ with $\beta \neq 0$. Clearly $u_0 v_0 = 0$.

But, we have the following more general result.

Proposition 2.6: Let λ_0 be a simple eigenvalue of the singular system (E, A, B, C) with $m = p = 1$ and $\operatorname{rank} \begin{pmatrix} \lambda E - A & B \\ C & 0 \end{pmatrix} = n + 1$. Then, there exist an associate eigenvector v_0 and an associate left eigenvector u_0 such that $u_0 E v_0 \neq 0$.

Proof: If λ_0 is a simple eigenvalue

$$\begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix} \neq 0$$

for all v_1 and w_1. So taking $v_1 = 0$ and $w_1 = 0$ we have that $Ev_0 \neq 0$.

Suppose now that $u_0 E v_0 = 0$, in this case we have that $0 \neq \begin{pmatrix} Ev_0 \\ 0 \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} = \operatorname{Im} \begin{pmatrix} \lambda_0 E - A & B \\ C & 0 \end{pmatrix}$. Then, $\begin{pmatrix} Ev_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_0 A - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$ for some $(v_1, w_1) \neq (v_0, w_0)$ because $Ev_0 \neq 0$.

So

$$\begin{pmatrix} \lambda_0 E - A & B & 0 & 0 \\ C & 0 & 0 & 0 \\ E & 0 & \lambda_0 E - A & B \\ 0 & 0 & C & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix} = 0$$

ans λ_0 can not be simple. Therefore $u_0 E v_0 \neq 0$. ■

3. Analysis of perturbation of simple eigenvalues

A. Standard systems

We begin studying the case of standard systems in order to make more comprehensive the study. So, we consider systems in the form $\dot{x} = Ax + Bu$, $y = Cx$ with $A \in M_n(\mathbb{C})$, $B \in M_{n \times m}(\mathbb{C})$ and $C \in M_{m \times n}(\mathbb{C})$ represented as a triple of matrices (A, B, C).

Let (A, B, C) be a linear system and assume that the matrices A, B, C smoothly depend on the vector $p = (p_1, \dots, p_r)$ of real parameters. The function $(A(p), B(p), C(p))$ is called a multi-parameter family of linear systems. Eigenvalues of linear system functions are continuous functions $\lambda(p)$ of the vector of parameters. In this section, we are going to study the behavior of a simple eigenvalue of the family of linear systems $(A(p), B(p), C(p))$.

Let us consider a point p_0 in the parameter space and assume that $\lambda(p_0) = \lambda_0$ is a simple eigenvalue of $(A(p_0), B(p_0), C(p_0)) = (A_0, B_0, C_0)$, and $v(p_0) = v_0$ is an eigenvector, i.e. there exists $w_0 \in M_{m \times 1}(\mathbb{C})$ such that

$$\left.\begin{aligned} A_0 v_0 - B_0 w_0 &= \lambda_0 v_0 \\ C_0 v_0 &= 0 \end{aligned}\right\}.$$

Equivalently

$$\left.\begin{aligned} (A_0 + B_0 F_A^B) v_0 - B_0 w_0 &= \lambda_0 v_0 \\ C_0 v_0 &= 0 \end{aligned}\right\},$$

$\forall F_A^B \in M_{m \times n}(\mathbb{C})$.

Now, we are going to review the behavior of a simple eigenvalue $\lambda(p)$ of the family of standard linear systems.

The eigenvector $v(p)$ corresponding to the simple eigenvalue $\lambda(p)$ determines a one-dimensional null-subspace of the matrix operator $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ smoothly dependent on p. Hence, the eigenvector $v(p)$ (and corresponding $w(p)$) can be chosen as a smooth function of the parameters. We will try to obtain an approximation by means of their derivatives.

We write the eigenvalue problem as

$$\left.\begin{aligned} A(p)v(p) - B(p)w(p) &= \lambda(p)v(p) \\ C(p)v(p) &= 0. \end{aligned}\right\}. \tag{3}$$

Taking the derivatives with respect to p_i, we have

$$\left.\begin{aligned} \left(\frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i}\right) v(p) + \frac{\partial B(p)}{\partial p_i} w(p) = \\ (A(p) - \lambda(p) I) \frac{\partial v(p)}{\partial p_i} - B(p) \frac{\partial w(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} v(p) = -C(p) \frac{\partial v(p)}{\partial p_i} \end{aligned}\right\}.$$

At the point p_0, we obtain

$$\left.\begin{aligned} \left(\frac{\partial \lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i}\right)_{|p_0} v_0 + \frac{\partial B(p)}{\partial p_i}_{|p_0} w_0 = \\ (A_0 - \lambda_0 I) \frac{\partial v(p)}{\partial p_i}_{|p_0} - B_0 \frac{\partial w(p)}{\partial p_i}_{|p_0} \\ \frac{\partial C(p)}{\partial p_i}_{|p_0} v_0 = -C_0 \frac{\partial v(p)}{\partial p_i}_{|p_0} \end{aligned}\right\}. \tag{4}$$

This is a linear equation system for the unknowns

$$\frac{\partial \lambda(p)}{\partial p_i}, \ \frac{\partial v(p)}{\partial p_i} \text{ and } \frac{\partial w(p)}{\partial p_i}.$$

Lemma 3.1: Let v_0 and u_0 be an eigenvector and a left eigenvector respectively, corresponding to the simple eigenvalue λ_0 of the system (E, A, B, C). Then, the matrix

$$T = \begin{pmatrix} \lambda_0 I - A_0 & B_0 \\ C_0 & 0 \end{pmatrix} + \begin{pmatrix} v_0 u_0 & 0 \\ 0 & 0 \end{pmatrix}$$

has full rank.

Proof: It suffices to consider the system in the reduced form $\left(\begin{pmatrix} A_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 \end{pmatrix}\right)$. ■

Proposition 3.1: The system (4) has a solution if and only if

$$\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0 \tag{5}$$

where u_0 is a left eigenvector for the simple eigenvalue λ_0 of the system (A_0, B_0, C_0).

Proof: The system (4) can be rewritten as

$$\begin{array}{c} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \\ \begin{pmatrix} A_0 - \lambda_0 I & -B_0 \\ -C_0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix}_{|p_0}. \end{array} \tag{6}$$

We have that (4) has a solution if and only if (6) has a solution.

Premultiplying both sides of the equation (6), by (u_0, ω_0)

$$\begin{array}{l} \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} I - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \\ \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} I & 0 \\ 0 & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} - \\ \qquad \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0. \end{array}$$

We obtain a solution for $\frac{\partial\lambda(p)}{\partial p_i}_{|(\lambda_0;p_0)}$.

$$\frac{\partial\lambda(p)}{\partial p_i}_{|p_0} = \frac{\begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}}{u_0 v_0}.$$

Using the normalization condition, that is to say, taking v_0 such that $u_0 v_0 = 1$, we have:

$$\frac{\partial\lambda(p)}{\partial p_i}_{|p_0} = \begin{pmatrix} u_0 & \omega_0 \end{pmatrix} \begin{pmatrix} \frac{\partial A(p)}{\partial p_i} & -\frac{\partial B(p)}{\partial p_i} \\ -\frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}.$$

Knowing $\frac{\partial\lambda(p)}{\partial p_i}_{|p_0}$ we can deduce $\frac{\partial v(p)}{\partial p_i}_{|p_0}$.

First of all, we observe that if $u_0 v_0 = 1$, then $u_0 v(p) \neq 0$ and we can take $v(p)$ such that $u_0 v(p) = 1$ (normalization condition, it suffices to take as $v(p)$ the vector $\frac{1}{u_0 v(p)} v(p)$). So

$$\frac{\partial u_0 v(p)}{\partial p_i} = u_0 \frac{\partial v(p)}{\partial p_i} = 0.$$

Consequently we can consider the compatible equivalent system:

$$\begin{array}{l} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \\ \begin{pmatrix} A_0 - \lambda_0 I + v_0 u_0 & -B_0 \\ -C_0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix}_{|p_0} \end{array} \tag{7}$$

In our particular case where $m = p$, the system has a unique solution

$$\begin{pmatrix} \frac{\partial v(p)}{\partial p_i} \\ \frac{\partial w(p)}{\partial p_i} \end{pmatrix}_{|p_0} = T^{-1} \begin{pmatrix} \frac{\partial\lambda(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} & 0 \end{pmatrix}_{|p_0} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}.$$

■

Taking the partial derivative $\partial^2/\partial p_i \partial p_j$ on both sides of both equations in the eigenvalue problem (3), we can obtain a second order approximation for eigenvalues.

B. Singular systems

Now we consider singular systems as in (1) written as quadruple of matrices (E, A, B, C).

In this case the eigenvalue problem is written as

$$\left.\begin{array}{rl} A(p)v(p) - B(p)w(p) & = \lambda(p)E(p)v(p) \\ C(p)v(p) & = 0. \end{array}\right\}. \tag{8}$$

Taking the derivatives with respect to p_i, we have

$$\left.\begin{array}{r} \left(\frac{\partial\lambda(p)}{\partial p_i} E(p) + \lambda(p) \frac{\partial E(p)}{\partial p_i} - \frac{\partial A(p)}{\partial p_i}\right) v(p) + \frac{\partial B(p)}{\partial p_i} w(p) \\ = (A(p) - \lambda(p)E(p)) \frac{\partial v(p)}{\partial p_i} - B(p) \frac{\partial w(p)}{\partial p_i} \\ \frac{\partial C(p)}{\partial p_i} v(p) = -C(p) \frac{\partial v(p)}{\partial p_i} \end{array}\right\}.$$

At the point p_0, we obtain

$$\left.\begin{array}{r}\left(\frac{\partial\lambda(p)}{\partial p_i}E_0+\lambda_0\frac{\partial E(p)}{\partial p_i}-\frac{\partial A(p)}{\partial p_i}\right)_{|p_0}v_0+\frac{\partial B(p)}{\partial p_i}_{|p_0}w_0\\ =(A_0-\lambda_0E)\frac{\partial v(p)}{\partial p_i}_{|p_0}-B_0\frac{\partial w(p)}{\partial p_i}_{|p_0}\\ \frac{\partial C(p)}{\partial p_i}_{|p_0}v_0=-C_0\frac{\partial v(p)}{\partial p_i}_{|p_0}\end{array}\right\}. \tag{9}$$

This is a linear equation system for the unknowns $\frac{\partial\lambda(p)}{\partial p_i}$, $\frac{\partial v(p)}{\partial p_i}$ and $\frac{\partial w(p)}{\partial p_i}$.

Suppose now, systems (E,A,B,C) with $m=p=1$ and rank $\begin{pmatrix}\lambda E-A & B\\ C & 0\end{pmatrix}=n+1$.

Lemma 3.2: Let v_0 and u_0 be an eigenvector and a left eigenvector respectively, corresponding to the simple eigenvalue λ_0 of the system (E,A,B,C). Then, the matrix

$$T=\begin{pmatrix}\lambda_0E-A_0 & B_0\\ C_0 & 0\end{pmatrix}+\begin{pmatrix}E_0v_0u_0E_0 & 0\\ 0 & 0\end{pmatrix}$$

has full rank.

Proof: First of all we proof that $E_0v_0u_0E_0\neq 0$.

$$\begin{pmatrix}u_0 & \omega_0\end{pmatrix}\begin{pmatrix}\lambda_0E_0-A_0+E_0v_0u_0E_0 & B_0\\ C_0 & 0\end{pmatrix}\begin{pmatrix}v_0\\ w_0\end{pmatrix}$$
$$=\begin{pmatrix}u_0 & \omega_0\end{pmatrix}\begin{pmatrix}E_0v_0u_0E_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_0\\ w_0\end{pmatrix}=$$
$$(u_0E_0v_0)^2\neq 0,$$

so, $E_0v_0u_0E_0\neq 0$.

In the other hand $v_0\notin \operatorname{Ker} E_0v_0u_0E_0$, because $0\neq(u_0E_0v_0)^2=u_0(E_0v_0u_0E_0v_0)$.

Suppose now, that

$$\begin{pmatrix}\lambda_0E_0-A_0+E_0v_0u_0E_0 & B_0\\ C_0 & 0\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix}=0$$

for some vectors v and w.

Then

$$0=\begin{pmatrix}u_0 & \omega_0\end{pmatrix}\begin{pmatrix}\lambda_0E_0-A_0+E_0v_0u_0E_0 & B_0\\ C_0 & 0\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix}=$$
$$\begin{pmatrix}u_0 & \omega_0\end{pmatrix}\begin{pmatrix}E_0v_0u_0E_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v\\ w\end{pmatrix}=u_0E_0v_0u_0E_0v.$$

Taking into account that $u_0E_0v_0\neq 0$ we have that $u_0E_0v=0$, so $E_0v_0u_0E_0v=0$, then v is an eigenvector of the system corresponding to the eigenvalue λ_0 linearly independent of v_0, but λ_0 is simple. ■

Proposition 3.2: The system (9) has a solution if and only if

$$\begin{pmatrix}u_0 & \omega_0\end{pmatrix}\begin{pmatrix}\frac{\partial\lambda(p)}{\partial p_i}E_0+\lambda_0\frac{\partial E}{\partial p_i}-\frac{\partial A(p)}{\partial p_i} & \frac{\partial B(p)}{\partial p_i}\\ \frac{\partial C(p)}{\partial p_i} & 0\end{pmatrix}_{|p_0}\begin{pmatrix}v_0\\ w_0\end{pmatrix}=0 \tag{10}$$

where u_0 is a left eigenvector for the simple eigenvalue λ_0 of the system (E_0,A_0,B_0,C_0).

Proof: Analogously to the proof of proposition 3.1 we observe that proposition 2.6 permits to clear the unknown $\frac{\partial\lambda(p)}{\partial p_i}$ from equation (10).

On the other hand, taking into account that $u_0E_0v_0\neq 0$, we have that $u_0E(p)v(p)\neq 0$ in a neighborhood of the origin. So, $u_0E_0\frac{\partial v(p)}{\partial p}=0$. Lemma 3.2 permits to obtain $\frac{\partial v(p)}{\partial p_i}$ and $\frac{\partial w(p)}{\partial p_i}$. ■

Example 3.1: Consider now, the following two-parameter differentiable family of systems $(E(p),A(p),B(p),C(p))$ with

$$E(p)=I,\ A(p)=\begin{pmatrix}p_1 & 0\\ 1 & 2+p_2\end{pmatrix},$$

$$B(p)=\begin{pmatrix}3+p_1+p_2\\ p_1\end{pmatrix},\ C(p)=\begin{pmatrix}1+p_1, 1+p_2\end{pmatrix}.$$

At $p_0=(0,0)$ we have that $\lambda_0=1$ is a simple eigenvalue, $v_0=\begin{pmatrix}-3 & 3\end{pmatrix}^t$ a right eigenvalue (with $w_0=(1)$) and $u_0=(0,1)$ (with $(\omega_0=(1))$ a left eigenvector. Then

$$\frac{\partial\lambda}{\partial p_1}=\frac{\begin{pmatrix}0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}-3\\ 3\\ 1\end{pmatrix}}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}-3 & 3\end{pmatrix}^t}=-\frac{2}{3},$$

$$\frac{\partial\lambda}{\partial p_2}=\frac{\begin{pmatrix}0 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\ 3\\ 1\end{pmatrix}}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}-3 & 3\end{pmatrix}^t}=3.$$

4. Conclusion

In this work families of singular systems in the form $E(p)\dot{x}=A(p)x+B(p)u$, $y=C(p)x$ smoothly dependent on a vector of real parameters $p=(p_1,\dots,p_n)$ have been considered. A study of the behavior of a simple eigenvalue of this family of singular linear system is analyzed and a description of a first approximation of the eigenvalues and corresponding eigenvectors have been obtained.

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