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Maple Implementation of A Symbolic Method for Fully Inhomogeneous Boundary Value Problems

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Abstract—In this paper, we discuss the maple implementation of a symbolic method for solving a boundary value problem with inhomogeneous Stieltjes boundary conditions over integro-differential algebras. The implementation includes computing the Green's operator and the Green's function of a given boundary value problem. Sample computations are presented to illustrate the Maple implementation.

Keywords—Maple implementation, Boundary value problems, Stieltjes conditions, Green's function, Green's operator, symbolic method.

I. Introduction

Generally, the applications of boundary value problems (BVPs) with inhomogeneous Stieltjes boundary conditions are arise in many areas of science and engineering. For the last five decades, many researchers and engineers have been actively developing applications of general BVPs of higher order ordinary differential equations with general boundary conditions. Symbolic analysis of BVPs, and the formulation of Green's operator and Green's function of semi-inhomogeneous BVPs (inhomogeneous differential equation with homogeneous boundary conditions), were first attempted by Markus Rosenkranz et al. in 2004 [4], also see [5]–[9], [11]–[16]. Extension of the symbolic method for semi-inhomogeneous BVPs to fully inhomogeneous BVPs (inhomogeneous differential equation with inhomogeneous boundary conditions) over integro-differential algebras has been presented by S. Thota in 2019 [10].

In this paper, for simplicity, we first recall the symbolic methods for semi-inhomogeneous BVPs and fully inhomogeneous BVPs over integro-differential algebras. After that, we present the Maple implementation of symbolic method for fully inhomogeneous BVPs. Sample computations are presented to illustrate the Maple implementation. Rest of the paper is arranged as follows. Section II recalls the basic concepts of integro-differential algebras, symbolic methods for semi-inhomogeneous BVPs and fully inhomogeneous BVPs over integro-differential algebras; and Section III presents the Maple implementation of the symbolic method for fully inhomogeneous BVPs with sample computations in Section III-A.

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II. Symbolic Method for Boundary Value Problems

In this section, we recall some basic concepts of integro-differential algebras, see [4], [7], [12] for further details, and also the symbolic methods for semi-inhomogeneous BVPs [4] and fully inhomogeneous BVPs [10] over integro-differential algebras.

A. Algebra of Integro-differential Operators

To express the BVP, Green's operator and Green's function in operator based notations, we recall the basic concepts of integro-differential algebras and the algebra of integro-differential operators. Throughout this section $\mathbb{K}$ denotes the field of characteristic zero and $\mathcal{F} = C^\infty[a, b]$ for simplicity.

Definition 1. *[4], [8], [12] An algebraic structure $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ is called an* integro-differential algebra *over $\mathbb{K}$ if $\mathcal{F}$ is a commutative $\mathbb{K}$-algebra with $\mathbb{K}$-linear operators $\mathtt{D}$ and $\mathtt{A}$ such that the following conditions are satisfied*

$\star\ \mathtt{D}(\mathtt{A}f) = f,$

$\star\ \mathtt{D}(fg) = (\mathtt{D}f)g + f(\mathtt{D}g),$

$\star\ (\mathtt{AD}f)(\mathtt{AD}g) + \mathtt{AD}(fg) = (\mathtt{AD}f)g + f(\mathtt{AD}g).$

Here $\mathtt{D} : \mathcal{F} \to \mathcal{F}$ and $\mathtt{A} : \mathcal{F} \to \mathcal{F}$ are two maps defined by $\mathtt{D} = \frac{d}{dx}$, a derivation, and $\mathtt{A} = \int_a^x dx$, a $\mathbb{K}$-linear right inverse of $\mathtt{D}$, i.e. $\mathtt{D} \circ \mathtt{A} = 1$ (the identity map). The map $\mathtt{A}$ is called an integral *for $\mathtt{D}$ and $\mathtt{A} \circ \mathtt{D} = 1 - \mathtt{E}$, where $\mathtt{E}$ is called the* evaluation *operator of $\mathcal{F}$ defined as $\mathtt{E} : f \mapsto f(a)$, evaluates at initial point a. An integro-differential algebra over $\mathbb{K}$ is called* ordinary *if $Ker(\mathtt{D}) = \mathbb{K}$.*

For an ordinary integro-differential algebra, the evaluation can be treated as a multiplicative linear functional $\mathtt{E} : \mathcal{F} \to \mathbb{K}$, i.e., $\mathtt{E}(fg) = (\mathtt{E}f)(\mathtt{E}g)$, for all $f, g \in \mathcal{F}$. Let $\Phi \subseteq \mathcal{F}^*$ be a set of all multiplicative linear functionals including $\mathtt{E}$. To specify a BVP, we also need a collection of "point evaluations" as new generators. For example, the boundary conditions $u(2) = 1, u'(1) = 5$, $\int_0^2 u\ dx = 0$ on a function $u \in \mathcal{F} = C^\infty[a, b]$ gives rise to the functional $\mathtt{E}_2 u = 1$, $\mathtt{E}_1 \mathtt{D}u = 5$, $\mathtt{E}_2 \mathtt{A}u = 0 \in \mathcal{F}^*$.

Definition 2. *[4], [8] Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathcal{F}^*$. The* integro-differential operators *$\mathcal{F}[\mathtt{D}, \mathtt{A}]$ are defined as the $\mathbb{K}$-algebra generated by the symbols $\mathtt{D}$ and $\mathtt{A}$, the functions $f \in \mathcal{F}$ and the characters (functionals) $\mathtt{E}_c, \phi, \chi \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table I.*

TABLE I
REWRITE RULES FOR INTEGRO-DIFFERENTIAL OPERATORS

$fg \to f \cdot g$	$\mathtt{D}f \to f\mathtt{D} + f'$	$\mathtt{A}f\mathtt{A} \to (\mathtt{A}f)\mathtt{A} - \mathtt{A}(\mathtt{A}f)$
$\chi\phi \to \phi$	$\mathtt{D}\phi \to 0$	$\mathtt{A}f\mathtt{D} \to f - \mathtt{A}f' - (\mathtt{E}f)\mathtt{E}$
$\phi f \to (\phi f)\phi$	$\mathtt{D}\mathtt{A} \to 1$	$\mathtt{A}f\phi \to (\mathtt{A}f)\phi$

For an integro-differential algebra $\mathcal{F}$, a fully inhomogeneous BVP is given by a monic *differential operator* $L = \mathtt{D}^n + a_{n-1}\mathtt{D}^{n-1} + \cdots + a_1\mathtt{D} + a_0$ and the *boundary conditions* $b_1, \ldots, b_n \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ with *boundary data* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Given a forcing function $f \in \mathcal{F}$ and a set of boundary data $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, we want to find $u \in \mathcal{F}$ such that

$$\begin{aligned} Lu &= f, \\ b_1u &= \alpha_1, \ldots, b_nu = \alpha_n. \end{aligned} \tag{1}$$

The quantities $\{f; \alpha_1, \ldots, \alpha_n\}$ are known collectively as the *data* for the BVP. One can decompose the data as

$$\{f; \alpha_1, \ldots, \alpha_n\} = \{f; 0, \ldots, 0\} + \{0; \alpha_1, \ldots, \alpha_n\}. \tag{2}$$

The BVP with data $\{f; 0, \ldots, 0\}$ is an inhomogeneous differential equation with homogeneous boundary conditions; the BVP with data $\{0; \alpha_1, \ldots, \alpha_n\}$ is a homogeneous differential equation with inhomogeneous boundary conditions. Symbolically, we can write the solution u of (1) as

$$u = G(f, \alpha_1, \ldots, \alpha_n),$$

where G is a linear operator that transforms the data into the solution. Hence we regard G as the inverse operator of L.

B. Symbolic Method Semi-inhomogeneous BVPs

The algorithm for computing the solution, $G(f, 0, \ldots, 0)$, of a semi-inhomogeneous BVP is described in [4] with details, and also [7], [12]. Consider a semi-inhomogeneous BVP

$$\begin{aligned} Lu &= f, \\ b_1u &= 0, \ldots, b_nu = 0, \end{aligned} \tag{3}$$

where L is a surjective linear map and $B = \{b_1, \ldots, b_n\} \subseteq \mathcal{F}^*$ is a closed subspace of the dual space. The main steps [4] to determine the solution $G(f, 0, \ldots, 0)$ of a semi-inhomogeneous BVP and the corresponding Green's operator G are:

I. Compute the fundamental right inverse $L^\diamond \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ from a given fundamental system as in (4).
II. Compute the projector $P \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ onto $\mathrm{Ker}(L)$ along $B^\perp$ as in (5).
III. Now the Green's operator G is computed as $G = L^\diamond - PL^\diamond$, and the solution is $u = G(f, 0, \ldots, 0) = (L^\diamond - PL^\diamond)f$.

The differential operator L is always surjective and the $\dim \mathrm{Ker}(L) = n < \infty$. Moreover, we can find the right inverse of L using variation of parameters [2] as follows: Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra and let $L \in \mathcal{F}[\mathtt{D}]$ be monic with regular fundamental system $u_1, \ldots, u_n$. Then the *fundamental right inverse* of L is [2], [4] given by

$$L^\diamond = \sum_{i=1}^{n} u_i \mathtt{A} d^{-1} d_i \in \mathcal{F}[\mathtt{D}, \mathtt{A}], \tag{4}$$

where d is the determinant of Wronskian matrix W for $u_1, \ldots, u_n$ and d_i the determinant of W_i obtained from W by replacing the i-th column by the n-th unit vector. If $\{b_1, \ldots, b_n\}$ and $\{u_1, \ldots, u_n\}$ are bases for B and $\mathrm{Ker}(L)$ respectively with b_i biorthogonal to u_i, then the projector operator P is [4, p. 26] determined by

$$P = \sum_{i=1}^{n} u_i b_i. \tag{5}$$

The regularity of a BVP can be tested algorithmically [4], [7], [10], [12] as follows: If $u_1, \ldots, u_n$ is a basis for $\mathrm{Ker}(L)$ and $\{b_1, \ldots, b_n\}$ is a basis for B, then the BVP is regular if and only if the *evaluation matrix*

$$b(u) = \begin{pmatrix} b_1(u_1) & \ldots & b_1(u_n) \\ \vdots & \ddots & \vdots \\ b_n(u_1) & \ldots & b_n(u_n) \end{pmatrix} \tag{6}$$

is regular.

C. Symbolic Method Semi-homogeneous BVPs

In this section, we consider the following type of semi-homogeneous BVPs

$$\begin{aligned} Lu &= 0, \\ b_1u &= \alpha_1, \ldots, b_nu = \alpha_n. \end{aligned}$$

Let H be any function (not necessarily satisfying the differential operator L) such that $b_iH = \alpha_i$, for $i = 1, \ldots, n$. Since H is depending only on the boundary data, this amounts to an *interpolation problem* with Stieltjes boundary conditions [6], [12]–[14], presented in the following theorem. Note that the right inverse of B means the right inverse of each element of B such that $b_iH = \alpha_i$.

Theorem 1. *[10] Let $\{u_1, \ldots, u_n\} \subset \mathcal{F}$ and $\{b_1, \ldots, b_n\} \subset \mathcal{F}^*$ be bases for Ker(L) and B respectively, and $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$ be boundary data. Then there exists a unique right inverse H of B such that $b_iH = \alpha_i$ is given by*

$$H = u_v^T b(u)^{-1} \alpha_v, \tag{7}$$

where $u_v = (u_1, \ldots, u_n)$ and $\alpha_v = (\alpha_1, \ldots, \alpha_n)$ are column vectors and $b(u)$ is evaluation matrix as in equation (6).

In the following theorem, we present the solution of the given semi-homogeneous BVPs.

Theorem 2. *[10] Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra and $B = \{b_1, \ldots, b_n\} \subset \mathcal{F}^*$. Given a* *boundary data* $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$*, the BVP*

$$\begin{aligned} &Lu = 0, \\ &b_1 u = \alpha_1, \ldots, b_n u = \alpha_n. \end{aligned}$$

has the unique solution

$$u = G(0, \alpha_1, \ldots, \alpha_n) = PH,$$

where $P \in \mathcal{F}[D, A]$ *is a projector onto Ker*(L) *along* $B^{\perp}$ *as in equation* (5) *and* $H \in \mathcal{F}$ *such that* $b_i H = \alpha_i$ *as in equation* (7).

D. Symbolic Method Fully-Inhomogeneous BVPs

Finally, we compute the solution [10] of fully inhomogeneous BVP with data $\{f; \alpha_1, \ldots, \alpha_n\}$ as composition of two solutions

$$G(f, \alpha_1, \ldots, \alpha_n) = (M - PM)(f) + PH$$

and the corresponding Green's operator is

$$G = (M - PM) + PH.$$

One can easily check that the Green's operator G satisfies $LG = 1$ and $BG = \alpha$, and the solution $G(f, \alpha_1, \ldots, \alpha_n)$ satisfies $LG(f, \alpha_1, \ldots, \alpha_n) = f$ and $BG(f, \alpha_1, \ldots, \alpha_n) = \alpha$.

III. Maple Implementation of Symbolic Method

In this section, we preset the Maple implementation of symbolic method for fully inhomogeneous BVPs. As presented in Section II-B, we use the Maple package `IntDiffOp` implemented by Anja Korporal et al. [1] to compute the solution of the semi-homogeneous BVPs. The solution of the semi-inhomogeneous BVPs can be computed as follows. The following procedure produces the evaluation matrix as given in equation (6), where `BC` is boundary conditions matrix.

```
> EvaluationMat := proc (BC::Matrix)
> local r, c, elts, fs;
> r, c := LinearAlgebra[Dimension](BC);
> fs := Matrix(1, r, [seq(x^(i-1), i =
1 .. r)]);
> elts := seq(seq(ApplyOperator(BC[t,
1], fs[1, j]), j = 1 .. r), t = 1 .. r);
> return Matrix(r, r, [elts])
> end proc:
```

Using the following procedure SolSemiInhomBVP(BC,CM), one can obtain the polynomial interpolation, where BC is boundary conditions matrix and CM is associated values.

```
> SolSemiInhomBVP := proc (BC::Matrix,
CM::Matrix)
> local r, c, fs, evm, invevm, approx;
> r, c := LinearAlgebra[Dimension](BC);
> fs := Matrix(1, r, [seq(x^(i-1), i =
1 .. r)]);
> evm := EvaluationMat(BC);
> invevm := 1/evm;
> approx := fs.invevm.CM;
> return simplify(approx[1, 1])
> end proc:
```

A. Sample Computations

Example 1. *Consider one of the classical examples of the ordinary linear BVPs presented in [10, Example 3.1] to compute the solution using Maple implementation. Recall the BVP*

$$\begin{aligned} &\frac{d^2u}{dx^2} = f, \\ &u(0) = a_1,\ u(1) = a_2. \end{aligned} \tag{8}$$

We want to compute the solution u *using the Maple implementation. First, consider the semi-homogeneous BVP*

$$\begin{aligned} &\frac{d^2u}{dx^2} = f, \\ &u(0) = 0,\ u(1) = 0. \end{aligned}$$

The solution is computed as follows using Maple

```
> with(IntDiffOp):
> L := DIFFOP(0,0,1);
```

$$D\hat{}2$$

```
> b1 := BOUNDOP(EVOP(0, EVDIFFOP(1),
EVINTOP()));
> b2 := BOUNDOP(EVOP(1, EVDIFFOP(1),
EVINTOP()));
```

$$E[0]$$
$$E[1]$$

```
> bvp := BP(L, BC(b1, b2));
```

$$BP(D\hat{}2, BC(E[0], E[1]))$$

```
> G1 := GreensOperator(bvp);
```

$$x.A - A.x - x.E[1].A + x.E[1].A.x$$

```
> u1 := ApplyOperator(G1, f(x));
```

$$x\left(\int_0^x f(x)\ dx\right) - \left(\int_0^x x f(x)\ dx\right) - x\left(\int_0^1 f(x)\ dx\right) + x\left(\int_0^1 x f(x)\ dx\right)$$

Secondly, consider the semi-inhomogeneous BVP

$$\begin{aligned} &\frac{d^2u}{dx^2} = 0, \\ &u(0) = a_1,\ u(1) = a_2. \end{aligned}$$

The solution is computed as follows using Maple

```
> C := Matrix([[b1], [b2]]);
```

$$\begin{bmatrix} E[0] \\ E[1] \end{bmatrix}$$

```
> CM := Matrix([[a1], [a2]]);
```

$$\begin{bmatrix} a1 \\ a2 \end{bmatrix}$$

```
> u2 := SolSemiInhomBVP(C, CM);
```

$$-a1x + a2x + a1$$

Finally, we have complete solution of fully inhomogeneous BVP (8) *as follow.*

$$
\begin{aligned}
u = &\ u_1 + u_2 \\
= &\ x\left(\int_0^x f(x)\ dx\right) - \left(\int_0^x xf(x)\ dx\right) - x\left(\int_0^1 f(x)\ dx\right) \\
&+ x\left(\int_0^1 xf(x)\ dx\right) - a1x + a2x + a1.
\end{aligned}
$$

One can easily observe that the solution obtained by Maple implementation and the solution obtained by symbolic method presented in [10] are coincided.

Example 2. *Consider the following fully inhomogeneous BVP,*

$$
\frac{d^3u}{dx^3} - 3\frac{d^2u}{dx^2} + 3\frac{du}{dx} - u = e^x,
$$
$$
u(0) = 1, u'(0) = 0, u(1) = -1.
$$

Solution of the given BVP is computed using Maple as follows

```
> with(IntDiffOp):
> L := DIFFOP(-1,3,-3,1);
```

$$-1 + 3.D - 3D\hat{}2 + D\hat{}3$$

```
> b1 := BOUNDOP(EVOP(0, EVDIFFOP(1),
EVINTOP()));
> b2 := BOUNDOP(EVOP(0, EVDIFFOP(0,1),
EVINTOP()));
> b3 := BOUNDOP(EVOP(1, EVDIFFOP(1),
EVINTOP()));
```

$$E[0]$$
$$E[0].D$$
$$E[1]$$

```
> bvp := BP(L, BC(b1, b2, b3));
```

$$BP(-1 + 3.D - 3D\hat{}2 + D\hat{}3, BC(E[0], E[0].D, E[1]))$$

```
> f(x) := exp(x);
```

$$x \to e^x$$

```
> u1 := ApplyOperator(GreensOperator(bvp),
f(x));
```

$$\frac{1}{6}e^x x^3 - \frac{1}{6}e^x x^2$$

```
> C := Matrix([[b1], [b2], [b3]]);
```

$$\begin{bmatrix} E[0] \\ E[0].D \\ E[1] \end{bmatrix}$$

```
> CM := Matrix([[1], [0], [-1]]);
```

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

```
> u2 := SolSemiInhomBVP(C, CM);
```

$$-2x^2 + 1$$

$$u = u_1 + u_2 = \frac{1}{6}e^x x^3 - \frac{1}{6}e^x x^2 - 2x^2 + 1$$

IV. Conclusion

In this paper, we presented the Maple implementation of a symbolic method for BVP with inhomogeneous Stieltjes boundary conditions. In this Maple implementation, we computed the Green's operator and the Green's function of a given BVP. Full details about the algorithm/method are available at [10]. The main idea of the algorithm is the decomposition of the data as

$$\{f; \alpha_1, \ldots, \alpha_n\} = \{f; 0, \ldots, 0\} + \{0; \alpha_1, \ldots, \alpha_n\}.$$

The BVP with data $\{f; 0, \ldots, 0\}$ is an inhomogeneous differential equation with homogeneous boundary conditions; the BVP with data $\{0; \alpha_1, \ldots, \alpha_n\}$ is a homogeneous differential equation with inhomogeneous boundary conditions. Sample computations (couple of examples) are presented to illustrate the Maple implementation.

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