Poles optimization of MIMO ARX-Laguerre model using genetic algorithm

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***Abstract*—In this paper, we propose the poles otimization of the linear MIMO ARX-laguerre model using genetic algorithm. This model is obtained by decompsing the MIMO ARX model on orthonormal and independent Laguerre bases allowing the filtering of the inputs and outputs of the system using the orthonormal functions of Laguerre. The resulting model, called MIMO ARX-Laguerre, ensures a reduction in the parametric complexity with respect to the number of parameters with a simple recursive vector representation. However, this reduction is conditioned by an optimal choice of the Laguerre pole characterizing each base. To do this, we propose to optimize, by exploiting the genetic algorithm, the Laguerre poles of the MIMO ARX-laguerre model. The optimization of the Laguerre poles is validated by a numerical simulation to the CSTR Benchmark.**

***Index Terms*—MIMO ARX-laguerre model, laguerre poles, optimization, genetic algorithm, CSTR Benchmark**

I. INTRODUCTION

Recently, we have seen an increase in the availability and accuracy of experimental data in many areas of engineering. This increase in data has spawned a large number of complex mathematical models that have been developed to explain the mechanisms responsible for these observations. These models comprising an increasing number of parameters are characterized by a high order which complicates and makes tedious and expensive the analysis, synthesis and simulation of the concerned system.

In this regard, solutions have been brought to the representation of complex systems to develop reduced models representing the dynamic behavior of these systems. In this context, several works exploit the orthonormal bases such as the base of Laguerre [1, 2], the base of Kautz [3] and the generalized orthogonal base (BOG) [4] to reduce the complexity of the model and subsequently lighten the procedure of control. The obtained model has several advantages such as insensitivity to the choice of the sampling period, the no-need for a priori knowledge of the delay of the system, the linearity with respect to the Fourier coefficients and especially the reduction of the number of parameters thanks to an optimal identification of the pole or poles characterizing the considered orthogonal base. The orthogonal basis of Laguerre is considered as a particular case of that of Kautz and also of the generalized orthogonal base. Although the modeling on this base requires more filters than that developed on the Kautz base at the BOG base especially for the representation of oscillating systems or with several regimes, the orthogonal functions of Laguerre have the advantage of being calculated in a simpler recursive way and especially that they depend exclusively on a single real parameter called pole of Laguerre $|\xi| < 1$. As a result, the Laguerre modelling is more adequate for the representation of strongly damped systems having a dominant dynamic. In addition, optimal pole identification guarantees a good description of the system with a smaller number of parameters. Thanks to these properties, the Laguerre base has been applied to the identification and control of linear systems. The major drawback of using the Laguerre base lies in its incompatibility with complex systems. Indeed, the Laguerre model, characterized by the filtering of the input, representing these types of systems requires a very high number of parameters. To work around this problem Bouzrara et al. [5] proposed to use Laguerre's orthonormal functions for input and output filtering of the ARX model. This proposal is used by Garna et al. [6] for the parametric reduction of the bilinear model. The obtained results of this procedure of modelling are of great and encouraging interest for the representation of complex linear MIMO systems.

In this case, the use of the orthogonal functions of Laguerre has been proposed [7] for the filtering of the inputs and outputs of the MIMO ARX model and the obtained model is called MIMO ARX-Laguerre. The latter is characterized by a reduced number of parameters compared to that of the MIMO ARX model if the Laguerre poles are identified in an optimal way. For this reason, we propose in this paper the optimization of the Laguerre poles of the MIMO ARX-Laguerre model using the genetic algorithm [8, 9, 10, 11]. The latter has been widely adopted in recent years and presents a very powerful meta-heuristic of research inspired by Charles Darwin's theory of natural evolution. These algorithms reflect the process of natural selection where the most suitable individuals are selected for reproduction in order to produce the offspring or the next generation. Unlike other optimization algorithms, the process of identification using genetic algorithms is based on guided random search. This search can lead to an optimal solution by starting from a random initial cost function and then searching only in the cheapest space (in the guided direction). This is suitable when working with large and complex data sets.

The paper is organized as follows: in section 2 we let's present the MIMO ARX-Laguerre model. This model is obtained by breaking down the MIMO ARX model into MISO ARX subsystems and by developing the coefficients of the latter on two independent Laguerre bases. In this context, we present a recursive representation of the MIMO ARX-Laguerre model. Section 3 presents the identification of the Fourier coefficients of the model studied. In section 4 we propose an optimization algorithm of Laguerre poles by exploiting the genetic algorithm. Finally, section 4 evaluates, through the application to a CSTR Benchmark, the performances of the proposed algorithm.

II. MIMO ARX-Laguerre Model

In this section, we present a recent alternative for the modeling of linear MIMO systems [7]. This alternative consists in extending each linear MIMO ARX model on a set of orthonormal and independent Laguerre bases by filtering the inputs and outputs of the process using Laguerre orthonormal functions. The resulting model, called the MIMO ARX-Laguerre model, ensures the reduction in the number of parameters compared to the classic MIMO ARX model with a recursive and simple vector representation. However, this reduction strongly depends on the optimal choice of the Laguerre pole characterizing each base.

Generally, the problem of identifying a linear MIMO system can be divided into several problems of identifying systems with multiple inputs and single output (MISO). Indeed, a MIMO system having p inputs and m outputs can be represented by a collection of m MISO subsystems depicted by Fig. 1.

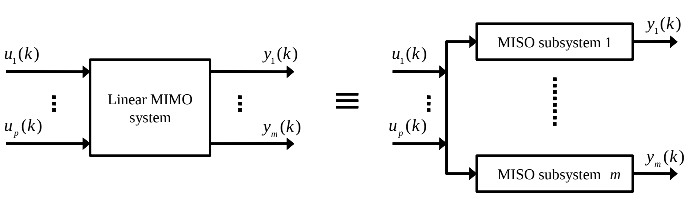

Fig. 1. Representation of a linear MIMO system by a set of MISO subsystems

In this case the output $y_i(k)$ of the i^{th} MISO subsystem can be represented by the MISO ARX model as follows [20]:

$$y_i(k) = \sum_{r=1}^{m}\sum_{j=1}^{n_a} h_{a_{ir}}(j)\, y_r(k-j) \;+\; \sum_{t=1}^{p}\sum_{j=1}^{n_b} h_{b_{it}}(j)\, u_t(k-j)$$
$$\text{pour};\quad i = 1, \ldots, m \tag{1}$$

where $u_t(k)$ and $y_r(k)$ are the t^{th} input and the r^{th} output of the system respectively; $h_{a_{ir}}(j)$ and $h_{b_{it}}(j)$ are the coefficients of the model; and n_a and n_b are the orders of the model. In the following, we present the decomposition of the coefficients $h_{a_{ir}}(j)$ and $h_{b_{it}}(j)$ on orthonormal and independent Laguerre bases. Thus, the development of the linear MIMO ARX-Laguerre model results from this decomposition of each MISO ARX model.

A. Principle of decomposition

According to the stability condition within the meaning of the BIBO (Bounded Input Bounded Output) criterion, the coefficients $h_{a_{ir}}(j)$ and $h_{b_{it}}(j)$ of the MISO ARX model (1) are absolutely summable:

$$\begin{cases} \sum_{j=1}^{\infty} |h_{a_{ir}}(j)| < \infty \\ \sum_{j=1}^{\infty} |h_{b_{it}}(j)| < \infty \end{cases} \tag{2}$$

The coefficients $h_{a_{ir}}(j)$ and $h_{b_{it}}(j)$ therefore belong to the Lebesgue space $\ell^2\,[0,\,+\infty)$ and they can be decomposed on orthonormal and independent Laguerre bases as follows:

$$\begin{cases} h_{a_{ir}}(j) = \sum_{n=0}^{N_a-1} g_{n,\,a_{ir}}\, \ell_n^{a_{ir}}(j, \xi_{a_{ir}}), \quad r = 1, \ldots, m \\ h_{b_{it}}(j) = \sum_{n=0}^{N_b-1} g_{n,\,b_{it}}\, \ell_n^{b_{it}}(j, \xi_{b_{it}}), \quad t = 1, \ldots, p \end{cases} \tag{3}$$

where N_a and N_b are the truncation orders, $\xi_{a_{ir}}$ and $\xi_{b_{it}}$ are the Laguerre poles and $\ell_n^{a_{ir}}(j, \xi_{a_{ir}})$ and $\ell_n^{b_{it}}(j, \xi_{b_{it}})$ represent the orthonormal and independent functions of the Laguerre bases [5]. So the decomposition (3) is characterized by a number of Laguerre poles equal to n_p:

$$n_p = m + p \tag{4}$$

The coefficients $g_{n,\,a_{ir}}$ and $g_{n,\,b_{it}}$ of the decomposition are the Fourier coefficients which can be defined by these equalities:

$$\begin{cases} g_{n,\,a_{ir}} = \sum_{j=0}^{n_a} h_{a_{ir}}(j)\, \ell_n^{a_{ir}}(j, \xi_{a_{ir}}) \\ g_{n,\,b_{it}} = \sum_{j=0}^{n_b} h_{b_{it}}(j)\, \ell_n^{b_{it}}(j, \xi_{b_{it}}) \end{cases} \tag{5}$$

By substituting the relation (3) in each MISO ARX model given by the equation (1), the resulting model, entitled MISO ARX-Laguerre is written [7, 12]:

$$y_i(k) = \sum_{r=1}^{m} \sum_{n=0}^{N_a-1} g_{n,\,a_{ir}}\, x^{\,i}_{n,\,y_r}(k) + \sum_{t=1}^{p} \sum_{n=0}^{N_b-1} g_{n,\,b_{it}}\, x^{\,i}_{n,\,u_t}(k) \tag{6}$$

with :

- $x^{\,i}_{n,\,y_r}$ is the output of the $(n+1)^{th}$ filter associated with the r^{th} output:

$$x^{\,i}_{n,\,y_r}(k) = \sum_{j=0}^{\infty} \ell_n^{a_{ir}}(j, \xi_{\,a_{ir}})\, y_r(k-j) \tag{7}$$

- $x^{\,i}_{n,\,u_t}$ is the output of the $(n+1)^{th}$ filter associated with the t^{th} input:

$$x^{\,i}_{n,\,u_t}(k) = \sum_{j=0}^{\infty} \ell_n^{b_{it}}(j, \xi_{\,b_{it}})\, u_t(k-j) \tag{8}$$

We note that the resulting model (6) is characterized by a number of Fourier coefficients equal to n_c:

$$n_c = m\, N_a + p\, N_b \tag{9}$$

Moreover, the model (6) is linear with respect to the Fourier coefficients $g_{n,\,a_{ir}}$ and $g_{n,\,b_{it}}$ therefore classical linear identification methods can be applied. However, it is nonlinear with respect to the Laguerre poles $\xi_{\,a_{ir}}$ and $\xi_{\,b_{it}}$, which requires the application of appropriate identification approaches.

B. Recursive representation of the MISO ARX-Laguerre model

According to [12], the MISO ARX-Laguerre model admits this recursive vector representation:

$$\begin{cases} \boldsymbol{X}^{\,i}_{y_r}(k+1) = \boldsymbol{A}^{\,i}_{y_r}\, \boldsymbol{X}^{\,i}_{y_r}(k) + \boldsymbol{b}^{\,i}_{y_r}\, y_r(k) \\ \boldsymbol{X}^{\,i}_{u_t}(k+1) = \boldsymbol{A}^{\,i}_{u_t}\, \boldsymbol{X}^{\,i}_{u_t}(k) + \boldsymbol{b}^{\,i}_{u_t}\, u_t(k) \\ y_i(k) = \boldsymbol{C}_i^T\, \boldsymbol{X}^i(k) \end{cases} \tag{10}$$

where for $i,\, r = 1, \ldots, m$ et $t = 1, \ldots, p$ on a :

- $\boldsymbol{X}^{\,i}_{y_r}$ and $\boldsymbol{X}^{\,i}_{u_t}$ are vectors containing the output of Laguerre filters:

$$\begin{cases} \boldsymbol{X}^{\,i}_{y_r}(k) = \left[x^{\,i}_{0,y_r}(k) \cdots x^{\,i}_{N_a-1,y_r}(k)\right]^T \in \Re^{N_a} \\ \boldsymbol{X}^{\,i}_{u_t}(k) = \left[x^{\,i}_{0,u_t}(k) \cdots x^{\,i}_{N_b-1,u_t}(k)\right]^T \in \Re^{N_b} \end{cases} \tag{11}$$

- $\boldsymbol{X}^{\,i}(k)$ is a vector consisting of $\boldsymbol{X}^{\,i}_{y_r}$ and $\boldsymbol{X}^{\,i}_{u_t}$:

$$\begin{aligned} \boldsymbol{X}^i(k) = \Big[&(\boldsymbol{X}^{\,i}_{y_1}(k))^T \cdots (\boldsymbol{X}^{\,i}_{y_m}(k))^T \\ &(\boldsymbol{X}^{\,i}_{u_1}(k))^T \cdots (\boldsymbol{X}^{\,i}_{u_p}(k))^T\Big]^T \in \Re^{n_c} \end{aligned} \tag{12}$$

- $\boldsymbol{C}_i$ is a vector containing the Fourier coefficients $g_{n,\,a_{ir}}$ and $g_{n,\,b_{it}}$ characterizing (6) :

$$\begin{aligned} \boldsymbol{C}_i = [&g_{0,\,a_{i1}} \cdots g_{N_a-1,\,a_{i1}} \cdots g_{0,\,a_{im}} \cdots g_{N_a-1,\,a_{im}} \\ &g_{0,\,b_{i1}} \cdots g_{N_b-1,\,b_{i1}} \cdots g_{0,\,b_{ip}} \cdots g_{N_b-1,\,b_{ip}}]^T \in \Re^{n_c} \end{aligned} \tag{13}$$

- $\boldsymbol{A}^{\,i}_{y_r}$ and $\boldsymbol{A}^{\,i}_{u_t}$ are square matrices of dimensions N_a and N_b, respectively whose components $\boldsymbol{A}^{\,i}_{y_r}(h,q)$ and $\boldsymbol{A}^{\,i}_{u_t}(h,q)$ are defined as follows:

$$\boldsymbol{A}^{\,i}_{y_r}(h,q) = \begin{cases} \xi_{\,a_{ir}} & si\ h = q \\ (-\xi_{\,a_{ir}})^{h-q-1}(1-\xi^2_{\,a_{ir}}) & si\ h > q \\ 0 & si\ h < q \end{cases}$$
$$\text{for } h,\, q = 1, \ldots, N_a \tag{14}$$

$$\boldsymbol{A}^{\,i}_{u_t}(h,q) = \begin{cases} \xi_{\,b_{it}} & si\ h = q \\ (-\xi_{\,b_{it}})^{h-q-1}(1-\xi^2_{\,b_{it}}) & si\ h > q \\ 0 & si\ h < q \end{cases}$$
$$\text{for } h,\, q = 1, \ldots, N_b \tag{15}$$

- $\boldsymbol{b}^{\,i}_{y_r}$ and $\boldsymbol{b}^{\,i}_{u_t}$ are vectors of dimensions N_a and N_b, respectively:

$$\begin{aligned} \boldsymbol{b}^{\,i}_{y_r} &= \sqrt{1-\xi^2_{\,a_{ir}}} \begin{bmatrix} 1 \\ (-\xi_{\,a_{ir}}) \\ (-\xi_{\,a_{ir}})^2 \\ \vdots \\ (-\xi_{\,a_{ir}})^{N_a-1} \end{bmatrix} \in \Re^{N_a} \\ \boldsymbol{b}^{\,i}_{u_t} &= \sqrt{1-\xi^2_{\,b_{it}}} \begin{bmatrix} 1 \\ (-\xi_{\,b_{it}}) \\ (-\xi_{\,b_{it}})^2 \\ \vdots \\ (-\xi_{\,b_{it}})^{N_b-1} \end{bmatrix} \in \Re^{N_b} \end{aligned} \tag{16}$$

From the relation (10), each MISO ARX-Laguerre model can be described by this compact vector representation:

$$\begin{cases} \boldsymbol{X}^i(k+1) = \boldsymbol{A}^i\, \boldsymbol{X}^i(k) + \boldsymbol{B}^{\,i}_{u}\, \boldsymbol{u}(k) + \boldsymbol{B}^{\,i}_{y}\, \boldsymbol{y}(k) \\ y_i(k) = \boldsymbol{C}_i^T\, \boldsymbol{X}^i(k) \end{cases} \tag{17}$$

with:

$$\boldsymbol{A}^i = bloc\_diag\left\{\boldsymbol{A}^{\,i}_{y_1}, \ldots, \boldsymbol{A}^{\,i}_{y_m},\, \boldsymbol{A}^{\,i}_{u_1}, \ldots, \boldsymbol{A}^{\,i}_{u_p}\right\} \in \Re^{n_c \times n_c} \tag{18}$$

$$\boldsymbol{B}^{\,i}_{y} = \begin{bmatrix} \underline{\boldsymbol{B}}^{\,i}_{y} \\ \boldsymbol{0}_{\,p\,N_b,\,m} \end{bmatrix} \in \Re^{n_c \times m},\quad \boldsymbol{B}^{\,i}_{u} = \begin{bmatrix} \boldsymbol{0}_{\,m\,N_a,\,p} \\ \underline{\boldsymbol{B}}^{\,i}_{u} \end{bmatrix} \in \Re^{n_c \times p} \tag{19}$$

$$\begin{cases} \underline{\boldsymbol{B}}^{\,i}_{y} = bloc\_diag\left\{\boldsymbol{b}^{\,i}_{y_1}, \ldots, \boldsymbol{b}^{\,i}_{y_m}\right\} \in \Re^{m\,N_a \times m} \\ \underline{\boldsymbol{B}}^{\,i}_{u} = bloc\_diag\left\{\boldsymbol{b}^{\,i}_{u_1}, \ldots, \boldsymbol{b}^{\,i}_{u_p}\right\} \in \Re^{p\,N_b \times p} \end{cases} \tag{20}$$

$$\boldsymbol{u}(k) = [u_1(k) \ldots u_p(k)]^T \;;\quad \boldsymbol{y}(k) = [y_1(k) \ldots y_m(k)]^T \tag{21}$$

where $\boldsymbol{0}_{\,i,\,j}$ of dimension $i \times j$.

C. Recursive representation of the MIMO ARX-Laguerre linear model

Since the MIMO ARX-Laguerre model can be decomposed into a set of MISO ARX-Laguerre models, the resulting MIMO ARX Laguerre model can therefore be formulated from the relation (17) by this compact recursive vector representation [12]:

$$\begin{cases} \boldsymbol{X}(k+1) = \boldsymbol{A}\,\boldsymbol{X}(k) + \boldsymbol{B}_u\,\boldsymbol{u}(k) + \boldsymbol{B}_y\,\boldsymbol{y}(k) \\ \boldsymbol{y}(k) = \boldsymbol{C}^T \boldsymbol{X}(k) \end{cases} \tag{22}$$

where:

$$\boldsymbol{X}(k) = [(\boldsymbol{X}^1(k))^T \dots (\boldsymbol{X}^m(k))^T]^T \in \Re^{N_c} \tag{23}$$

and $\boldsymbol{A}$, $\boldsymbol{B}_u$, $\boldsymbol{B}_y$ and $\boldsymbol{C}$ are matrices of dimensions $(N_c \times N_c)$, $(N_c \times p)$, $(N_c \times m)$ and $(N_c \times m)$, respectively, with:

$$N_c = m\,n_c = m\,(m\,N_a + p\,N_b) \tag{24}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}^1 & \boldsymbol{0}_{n_c,n_c} & \cdots & \boldsymbol{0}_{n_c,n_c} \\ \boldsymbol{0}_{n_c,n_c} & \boldsymbol{A}^2 & \cdots & \boldsymbol{0}_{n_c,n_c} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}_{n_c,n_c} & \boldsymbol{0}_{n_c,n_c} & \cdots & \boldsymbol{A}^m \end{bmatrix} \in \Re^{N_c \times N_c}$$
$$\boldsymbol{B}_u = \begin{bmatrix} \boldsymbol{B}_u^1 \\ \boldsymbol{B}_u^2 \\ \vdots \\ \boldsymbol{B}_u^m \end{bmatrix} \in \Re^{N_c \times p},\; \boldsymbol{B}_y = \begin{bmatrix} \boldsymbol{B}_y^1 \\ \boldsymbol{B}_y^2 \\ \vdots \\ \boldsymbol{B}_y^m \end{bmatrix} \in \Re^{N_c \times m} \tag{25}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & \underline{\boldsymbol{0}}_{n_c} & \cdots & \underline{\boldsymbol{0}}_{n_c} \\ \underline{\boldsymbol{0}}_{n_c} & \boldsymbol{C}_2 & \cdots & \underline{\boldsymbol{0}}_{n_c} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\boldsymbol{0}}_{n_c} & \underline{\boldsymbol{0}}_{n_c} & \cdots & \boldsymbol{C}_m \end{bmatrix} \in \Re^{N_c \times m} \tag{26}$$

such that $\underline{\boldsymbol{0}}_{n_c}$ is the null column vector of dimension n_c.

We note that the MIMO ARX-Laguerre model defined by (22) is characterized by N_c Fourier coefficients and N_p Laguerre poles defining the vector $\boldsymbol{\xi}_{m,p}$ as follows:

$$\boldsymbol{\xi}_{m,p} = \underbrace{[\xi_{a_{11}}, \dots, \xi_{a_{1p}}, \xi_{b_{11}}, \dots, \xi_{b_{1m}}}_{for\ the\ 1^{st}\ output}, \dots, \underbrace{\xi_{a_{m1}}, \dots, \xi_{a_{mp}}, \xi_{b_{m1}}, \dots, \xi_{b_{mm}}]}_{for\ the\ m^{th}\ output} \in \Re^{N_p} \tag{27}$$

avec :

$$N_p = m\,n_p = m\,(m + p) \tag{28}$$

where n_p is defined by (4).

III. Identification of the Fourier coefficients of the ARX-Laguerre MIMO model

From the representation (22) the MIMO ARX-Laguerre model is represented by a set of MISO ARX-Laguerre representations given by (17), the proposed method of recursive identification of the Fourier coefficients constituting $\boldsymbol{C}_i$ associated with the i^{th} output $y_i(k)$ of the MIMO ARX-Laguerre model, consists in minimizing the following criterion:

$$J_i = \sum_{k=1}^{h} \left(y_{o,i}(k) - \boldsymbol{C}_i^T\,\boldsymbol{X}^i(k)\right)^2 ; \quad i = 1, \dots, m \tag{29}$$

where $y_{o,i}(j)$ is the measured i^{th} output of the system. As each MISO ARX-Laguerre model is linear and the criterion J_i is quadratic with respect to $\boldsymbol{C}_i$, respectively, its minimization provides a global minimum. Therefore, standard parameter estimation methods such as least mean square (LMS) or recursive least squares (RLS) methods can be used to calculate the estimated parameter vectors $\boldsymbol{C}_i$.

IV. Optimization of the Laguerre poles of the ARX-Laguerre MIMO model

The MIMO ARX-Laguerre model is a set of ARX-Laguerre MISO models given by (10), each characterized by n_c Fouurier cooefficients and n_p Laguerre poles. As shown by Bouzrara et al. [5], we obtain a significant reduction in the number of parameters when each Laguerre base is characterized by an optimal value of the Laguerre pole. Thus, we propose the optimization of the Laguerre poles using the genetic algorithm.

A. Genetic algorithm

The first step of the genetic algorithm is fixing the expression of the objective or evaluation function to optimize.Then, probabilistic steps are involved to create an initial population of N_{ind} individuals. The initial population will undergo genetic operations (evaluation, mutation and crossover ...) to converge towards the optimal solution of the problem which minimizes the objective function [13]. The optimization based on the genetic algorithm is summarized in these steps:

- **Initiation**
  The process begins with a collection of individuals called initial population. Each individual is a solution to the problem to be solved. This step is generally random.
- **Evaluation**
  This step consists in evaluating the performance (fitness) of each individual of the initial population vis--vis the objective function. This step assigns each individual a score where the highest is assigned to the individual who minimizes the objective function. Based on these scores, a new population of potential solutions is created using simple evolutionary operators: selection, crossover and mutation.
- **Selection**
  The idea is to select the most suitable individuals and to

let them pass on their genes to the next generation [14, 15, 16].

- **Crossover**
  The step of genetic crossover creates new individuals. For each pair of parents, a crossover point is chosen randomly in the genes. The offspring are created by exchanging the genes of the parents with each other until they reach the crossover point. The new offspring is added to the population and this step only affects a limited number of individuals established by the crossover rate Pc ($\leq 100\%$).
- **Mutation**
  The mutation consists in bringing a small disturbance to a number of individuals established by the mutation rate Pm ($\leq 100\%$). This operation has the effect of counteracting the attraction exerted by the best individuals, which makes it possible to explore other areas of the search space.
- **Termination**
  The algorithm ends if the population has converged and does not produce offspring significantly different from the previous generation.

B. Optimization of the Laguerre poles

In this case, the Laguerre filters depend essentially on the Laguerre poles expressed in a non-linear way (unlike the Fourier coefficients). Consequently, in order to better locate the dynamics of the system and to allow an optimal and adequate choice of the poles for a significant parametric reduction of the MIMO ARX-Laguerre model (22), we propose the optimization of these poles using the genetic algorithm.

In this regard, the optimal pole values of the MIMO ARX-Laguerre multimodel are identified by minimizing the normalized mean squared error ($EQMN$). The latter is chosen as an objective function and consists in calculating the cumulative error between the measured outputs vector $\boldsymbol{y}_o(k)$ and the outputs vector of the MIMO ARX-Laguerre model $bmy(k)$ on a measurement window H:

$$EQMN = \frac{\|\boldsymbol{y}_o(k) - \boldsymbol{y}(k)\|^2}{\|\boldsymbol{y}_o(k)\|^2} = \frac{\sum_{i=1}^{m}\sum_{k=1}^{H}(y_{o,i}(k) - y_i(k))^2}{\sum_{i=1}^{m}\sum_{k=1}^{H}(y_{o,i}(k))^2} \tag{30}$$

After fixing the objective function, we generate an initial population of individuals Ind containing the Laguerre poles of the MIMO ARX-Laguerre model.

In what follows, in order not to complicate and reduce the calculation time of the genetic algorithm, we propose to reduce the number of Laguerere poles to be optimized by taking equal the poles $\xi_{a_{ir}} = \xi_{a_i}$ and $\xi_{b_{it}} = \xi_{b_i}$ for $i, r = 1, \ldots, m$ et $t = 1, \ldots, p$. In this case, the dimension of the poles vector $\boldsymbol{\xi}_{m,p}$ defined by (27) is equal to $2m$ and not $m(m+p)$ meaning $\boldsymbol{\xi}_{m,p} \in \Re^{2m}$ thus forming the vector of population of individuals Ind :

$$Ind = \boldsymbol{\xi}_{m,p} = \left[\underbrace{\xi_{a1}, \xi_{b1}}_{for\ the\ 1^{st}\ output}, \ldots, \underbrace{\xi_{am}, \xi_{bm}}_{for\ the\ m^{th}\ output}\right] \in \Re^{2m} \tag{31}$$

The performances of the individuals forming Ind are evaluated using the criterion $EQMN$ as an evaluation function. The individuals with the highest performances are selected to undergo different genetic operations (crossover, mutation and selection). After a defined number of iterations, the genetic algorithm converges towards the optimal values. The strategy for identifying the Fourier coefficients and optimizing the Laguerre poles of the MIMO ARX-Laguerre model using the genetic algorithm is summarized in algorithm 1.

Algorithm 1: **Identification of the parameters of the MIMO ARX-Laguerre model**

1) *Suppose we have H input/output observations $(\boldsymbol{u}(k), \boldsymbol{y}_o(k))$.*
2) *Fix the truncation orders N_a and N_b.*
3) *Fix the size of the population N_{ind} as well as the crossover and mutation rates Pc and Pm, respectively and the number of iterations G_{max}.*
4) *Initialization: Generate randomly an initial population of N_{ind} vectors of parameters $Ind^e = \boldsymbol{\xi}^e_{m,p}$, $e = 1, \ldots, N_{ind}$ and a $counter = 1$*
5) *While $counter < G_{max}$ do :*
   5.1 *Evaluation : For each individual estimate the Fourier coefficients then evaluate the objective function $EQMN$ given by (30)*
   5.2 *Selection: Choose $N_{ind} \times Pc$ individuals of the current population according to the evaluation of the values of the evaluation function.*
   5.3 *Crossover and mutation: Apply the crossover to the chosen individuals then apply the mutation with a probability Pm to generate new N_{ind} possible solutions.*
   5.4 *Evaluation : For each new solution, estimate the Fourier coefficients then evaluate the objective function $EQMN$.*
   5.5 *Termination of the genetic algorithm when the number of iterations $compteur = G_{max}$ is reached. If not:*
   5.6 *$compteur = compteur + 1$ and return to step 5.*
6) *The individual of the last population Ind^e such that $EQMN_{min} = \min_{e=1 \ldots N_{ind}} (EQMN^e)$ corresponds to the optimal Laguerre poles of the MIMO ARX-Laguerre model.*

V. Application to the CSTR Benchmark

To validate algorithm 1 of the identification of the parameters of the MIMO ARX-Laguerre model, the application to the CSTR Benchmark is proposed. The studied reactor involves two reactants $B1$ and $B2$ with concentrations C_{B1}

and C_{B2} and flow rates w_1 and w_2, respectively. These flows are controlled by using the two control valves $CV1$ and $CV2$ respectively. The main use of the system is the chemical reaction between the two reactants $B1$ and $B2$ at a level h of the liquid in the reactor at a concentration C_B. The mathematical model of the nonlinear CSTR is given by Beale et al. [17]:

$$\begin{cases} \dfrac{dh(t)}{dt} = w_1(t) + w_2(t) - 0.2\sqrt{h(t)} \\ \dfrac{dC_B(t)}{dt} = (C_{B1} - C_B(t))\dfrac{w_1(t)}{h(t)} + (C_{B2} - C_B(t))\dfrac{w_2(t)}{h(t)} \\ \qquad - \dfrac{k_1 C_B(t)}{(1 + k_2 C_B(t))} \end{cases} \quad (32)$$

where k_1 and k_2 are the constants associated with the rate of consumption of the chemical reaction. In what follows, the reactor is assumed to be a MIMO process with two inputs w_1 and C_{B1} which are the flow rate and the concentration of the reactant $B1$ and two outputs which are the height h of the fluid in the reactor and its concentration C_B. In this case, we fix the feed rate k_1, k_2 and the rate w_2 and the concentration C_{B2} of the reactant $B2$. These parameters are defined in TABLE I.

TABLE I
FIXED PARAMETERS OF THE CSTR BENCHMARK

Fixed parameters	Values
w_2	0.3 $L\ min^{-1}$
C_{B2}	5 $kmol\ m^{-3}$
k_1, k_2	1

For the identification phase, the value of the sampling period is set to $T_e = 0.5\,s$ and the inputs w_1 and C_{B1} are two pseudo-random sequences with amplitudes varying in the intervals $[0.35,\ 0.7]$ and $[0,\ 30]$ as illustrated in Fig. 2. The output signals h and C_B are shown in Fig. 3.

The truncation orders are taken equal for all the sub-models in order to reduce the parametric complexity:

$$N_a = N_b = 1 \quad (33)$$

By applying algorithm 1, we proceed to the identification of the MIMO ARX-Laguerre model. To do this, we consider an initial population of size $N_{ind} = 60$ while fixing $G_{max} =$ 1000 generations and the values of the crossover and mutation rates at $Pc = 0.9$ and $Pm = 0.033$ respectively.

In Fig. 4 we present the evolution of the evaluation function $EQMN$ defined by (30) calculated for each iteration of algorithm 1. Moreover, we also present in Fig. 5 the evolution of the Laguerre poles and in Fig. 6 the evolution of the Fourier coefficients characterizing the MIMO ARX-Laguerre model for $N_a = N_b = 1$.

Thus for the last population corresponding to the number of iterations equal to 1000, meaning when we have $counter =$ 1000, the optimal Laguerre poles are identified. These are summarized in TABLE II and moreover the optimal values of

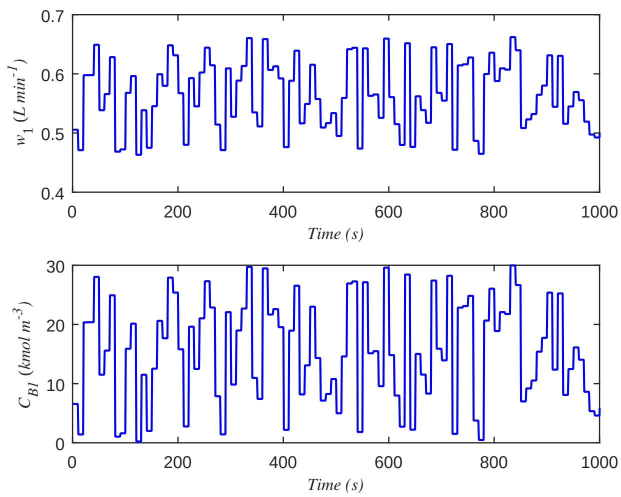

Fig. 2. Identification phase input signals applied to the CSTR Benchmark

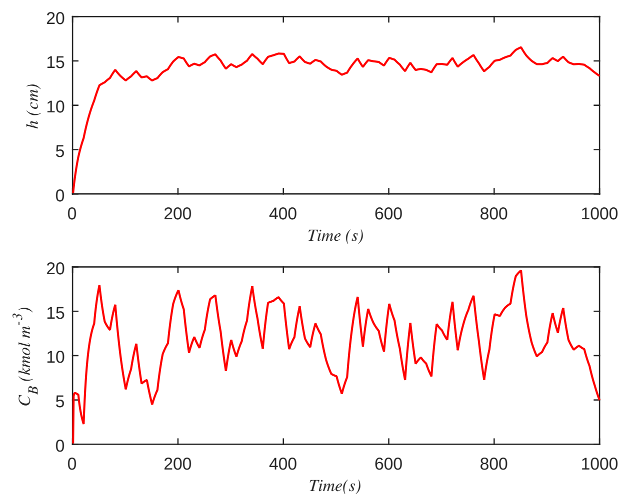

Fig. 3. Identification phase output signals of the CSTR Benchmark

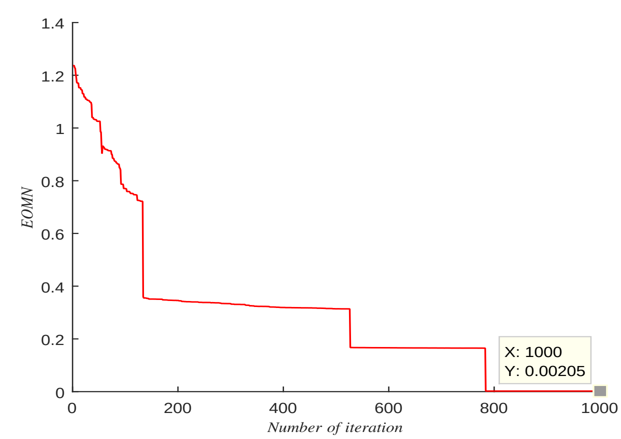

Fig. 4. Application to the CSTR Benchmark: Evolution of the evaluation function $EQMN$

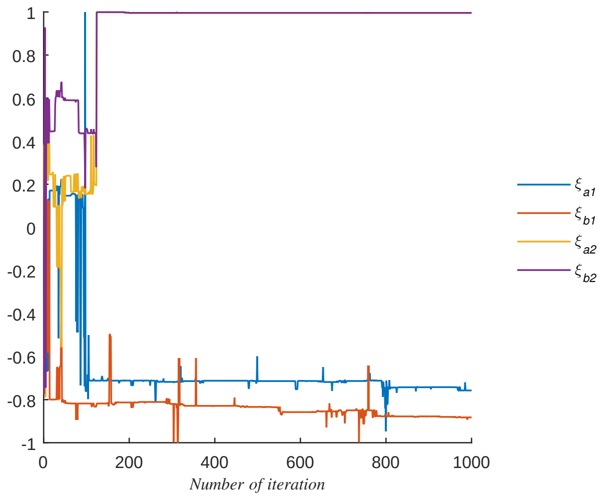

Fig. 5. Application to the CSTR Benchmark: Evolution of the Laguerre poles of the MIMO ARX-Laguerre model

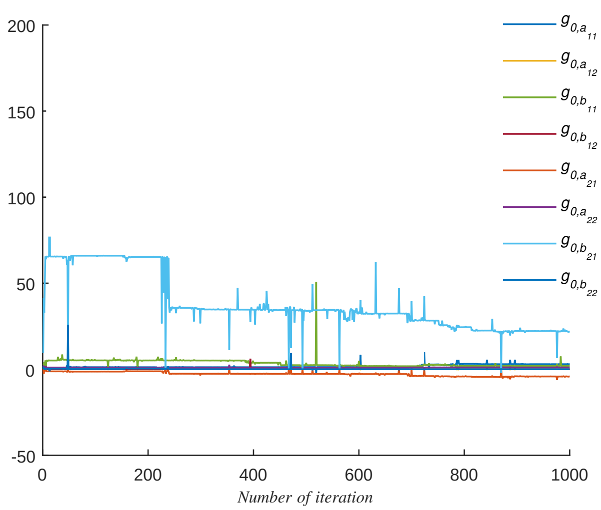

Fig. 6. Application to the CSTR Benchmark: Evolution of the Fourier coefficients of the MIMO ARX-Laguerre model

the Fourier coefficients of the MIMO ARX-Laguerre model are presented in TABLE III.

TABLE II
IDENTIFIED LAGUERRE POLES OF THE MIMO ARX-LAGUERRE MODEL

ξ^s_{a1}	ξ^s_{b1}	ξ^s_{a2}	ξ^s_{b2}
-0.7573	-0.8823	0.9959	0.9956

According to the figure 4, we note that the results of the parametric identification of the MIMO ARX-Laguerre model are obtained for a value of the evaluation function equal to $EQMN_{min} = 0.0020\%$. In this regard, we present in figure 7 the evolution of the outputs of the system and of the outputs of the MIMO ARX-Laguerre model. We note the agreement

TABLE III
IDENTIFIED FOURIER COEFFICIENTS OF THE ARX-LAGUERRE MIMO

$g_{0,a_{11}}$	$g_{0,a_{12}}$	$g_{0,b_{11}}$	$g_{0,b_{12}}$
0.5660	0.0088	10.6732	0.0014
$g_{0,a_{21}}$	$g_{0,a_{22}}$	$g_{0,b_{21}}$	$g_{0,b_{22}}$
-0.3534	0.5059	71.7500	0.0797

between the outputs of the Benchmark (h, C_B) and those of the MIMO ARX-Laguerre model (y_1, y_2).

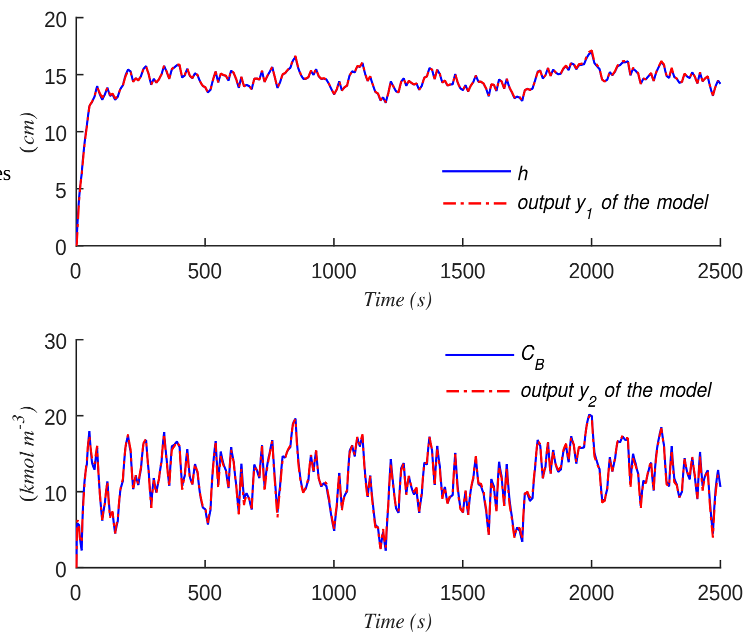

Fig. 7. Evolution of the benchmark outputs and the MIMO ARX-Laguerre model outputs

VI. CONCLUSION

In this paper, we have proposed an algorithm for the use of the genetic algorithm to optimize the Lagurre poles of the MIMO ARX-Laguerre model. This model is obtained by decomposing the MIMO ARX model on the Laguerre orthonormal. The resulting model guarantees a significant reduction in the number of parameters with a simple recursive representation. Parametric reduction is guaranteed if the optimal values of the Laguerre poles are selected. The proposed algorithm is validated through a numerical simulation to the CSTR Benchmark.

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