Hybrid Synchronization of Hyperchaotic Qi Systems via Sliding Mode Control

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Abstract—**This study investigates the hybrid synchronization of hyperchaotic Qi systems (2008) via sliding mode control. The stability results for the hybrid synchronization schemes derived in this paper are established using the sliding mode control theory and the Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the sliding controller design is very effective and convenient to achieve global hybrid synchronization of the identical hyperchaotic Qi systems. Numerical simulations are presented to demonstrate the effectiveness of the synchronization results derived in this paper.**

Keywords- Sliding control, hyperchaos, hybrid synchronization, hyperchaotic Qi system.

I. INTRODUCTION

Chaotic systems are nonlinear dynamical systems which are highly sensitive to initial conditions. This sensitivity is popularly known as the *butterfly effect* [1]. Chaos is an interesting nonlinear phenomenon and has been studied well in the last three decades. Chaos theory has wide applications in several fields like physical systems [2], chemical systems [3], ecological systems [4], secure communications [5-7], etc.

Hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. Since hyperchaotic system has the characteristics of high capacity, high security and high efficiency, it has the potential of broad applications in nonlinear circuits, secure communications, lasers, neural networks, biological systems and so on. Thus, the studies on hyperchaotic systems, viz. control, synchronization and circuit implementation are very challenging works in the chaos literature.

Chaos synchronization is a phenomenon that may occur when two or more chaotic oscillators are coupled or a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a challenging research problem.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the chaos synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll ([8], 1990), chaos synchronization problem has been studied intensively and extensively in the literature [8-30]. In the last two decades, various schemes have been successfully applied for chaos synchronization such as OGY method [9], active control method [10-13], adaptive control method [14-17], time-delay feedback method [18], backstepping design method [19-21], sampled-data feedback synchronization method [22], etc.

So far, many types of synchronization phenomenon have been presented such as complete synchronization [8], generalized synchronization [23], anti-synchronization [24], projective synchronization [25], generalized projective synchronization [26], etc.

Complete synchronization (CS) is characterized by the equality of state variables evolving in time, while anti-synchronization (AS) is characterized by the disappearance of the sum of relevant state variables evolving in time.

Projective synchronization (PS) is characterized by the fast that the master and slave systems could be synchronized up to a scaling factor. In generalized projective synchronization (GPS), the responses of the synchronized dynamical states synchronize up to a constant scaling matrix α. It is easy to see that the complete synchronization and anti-synchronization are special cases of the generalized projective synchronization where the scaling matrix $\alpha = I$ and $\alpha = -I$, respectively.

In hybrid synchronization of two chaotic systems [27-29], one part of the systems is completely synchronized and the other part is anti-synchronized so that the complete synchronization (CS) and anti-synchronization (AS) co-exist in the systems.

In control theory, sliding mode control, or SMC, is a nonlinear control method that alters the dynamics of a nonlinear system by application of a discontinuous control signal that forces the system to "slide" along a cross-section of the system's normal behaviour. The state-feedback control law is not a continuous function of time. Instead, it can switch from one continuous structure to another continuous structure based on the current position in the state space. Hence, sliding mode control is a variable structure control method.

In robust control systems, sliding mode control is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and disturbances.

In this paper, we derive new results based on the sliding mode control [30-33] for the hybrid chaos synchronization of identical hyperchaotic Qi systems ([34], 2008). Our stability results have been established using the Lyapunov stability theory [35].

This paper has been organized as follows. In Section II, we describe the problem statement and our methodology. In Section III, we discuss the hybrid synchronization of identical hyperchaotic Qi systems (2008). In Section IV, we summarize the main results obtained in this paper.

II. Problem Statement and Our Methodology using Sliding Mode Control

In this section, we discuss the master-slave synchronization of identical chaotic systems and our methodology of achieving hybrid synchronization using sliding mode control (SMC).

Consider the chaotic system described by the dynamics

$$\dot{x} = Ax + f(x) \tag{1}$$

where $x \in R^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters and $f : R^n \to R^n$ is the nonlinear part of the system. We consider the system (1) as the *master* or *drive* system.

As the *slave* or *response* system, we consider the following chaotic system described by the dynamics

$$\dot{y} = Ay + f(y) + u \tag{2}$$

where $y \in R^n$ is the state of the system and $u \in R^m$ is the controller to be designed.

In hybrid synchronization, we define the synchronization error so that the odd states of the systems (1) and (2) are completely synchronized and the even states of the systems (1) and (2) are anti-synchronized.

Thus, we define the *hybrid synchronization error* as

$$e_i = \begin{cases} y_i - x_i, & \text{if } i \text{ is odd.} \\ y_i + x_i, & \text{if } i \text{ is even.} \end{cases} \tag{3}$$

then the error dynamics can be expressed in the form

$$\dot{e} = Ae + \eta(x, y) + u \tag{4}$$

The objective of the global chaos synchronization problem is to find a controller u such that

$$\lim_{t \to \infty} \|e(t)\| = 0 \quad \text{for all } e(0) \in R^n. \tag{5}$$

To solve this problem, we first define the control u as

$$u = -\eta(x, y) + Bv \tag{6}$$

where B is a constant gain vector selected such that (A, B) is controllable.

Substituting (6) into (4), the error dynamics simplifies to

$$\dot{e} = Ae + Bv \tag{7}$$

which is a linear time-invariant control system with single input v.

Thus, the original hybrid chaos synchronization problem can be replaced by an equivalent problem of stabilizing the zero solution $e = 0$ of the system (7) by a suitable choice of the sliding mode control.

In the sliding mode control, we define the variable

$$s(e) = Ce = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \tag{8}$$

where

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

is a constant vector to be determined.

In the sliding mode control, we constrain the motion of the system (7) to the sliding manifold defined by

$$S = \left\{ x \in R^n \mid s(e) = 0 \right\}$$

which is required to be invariant under the flow of the error dynamics (7).

When in sliding manifold S, the system (7) satisfies the following conditions:

$$s(e) = 0 \tag{9}$$

which is the defining equation for the manifold S and

$$\dot{s}(e) = 0 \tag{10}$$

which is the necessary condition for the state trajectory $e(t)$ of (7) to stay on the sliding manifold S.

Using (7) and (8), the equation (10) can be rewritten as

$$\dot{s}(e) = C\left[Ae + Bv\right] = 0 \tag{11}$$

Solving (11) for v, we obtain the equivalent control law

$$v_{\text{eq}}(t) = -(CB)^{-1} CA\, e(t) \tag{12}$$

where C is chosen such that $CB \neq 0$.

Substituting (12) into the error dynamics (7), we obtain the closed-loop dynamics as

$$\dot{e} = \left[I - B(CB)^{-1} C\right] Ae \tag{13}$$

The row vector C is selected such that the system matrix of the controlled dynamics $\left[I - B(CB)^{-1} C\right] A$ is Hurwitz, *i.e.* it has all eigenvalues with negative real parts. Then the controlled system (13) is globally asymptotically stable.

To design the sliding mode controller for (7), we apply the constant plus proportional rate reaching law

$$\dot{s} = -q \operatorname{sgn}(s) - ks \tag{14}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function and the gains $q > 0$, $k > 0$ are determined such that the sliding condition is satisfied and sliding motion will occur.

From equations (11) and (14), we can obtain the control $v(t)$ as

$$v(t) = -(CB)^{-1} \left[C(kI + A)e + q \operatorname{sgn}(s)\right] \tag{15}$$

which yields

$$v(t) = \begin{cases} -(CB)^{-1}\left[C(kI + A)e + q\right], & \text{if } s(e) > 0 \\ -(CB)^{-1}\left[C(kI + A)e - q\right], & \text{if } s(e) < 0 \end{cases} \tag{16}$$

Theorem 1. The master system (1) and the slave system (2) are globally and asymptotically hybrid-synchronized for all initial conditions $x(0), y(0) \in R^n$ by the feedback control law

$$u(t) = -\eta(x, y) + Bv(t) \tag{17}$$

where $v(t)$ is defined by (15) and B is a column vector such that (A, B) is controllable. Also, the sliding mode gains k, q are positive.

Proof. First, we note that substituting (17) and (15) into the error dynamics (4), we obtain the closed-loop error dynamics as

$$\dot{e} = Ae - B(CB)^{-1}\left[C(kI + A)e + q\,\text{sgn}(s)\right] \tag{18}$$

To prove that the error dynamics (18) is globally asymptotically stable, we consider the candidate Lyapunov function defined by the equation

$$V(e) = \frac{1}{2}s^2(e) \tag{19}$$

which is a positive definite function on R^n.

Differentiating V along the trajectories of (18) or the equivalent dynamics (14), we get

$$\dot{V}(e) = s(e)\dot{s}(e) = -ks^2 - q\,\text{sgn}(s)s \tag{20}$$

which is a negative definite function on R^n.

This calculation shows that V is a globally defined, positive definite, Lyapunov function for the error dynamics (18), which has a globally defined, negative definite time derivative $\dot{V}$.

Thus, by Lyapunov stability theory [35], it is immediate that the error dynamics (18) is globally asymptotically stable for all initial conditions $e(0) \in R^n$.

This means that for all initial conditions $e(0) \in R^n$, we have

$$\lim_{t\to\infty}\|e(t)\| = 0.$$

Hence, it follows that the master system (1) and the slave system (2) are globally and asymptotically hybrid synchronized for all initial conditions $x(0), y(0) \in R^n$.

This completes the proof. ■

III. Hybrid Synchronization of Identical Hyperchaotic Qi Systems

A. *Theoretical Results*

In this section, we apply the sliding mode control results derived in Section II for the hybrid synchronization of identical hyperchaotic Qi systems ([34], 2008).

Thus, the master system is described by the 4D Qi dynamics

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3 \\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3 \\
\dot{x}_3 &= -cx_3 - \varepsilon x_4 + x_1x_2 \\
\dot{x}_4 &= -dx_4 + fx_3 + x_1x_2
\end{aligned} \tag{21}$$

where x_1, x_2, x_3, x_4 are state variables and a, b, c, d, e, f are constant, positive parameters of the system.

The slave system is also described by the controlled 4D Qi dynamics

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + y_2y_3 + u_1 \\
\dot{y}_2 &= b(y_1 + y_2) - y_1y_3 + u_2 \\
\dot{y}_3 &= -cy_3 - \varepsilon y_4 + y_1y_2 + u_3 \\
\dot{y}_4 &= -dy_4 + fy_3 + y_1y_2 + u_4
\end{aligned} \tag{22}$$

where y_1, y_2, y_3, y_4 are state variables and u_1, u_2, u_3, u_4 are the controllers to be designed.

The 4D Qi systems (21) and (22) are hyperchaotic when

$$a = 50,\ b = 24,\ c = 13,\ d = 8,\ \varepsilon = 33,\ f = 30$$

Figure 1 illustrates the phase portrait of the hyperchaotic Qi system (21).

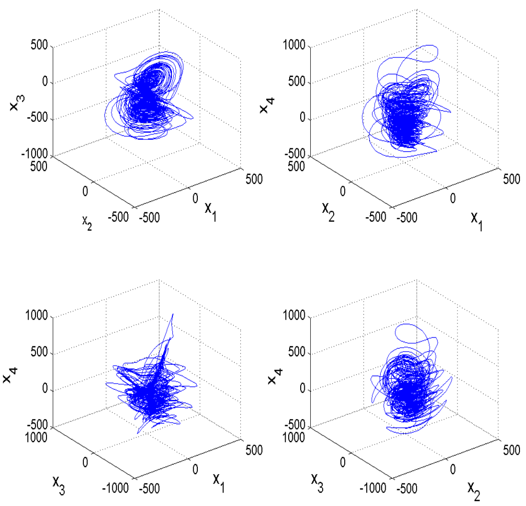

Figure 1. Phase Portrait of the Hyperchaotic Qi System

The hybrid synchronization error is defined by

$$\begin{aligned}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 + x_2 \\
e_3 &= y_3 - x_3 \\
e_4 &= y_4 + x_4
\end{aligned} \tag{23}$$

The error dynamics is easily obtained as

$$\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1) - 2ax_2 + y_2y_3 - x_2x_3 + u_1 \\
\dot{e}_2 &= b(e_1 + e_2) + 2bx_1 - y_1y_3 - x_1x_3 + u_2 \\
\dot{e}_3 &= -ce_3 - \varepsilon e_4 + 2\varepsilon x_4 + y_1y_2 - x_1x_2 + u_3 \\
\dot{e}_4 &= -de_4 + fe_3 + 2fx_3 + y_1y_2 + x_1x_2 + u_4
\end{aligned} \tag{24}$$

We write the error dynamics (24) in the matrix notation as

$$\dot{e} = Ae + \eta(x, y) + u \tag{25}$$

where

$$A = \begin{bmatrix} -a & a & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & -c & -\varepsilon \\ 0 & 0 & f & -d \end{bmatrix}, \tag{26}$$

$$\eta(x, y) = \begin{bmatrix} -2ax_2 + y_2y_3 - x_2x_3 \\ 2bx_1 - y_1y_3 - x_1x_3 \\ 2\varepsilon x_4 + y_1y_2 - x_1x_2 \\ 2fx_3 + y_1y_2 + x_1x_2 \end{bmatrix} \tag{27}$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{28}$$

The sliding mode controller design is carried out as detailed in Section II.

First, we set u as

$$u = -\eta(x, y) + Bv \tag{29}$$

where B is chosen such that (A, B) is controllable.

We take B as

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{30}$$

In the hyperchaotic case, the parameter values are

$$a = 50,\ b = 24,\ c = 13,\ d = 8,\ \varepsilon = 33,\ f = 30$$

The sliding mode variable is selected as

$$s = Ce = \begin{bmatrix} 8 & 1 & 1 & 1 \end{bmatrix} e \tag{31}$$

which makes the sliding mode state equation asymptotically stable.

We choose the sliding mode gains as $k = 6$ and $q = 0.3$.

We note that a large value of k can cause chattering and an appropriate value of q is chosen to speed up the time taken to reach the sliding manifold as well as to reduce the system chattering.

From Eq. (15), we can obtain $v(t)$ as

$$v = \begin{bmatrix} 29.818 & -39.091 & -2.091 & 3.182 \end{bmatrix} e - 0.0273 \,\mathrm{sgn}(s) \tag{32}$$

Thus, the required sliding mode controller is obtained as

$$u = -\eta(x, y) + Bv \tag{33}$$

where $\eta(x, y)$, B and $v(t)$ are defined as in the equations (27), (30) and (32).

By Theorem 1, we obtain the following result.

Theorem 2. The identical hyperchaotic Qi systems (21) and (22) are globally hybrid-synchronized for all initial conditions with the sliding controller u defined by (33). ■

B. *Numerical Results*

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-8}$ is used to solve the hyperchaotic Qi systems (21) and (22) with the sliding mode controller u given by (33) using MATLAB.

The initial values of the master system (21) are taken as

$$x_1(0) = 18,\ \ x_2(0) = 26,\ \ x_3(0) = 4,\ \ x_4(0) = 7$$

The initial values of the slave system (22) are taken as

$$y_1(0) = 28,\ y_2(0) = 10,\ y_3(0) = 16,\ y_4(0) = 25$$

Figure 2 illustrates the hybrid synchronization of the identical hyperchaotic Qi systems (21) and (22).

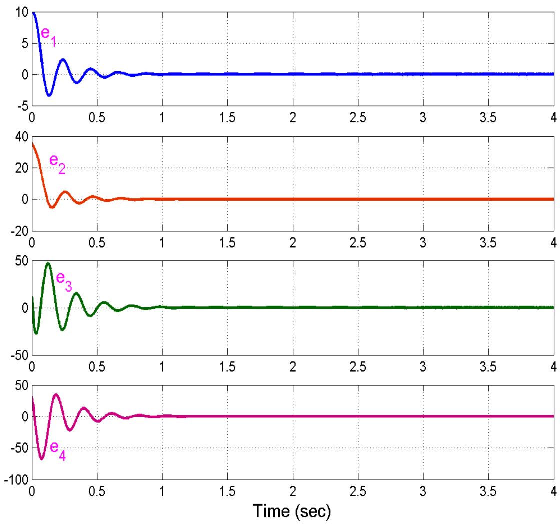

Figure 3. Hybrid Synchronization of the Hyperchaotic Qi Systems

IV. Conclusions

In this study, we have designed sliding controllers to achieve hybrid synchronization for the identical hyperchaotic Qi systems (2008). Our synchronization results based on the sliding mode control have been proved using the Lyapunov stability theory. Numerical simulations are also shown to validate synchronization results derived in this paper.

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