The Analysis of Covariance Matrix for Kalman Filter based SLAM with Intermittent Measurement

Nur Aqilah Othman and Hamzah Ahmad
Faculty of Electrical and Electronics Engineering
Universiti Malaysia Pahang (UMP)
Pekan Campus, 26600 Pekan
Pahang, Malaysia
Email: {aqilah, hamzah}@ump.edu.my

***Abstract*—This paper presents an analysis of the impact of intermittent measurement to the Simultaneous Localization and Mapping (SLAM) of a mobile robot. Intermittent measurement is a condition when the mobile robot lost its measurement data during observation due to sensor failure or imperfection of the system. This is crucial, since SLAM requires measurement data recursively for data update in estimating its current states. In this study, the analysis focused on the effect of intermittent measurement on the state error covariance matrix for two basic conditions; mobile robot is stationary and moving. The impact on the determinant of covariance matrix is observed. From the analysis, it can be concluded that intermittent measurement may lead to incorrect estimation of robot position and increment of state error covariance matrix.**

Keywords—intermittent measurement; SLAM; Kalman filter; covariance matrix

I. INTRODUCTION

A process of building a map of an environment whilst consequently estimating the location of a robot is known as ‘simultaneous localisation and mapping’ (SLAM). Using SLAM, a mobile robot has the fundamental ability to locate itself as well as environmental features (landmarks) without a known map. In 2D SLAM, the mobile robot is set to move in an environment comprising of population of landmarks. Proprioceptive and exteroceptive sensors are installed on the robot. The former sensor is used in measuring the robot’s own motion whilst the latter measures the relative location between the robot and nearby landmarks. Thus, the objective of SLAM is to estimate the position and orientation of the robot (robot pose) together with the locations of all the landmarks [1].

SLAM was first mathematically formulated as an estimation problem to understand the relationship between mobile robot and landmarks. All landmark positions and the robot pose were presented in a common state vector and a complete covariance matrix. A statistical basis for describing relationships between landmarks and manipulating geometric uncertainty was established prior to that, showing that there must be a high degree of correlation between estimates of the location of different landmarks and these correlations would grow with successive observations [1][2]. The correlations were crucial to achieve an efficient estimation. The more these correlations grow, the better the solution [3]. Stochastic estimation techniques such as the Kalman Filter (KF) [4], Particle Filter [5], $H\infty$ Filter [6] or Information Filter (IF) [7] have been used to solve the SLAM problem. Kalman filter is the most used method due to the simplicity of algorithm and lower computational cost compared to other filters [8].

In this paper, we have studied the KF-based SLAM behaviour under intermittent measurement. Intermittent measurement is a condition when the mobile robot lost its measurement data during observations due to sensor failure or imperfection of the system. The issue is important, since this condition may lead to erroneous result [9]. The research of intermittent measurement have been focused mainly for network system [10] [11] and there has been very limited studies on mobile robot application [12].

The paper is structured as follows. Section II presents the model of the system and the Kalman filter based algorithm to the SLAM problem. Section III shows the analysis of KF-based SLAM under normal and intermittent measurement conditions for stationary and moving robot. Finally section IV concludes the study.

II. KALMAN FILTER BASED SLAM

A. SLAM Model

SLAM is represented through discrete time dynamical system equation using process and observation model. The process model describes the motion of the mobile robot while the observation model defines the measurement of the map features or landmarks with respect to the mobile robot position. Fig. 1 shows the setup of the SLAM that is represented by these models.

For a linear system, the process model of SLAM from time k to time $k + 1$ is described as

$$X_{k+1} = F_k X_k + B_k u_k + G_k w_k \tag{1}$$

where X_k is the state of the mobile robot and landmarks, F_k is the state transition matrix, B_k is the control matrix, u_k is the control inputs, G_k is the noise covariance matrix and w_k is the zero-mean Gaussian process noise with covariance Q.

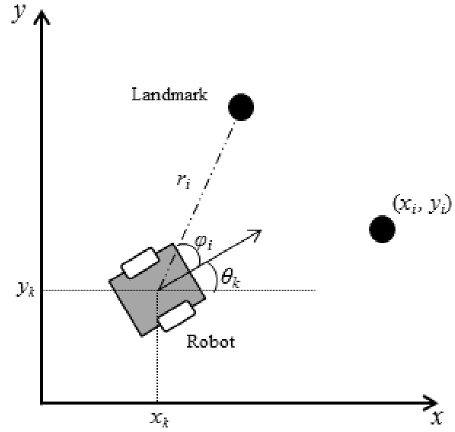

Fig. 1 : SLAM model

The state vector $X_k \in \Re^{3+2m}$ at time k is represented by a joint state-vector of robot X_r and landmarks X_m states as

$$X_k = \begin{bmatrix} X_r \\ X_m \end{bmatrix} = \begin{bmatrix} \theta_k \\ x_k \\ y_k \\ x_i \\ y_i \end{bmatrix} \tag{2}$$

where x_k and y_k are the coordinates of the centre of the mobile robot with respect to global coordinate frame and θ_k is the heading angle of the mobile robot. The landmarks are model as point landmarks and represented by Cartesian coordinate (x_i, y_i), i = 1, 2,…, m where m is number of landmarks.

In this study, a model of two-wheel mobile robot is used. $X_r = [\theta_k \;\; x_k \;\; y_k]^{\mathrm{T}}$ is used to represent the robot position or in this study sometimes we denote it as robot pose. The process model to describe the kinematic motion of mobile robot is defined as $X_{r(k+1)} = f(X_{r(k)}, u_k, \delta\omega, \delta\upsilon)$ and $u_k = [\omega_k \quad \upsilon_k]^{\mathrm{T}}$ in which

$$\begin{aligned} \theta_{k+1} &= \theta_k + (\omega_k + \delta\omega)T \\ x_{k+1} &= x_k + (\upsilon_k + \delta\upsilon)T\cos(\theta_k) \\ y_{k+1} &= y_k + (\upsilon_k + \delta\upsilon)T\sin(\theta_k) \end{aligned} \tag{3}$$

with control inputs ω_k is mobile robot angular acceleration and υ_k is its velocity with associated process noises, $\delta\omega$ and $\delta\upsilon$. T is the sampling rate or the time interval of one movement step.

The process model for the landmarks $X_m = [x_i \;\; y_i]^{\mathrm{T}}$ for i = 1, 2,…, m is unchanged with zero noise as landmarks are assumed to be stationary.

$$X_{m(k+1)} = X_{m(k)} \tag{4}$$

The state observation or measurement process is represents using observation model

$$z_k = \begin{bmatrix} r_i \\ \varphi_i \end{bmatrix} = H_k X_k + v_{r_i \varphi_i} \tag{5}$$

where H_k is the measurement matrix and $v_{r_i \varphi_i}$ is the zero-mean Gaussian noise with covariance matrix R. At time k + 1, the observation of i-th landmark is a range r_i and bearing φ_i which indicates relative distance and angle from mobile robot to any observed landmarks. It is assumed that the sensors on the robot are equipped with a range and bearing sensors that make the observations of the landmarks in the environment and also encoders at the wheels for the measurement of vehicle speed. Range and bearing are defined as

$$r_i = \sqrt{(y_i - y_{k+1})^2 + (x_i - x_{k+1})^2} + v_{r_i} \tag{6}$$

$$\varphi_i = \arctan\left(\frac{y_i - y_{k+1}}{x_i - x_{k+1}}\right) - \theta_{k+1} + v_{\theta_i} \tag{7}$$

where ($x_{k+1}, y_{k+1}, \theta_{k+1}$) is current robot position, (x_i, y_i) is position of observed landmark, v_{r_i} and v_{θ_i} are the noises on the measurements.

B. *State Error Covariance Matrix*

Generally the covariance of two variants is the measurement on how strongly these two variables are correlated. The correlation on the other hand is a concept used to measure the degree of linear dependencies between variables. The covariance matrix of a state estimation in SLAM is a combination matrix of robot and landmark position covariance matrixes and correlation between robot and landmarks. The covariance matrix in SLAM, P is defined as

$$P = \begin{bmatrix} P_{RR} & P_{RM} \\ P_{MR} & P_{MM} \end{bmatrix} \tag{8}$$

P_{RR} : Covariance matrix of the robot position

P_{MM} : Covariance matrix of the landmark position

P_{RM} : Cross-covariance matrix of the robot and landmark position or cross-correlation between them

In SLAM, the covariance matrix indicates the error associated with the robot and landmark state estimations. From the covariance matrix, researchers can observe the uncertainties and errors of the estimation either grow or decline, in which represent the precision and consistency of the estimation. Therefore the study on the behaviour of covariance matrix is one of the important issues in SLAM.

C. Kalman Filter Algortihm

Kalman filter is used to provide estimates of mobile robot pose and landmark location. Kalman filter recursively computes estimates for a state X_k according to the process and observation model in (1) and (5) respectively. The stages of Kalman filter algorithm are as follows:

- *Prediction* (time update) to estimate priori estimation of state and its error covariance matrix:

$$X_{k+1}^{-} = F_k X_k + B_k u_k \tag{9}$$

$$P_{k+1}^{-} = F_k P_k F_k^{T} + G_k Q_k G_k^{T} \tag{10}$$

- *Update* (measurement update) to provide a correction based on the measurement z_k to yield a posteriori state estimate and its error covariance:

$$\mu_{k+1} = z_{k+1} - H_{k+1} X_{k+1}^{-} \tag{11}$$

$$S_{k+1} = H_{k+1} P_{k+1}^{-} H_{k+1}^{T} + R_{k+1} \tag{12}$$

$$K_{k+1} = P_{k+1}^{-} H_{k+1}^{T} \left(S_{k+1}\right)^{-1} \tag{13}$$

$$X_{k+1} = X_{k+1}^{-} + K_{k+1}\mu_{k+1} \tag{14}$$

$$P_{k+1} = (I - K_{k+1} H_{k+1}) P_{k+1}^{-} \tag{15}$$

where μ_{k+1} is the difference between the measurement from the sensor and predicted measurement from Kalman filter (normally is called as innovation), S_{k+1} is associated covariance for the innovation and K_{k+1} is Kalman gain.

D. Kalman Filter Algorithm with Intermittent Measurement

The measurements from the sensor may be lost due to sensor failure or miscommunication between sensor and controller. During this condition, the estimator will not receive the observation, which could lead to estimation error. Intermittent measurement has always been modelled either as Markov chain or Bernoulli process [10]. Based on the representation through Bernoulli process, to compensate the missing measurements, the update algorithm for Kalman filter is modified into

$$X_{k+1} = X_{k+1}^{-} + \gamma_{k+1} K_{k+1}\mu_{k+1} \tag{16}$$

$$P_{k+1} = (I - \gamma_{k+1} K_{k+1} H_{k+1}) P_{k+1}^{-} \tag{17}$$

where γ_{k+1} is a Bernoulli random variable and has value either one or zero [10].

III. Analysis Of Intermittent Measurement

This paper attempts to prove that if there are some missing measurement data during robot observation, the estimations of mobile robot pose and landmarks locations are not correct [13] and the covariance of the estimation is increased, determined by the determinant of covariance matrix. The analysis was conducted during the intermittent measurement occurred at time $k = a$ and after intermittent measurement at time $k = a + n$. The impact on covariance matrix is observed and analysed.

Definition 1: Measurement data lost is defined whenever measurement data is not successfully retrieved after one sample time and occurred randomly in mobile robot observations [14].

The above definition describes that if a measurement is unavailable at time *k*, then the measurement matrix $H_k \cong [0]$, where $[0]$ denotes a zero matrix. We now demonstrate how the covariance matrix of state estimation behaves if this is partially happened during mobile robot observation.

A. Robot is stationary

For the first case of study, robot is considered to be stationary and observed one landmark in its environment for n times. Since robot is not moving, there are no control input for the mobile robot's motion, therefore $u_k = 0$. Under this scenario, two conditions are observed; with and without existence of process noise.

1) No process noise

Since mobile robot is stationary, some researchers assumed that there are no process noises for that moment as no movement is involved [15].

Proposition 1: Covariance of process noise Q_k is the summation of the covariance of control noise $\delta\omega$ and $\delta\upsilon$. Since the mobile robot is assumed to remain stationary, so no control noise is injected to the system, therefore Q_k is assumed to be zero. Thus under this assumption, covariance matrix under intermittent measurement is larger than covariance matrix under normal condition, in which the measurement data is consistently available.

Proof: The state error covariance matrix is predicted through (10). Since the robot is stationary, the state transition matrix F_k possesses normally an identity matrix. Thus, priori covariance matrix at time $k + 1$ is equal to posterior covariance matrix at time k, since no process noise is added to the system.

$$\begin{aligned} P_{k+1}^{-} &= F_k P_k F_k^{T} + G_k Q_k G_k^{T} \\ &= I P_k I^{T} + 0 \\ &= P_k \end{aligned} \tag{18}$$

Matrixes in the algorithm are positive semidefinite (psd) matrix [4]. If the measurement data is consistently available at $1 < k < \infty$ time, updated covariance at time $k + 1$ has smaller value than priori covariance because of the correction done by Kalman filter. From (15) and (18)[1]

$$P_{n(k+1)} = P_{k+1}^{-} - K_{k+1} H_{k+1} P_{k+1}^{-} \tag{19}$$

$$P_{n(k+1)} \le P_{k+1}^{-} \tag{20}$$

$$P_{n(k+1)} \le P_k \tag{21}$$

If intermittent measurement occurred, there is no observation available, hence from the explanation of *Definition 1* measurement matrix $H_{k+1} \rightarrow [0]$ and $\gamma_{k+1} = 0$ in (17). Under this assumption, posterior covariance matrix with intermittent measurement is equal to the priori covariance matrix, which is similar to the covariance matrix at time *k*.

[1]Subscript *i* and *n* denote a parameter during intermittent measurement and under normal condition respectively.

$$\begin{aligned} P_{i(k+1)} &= P_{k+1}^{-} - \gamma_{k+1} K_{k+1} H_{k+1} P_{k+1}^{-} \\ P_{i(k+1)} &= P_{k+1}^{-} \\ P_{i(k+1)} &= P_{k} \\ P_{i(k+1)} &\geq P_{n(k+1)} \end{aligned} \tag{22}$$

Suppose in measurement update, the covariance is corrected through Kalman gain, but this cannot be done due to unavailability of measurement data. □

Definition 2: The determinant of the state error covariance matrix is a measure of the volume of the uncertainty ellipsoid associated with the state estimate [4].

From (21) and (22), it shows that determinant of state error covariance during intermittent measurement is larger than determinant of state error covariance under normal condition.

$$\begin{aligned} &\det\left(P_{n(k+1)}\right) \leq \det\left(P_{k}\right) \;\; \cap \;\; \det\left(P_{i(k+1)}\right) = \det\left(P_{k}\right) \\ &\Rightarrow \det\left(P_{n(k+1)}\right) \leq \det\left(P_{i(k+1)}\right) \end{aligned} \tag{23}$$

This denotes the total uncertainty is increasing if the measurement data was suddenly unavailable, which indicates imprecise estimation of current state. Thus, under intermittent measurement the mobile robot may incorrectly estimates its current position.

2) With process noise

In a real situation, it is hard to obtain a noise-free system. Although the mobile robot is not moving, process noise may also exist in the SLAM system, e.g. noises from the environment or encoder attached to the robot. In this section, the analysis is continued with $Q_k \neq 0$.

From (10) priori covariance matrix at time $k + 1$ is no longer equal to posterior covariance matrix at time k, since process noise is added to the system. The priori covariance at $k + 1$ is larger than covariance at k.

$$\begin{aligned} P_{k+1}^{-} &= F_k P_k F_k^{T} + G_k Q_k G_k^{T} \\ &= I P_k I^{T} + G_k Q_k G_k^{T} \end{aligned} \tag{24}$$

$$P_{k+1}^{-} > P_k \tag{25}$$

This is true for both cases; normal and intermittent measurement. Under normal condition, priori covariance matrix is updated using (12) – (15) and possesses smaller value of posterior covariance matrix. This is true as $k \to \infty$ covariance matrix is decreasing and converging.

However, the effect is not similar if measurements are not available. Using (17) with $H_{k+1} \to [0]$ and $\gamma_{k+1} = 0$, covariance matrix is not able to be updated and remains with the value of priori covariance. The covariance matrix will accumulate if the measurement is still not available, since process noise is injected to the system for each time update (10).

Since covariance matrix under normal condition is decreasing and converging as $k \to \infty$ where else covariance matrix with intermittent measurement is increasing as long as measurement is not available, the determinant for both covariances have a similar trend;

$$\det\left(P_{n(k+1)}\right) \leq \det\left(P_{i(k+1)}\right). \tag{26}$$

This proves that uncertainty of the state estimation when data is not available is higher than normal condition, indicates erroneous prediction of robot pose. Therefore some sort of control strategies should be proposed to compensate this error.

B. Robot is moving

The mobile robot moves from stationary position and observes one landmark in its environment for n times. Since robot is moving, there is control input applied to the system for mobile robot's motion, therefore $u_k \neq 0$ and $Q_k \neq 0$.

Since control input is concerned only in the prediction of priori state estimates (9), the effect on the covariance matrix is not significant. However, the process noise under this condition is possibly larger than process noise under stationary condition, due to existence of $\delta\omega$ and $\delta\upsilon$. Therefore covariance matrix when mobile robot moves is greater than covariance matrix under stationary situation.[2]

$$\begin{aligned} &P_{k+1}^{-} = F_k P_k F_k^{T} + G_k Q_k G_k^{T} \\ &\textit{If} \;\; Q_{k(u)} > Q_{k(s)} \;\; \textit{thus} \;\; P_{k+1(u)}^{-} > P_{k+1(s)}^{-} \\ &\textit{Hence from } (15) \;\; P_{k+1(u)} > P_{k+1(s)} \end{aligned} \tag{27}$$

The behaviour of covariance matrix under normal and intermittent measurement when robot moves is similar with the behaviour when robot is stationary. Covariance matrix when measurement data is not available is higher than normal condition. The characteristic is also analogous for the determinant of the covariance matrix. This analysis can be proved in a similar way to the case one i.e. under stationary condition. Therefore the details of the proof are omitted.

IV. CONCLUSION

This paper presented the analysis of Kalman filter-based SLAM for the instant that measurements data may be randomly unavailable. It has been shown that although the measurements data is not available intermittently during mobile robot observation, the estimation is still possible, but possesses erroneous result. This is proved by the increment of state error covariance matrix and its determinant, in comparison to the normal condition. The analysis proved that the measurement matrix H_k highly affects the performance of KF based SLAM during intermittent measurement. Therefore a control strategy for this situation should be implemented to compensate the error. As future works we are planning to prove the analysis in this study through simulation and investigate the effect of intermittent measurement on the correlation between mobile robot and landmarks.

ACKNOWLEDGMENT

The authors would like to thank the Research and Innovation Department of UMP for the supports. This research is funded under Postgraduate Research Grant Scheme (PRGS).

[2]Subscript u and s denote a parameter for the case of stationary and moving robot respectively.

REFERENCES

[1] R. C. Smith and P. Cheeseman, "On the representation and estimation of spatial uncertainty," *The International Journal of Robotics Research*, vol. 5, no. 4, pp. 56–68, 1987.

[2] H. Durrant-Whyte, "Uncertain geometry in robotics," *Robotics and Automation, IEEE Journal of*, vol. 4, no. 1, pp. 23 –31, Feb 1988.

[3] H. Durrant-Whyte and T. Bailey, "Simultaneous localization and mapping: Part I, *Robotics Automation Magazine*, IEEE, vol. 13, no. 2, pp. 99 –110, June 2006.

[4] M. Dissanayake, P. Newman, S. Clark, H. Durrant-Whyte, and M. Csorba, "A solution to the simultaneous localization and map building (SLAM) problem," *Robotics and Automation, IEEE Transactions on*, vol. 17, no. 3, pp. 229 –241, Jun 2001.

[5] M. Montemerlo, S. Thrun, D. Koller, and B. Wegbreit, "Fastslam: A factored solution to the simultaneous localization and mapping problem," in In *Proceedings of the AAAI National Conference on Artificial Intelligence*. AAAI, 2002, pp. 593–598.

[6] H. Ahmad and T. Namerikawa, "H infinity filtering convergence and it's application to slam," *in ICCAS-SICE*, 2009, Aug. 2009, pp. 2875 – 2880.

[7] Thrun, S., Koller, D., Ghahramani, Z., Durrant-Whyte, H. and N. A.Y., "Simultaneous mapping and localization with sparse extended information filter: theory and initial results". In: *Proc. Of the fifth international workshop on algorithmic foundation of robotics*, Nice 2002.

[8] H. Ahmad and T. Namerikawa, "Covariance Bounds Analysis during Intermittent Measurement for EKF-based SLAM," *International Journal of Integrated Engineering*, vol. 4 no. 3 pp. 19-25, 2012.

[9] H. Ahmad and T. Namerikawa, "Intermittent Measurement in Robotic Localization and Mapping with FIM Statistical Bounds," *IEEJ Transactions on Electronics, Information and Systems*, vol. 131 no. 6 pp. 1223-1232, 2011.

[10] Sinopoli Schento B., L., Franceschentti M., Poolla K., Jordan M.I, Sastry S.s, "Kalman Filtering with Intermittent Measurements", *IEEE Transaction on Automatic Control*, Vol. 49, Issue 9, pp.1453-1464, 2004.

[11] S. Kar, B. Sinopoli, J.M.F. Moura, "Kalman Filtering with Intermittent Observations: Weak Convergence to a Stationary Distribution", *IEEE Trans. On Automatic Control*, Vol. 57, No. 2, 2012.

[12] P. Muraca, P. Pugliese, G. Rocca, "An Extended Kalman Filter for the State Estimation of a Mobile Robot from Intermittent Measurements", *Mediterranean Conference on Control and Automation Congress Center*, France, 2008.

[13] N.A.Othman, H.Ahmad, S.Razali,"Analysis of Intermittent Measurement for KF-based SLAM", *2nd International Conference on Electrical, Control and Computer Engineering*, Malaysia, August 2013.

[14] H. Ahmad, T. Namerikawa, "H∞ Filter-based Robotic Localization and Mapping with Intermittent Measurements", 8^{th} France-Japan and 6^{th} Europe-Asia Congress on Mechatronics, Japan, 2010.

[15] Julier, S.J.; Uhlmann, J.K., "A counter example to the theory of simultaneous localization and map building," *Robotics and Automation, 2001. Proceedings 2001 ICRA. IEEE International Conference on* , vol.4, no., pp.4238,4243 vol.4, 2001.