Numerical Simulation and Analysis of the Duffing System using Python

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Abstract: - In this paper, we are presenting a numerical investigation of Duffing's nonlinear system. The Duffing's system is described by a nonlinear second-order differential equation, which shares he same structure like the damped and driven oscillator systems. The numerical integration of the system is done using Python scientific libraries such as NumPy and SciPy. The proposed technique is utilized to construct the bifurcation diagram and the Poincaré section and to analyze the behavior of the system by changing one parameter. This proposed method is characterized by versatility and the extrapolation of these numerical techniques and algorithms in the study of other nonlinear dynamical systems.

Key-Words: - Duffing Oscillator, Nonlinear Dynamics,Bifurcation Diagram, Poincaré Section, Chaos Theory

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1 Introduction

A wide range of complex phenomena in fields like physics, engineering, chemistry, and economics have similar behavior with nonlinear circuits. For this reason, there is an increased interest in the study of those systems, [1]. Unlike in linear circuits, where the principle of linearity and superposition is omnipresent, nonlinear circuits exhibit a rich spectrum of behaviors, including periodicity, quasi-periodicity, and chaotic regimes, [2].

One of the examples of nonlinear circuits is the Duffing oscillator, invented by the German engineer Georg Duffing in 1918. This oscillator was one of the first that brought the nonlinear systems, which until then were only described by equations, to the laboratory, making this oscillator to serve as a fundamental model for exploring nonlinear dynamics, [3]. The Duffing oscillator is described by a second-order differential equation with a nonlinear restoring force.

This force can take various forms, such as cubic stiffness or a combination of linear and nonlinear terms. This oscillator is not only theoretically intriguing but also practically significant, as it models phenomena ranging from mechanical systems with nonlinear springs to electronic circuits exhibiting nonlinear inductance, [4], [5]. In addition, we have to forget the role of the Duffing oscillator in advancing our understanding of nonlinear systems, as their application in cryptography has sparked the keen interest of the research community, [6].

Chaotic systems have raised the interest of the research community, as this phenomenon is not only present in electronic systems but also in optics, with the main purpose of encoding information, [7].

In this paper, we are presenting the fundamentals of the Duffing's oscillator, by outlining the general characteristics and mathematical foundations, followed by a detailed examination of its operations. We simulate its behavior solving numerical the system trying to elucidate the significance of the Duffing oscillator in both theoretical research and practical applications.

In addition we focus on the numerical simulation of the Duffing system using Python's NumPy and SciPy libraries. Python has rapidly become one of the leading tools for studying dynamical systems due to its accessibility, flexibility, and the extensive ecosystem of scientific libraries, [8]. The bifurcation diagram and Poincaré section are presented to illustrate the system's dynamics under varying parameters.

2 Problem Formulation

The presence of chaotic behavior in nonlinear circuits has been theoretically and experimentally proven, [9], [10]. Apart from the well known Chua's circuit chaos has been appeared and in simple circuits with a non-linear element such as a transistor, [11]. At first, we will formulate theoretically the Duffing's systems, giving the circuit and the differential equations that govern its behavior, [12]. Figure 1 shows the nonlinear circuit that implements the Duffing equation. As shown in the figure, the circuit consists of a linear resistor, a linear capacitor, and a non-linear coil.

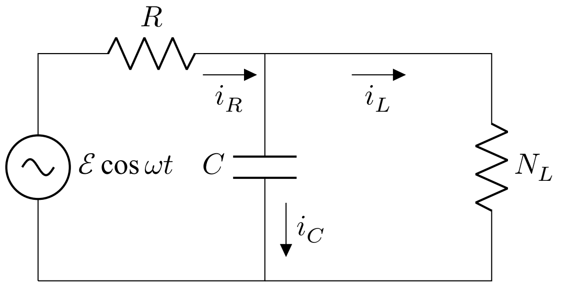

Fig. 1: The circuit that implements Duffing's equation

The coil has a ferromagnetic core and ignoring the phenomenon of magnetic hysteresis, the $i - \phi$ characteristic is approximated by the relation:

$$i = a_1\phi + a_3\phi^3 \quad (1)$$

where ϕ denotes the magnetic flux and a_1 and a_3 are constants. Applying Kirchhoff's first law to a node of the circuit we have.

$$i_R = i_C + i_L \quad (2)$$

where:

$$i_R = \frac{\mathcal{E}\cos\omega t - v_L}{R} \quad (3)$$

$$i_C = C\frac{dv_C}{dt} = C\frac{dv_L}{dt} \quad (4)$$

$$i_L = a_1\phi + a_3\phi^3 \quad \text{and} \quad v_L = \frac{d\phi}{dt} \quad (5)$$

By substituting the equations (3), (4) and (5) into the equation (2) we have:

$$\frac{\mathcal{E}\cos\omega t - v_L}{R} = C\frac{dv_L}{dt} + a_1\phi + a_3\phi^3 \quad (6)$$

Rearranging equation (6) and taking into consideration the equation (5)

$$\frac{d^2\phi}{dt^2} + \frac{1}{RC}\frac{d\phi}{dt} + \frac{a_1}{C}\phi + \frac{a_3}{C}\phi^3 = \frac{\mathcal{E}}{CR}\cos\omega t \quad (7)$$

By substituting, $\epsilon = \frac{1}{RC}$, $a = \frac{a_1}{C}$, $b = \frac{a_3}{C}$ and $B = \frac{\mathcal{E}}{CR}$ we have the Duffing's equation in its general form:

$$\ddot{x} + \epsilon\dot{x} + a * x + b * x^3 = B\cos(\omega t) \quad (8)$$

Duffing's equation is a second-order differential equation that includes a nonlinear term and is excited by a harmonic signal. It describes in general, the motion of a damped oscillator with a more complex restoring force than the simple harmonic oscillator.

Depending on parameters a and b, the Duffing's oscillator displays diverse dynamical behavior. For the case where $a = 0$ and $b = 1$, which was studied by Y.Ueda, the system is governed by a single-well potential; hence, under the influence of an external periodic driving force, a complex behavior could be observed. In his work, Ueda highlighted the sensitivity of the system to initial conditions, leading to the discovery of chaotic attractors, [13]. In contrast, in the case when $a = 1$ and $b = 1$, which was studied by Parlitz and Lauterborn, the system appears to have a double-well potential, [14]. In this case, the system has a more complex dynamic since the interplay between these parameters influences the system's ability to jump between potential wells and the chaotic transition is more probable.

3 Simulation

We aim to solve Duffing's equation numerically using Python, using powerful libraries such as NumPy, SciPy for the simulation, and Matplotlib for the visualization of the results. The simulation of the system will be carried out using the constants $\epsilon = 0.3$, $a = -1$, $b = 1$, $B = 0.37$, and $\omega = 1.2$. We selected initial conditions with $x = 0$ and $\dot{x} = 0$ and numerically solved the differential equation using the 'odeint' function, which is part of the SciPy library. The time for the simulation is defined from the value $t_i = 0$ to $t_f = 360$, choosing 1000 points in this interval. This was made possible by using the `linspace()` function provided in the NumPy library. The phase diagram of the system, where $\dot{x}$ versus x is presented, is constructed from the simulation results for the compared values and is shown in Figure 2.

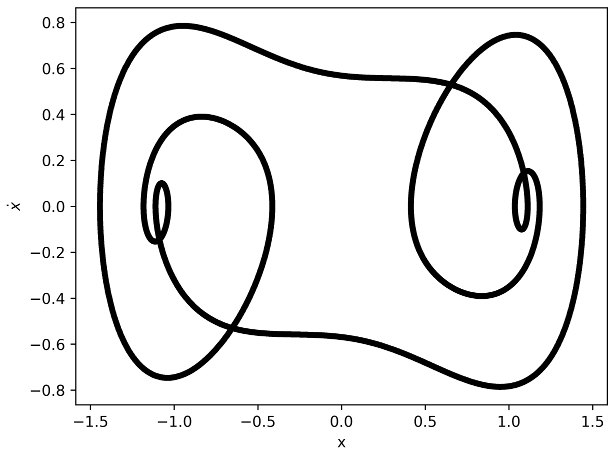

Fig. 2: Phase space for Duffing's oscilator with parameters $\epsilon = 0.3$, $a = -1$, $b = 1$, $B = 0.37$ and $\omega = 1.2$

The system, as illustrated in Figure 2, exhibits periodic behavior under the given parameters. This diagram is created once the system has reached a steady state. In our case, this is judged to have been achieved after the first 500 samples of the data obtained from solving the system. Next, we will focus on the chaotic transition of the circuit by varying the amplitude B of the signal provided by the source.

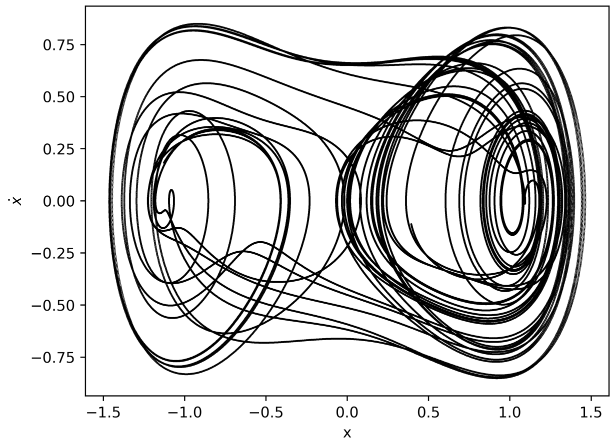

Fig. 3: Phase space for Duffing's oscilator with parameters $\epsilon = 0.3$, $a = -1$, $b = 1$, $B = 0.4$ and $\omega = 1.2$

By changing the amplitude to $B = 0.4$, the system's behavior changes dramatically. The system transitions from periodic operation to chaos, as shown in Figure 3. In this operating state we consider the system to have an infinite period. The plot of x versus time presents a strongly stochastic picture. This image is not shown here for brevity. A bifurcation diagram provides a representation of the system's behavior for multiple amplitude values. In this approach, we select a large number of samples - specifically, $N = 40.000$ -across a range of signal width values from $B = 0.25$ to $B = 0.5$. For each amplitude value of B, starting from the initial conditions $x = -1$ and $\dot{x} = 1$, we simulate the system over a time equal to the period of the external signal. The final state value of the system is retained and used as the initial value for the next simulation. For each value of the voltage amplitude, we perform 200 repetitions of the procedure described above.

If the system is periodic, it will return to the same state after a period has elapsed. This will be represented on the diagram as a point corresponding to the specific amplitude value. This can be seen in Figure 4 for values of $B < 0.265$. As the value of B increases, the system exhibits frequency doubling and transitions to chaotic behavior. This transition is most clearly illustrated in the Figure 5 for values of $B > 0.25$ and $B < 0.31$.

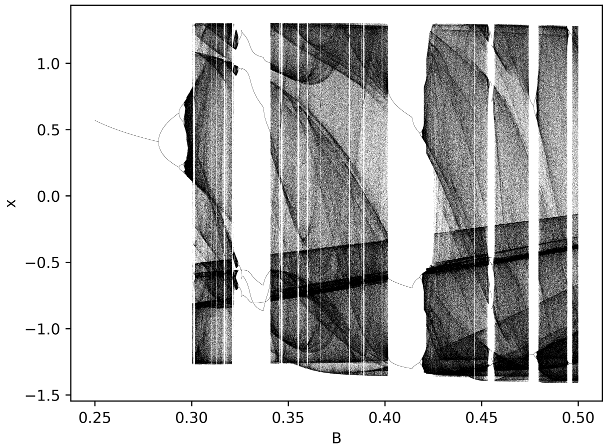

Fig. 4: Bifurcation diagram for Duffing's oscilator with parameters $\epsilon = 0.3$, $a = -1$, $b = 1$, $\omega = 1.2$ and $0.25 \leq B \leq 0.5$

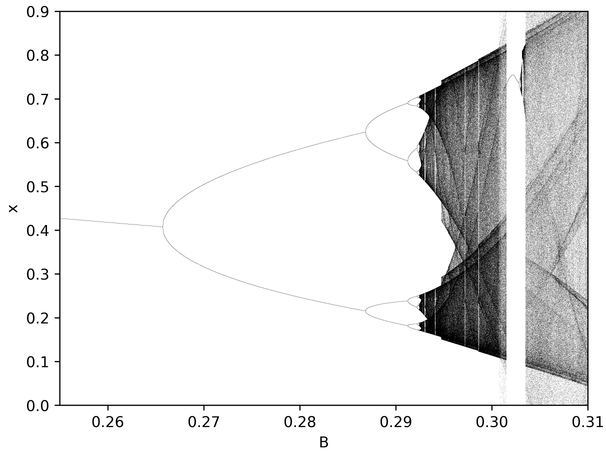

Fig. 5: Bifurcation diagram for Duffing's oscilator with parameters $\epsilon = 0.3$, $a = -1$, $b = 1$, $\omega = 1.2$ and $0.25 \leq B \leq 0.31$

Figure 4 illustrates the bifurcation diagram when varying the parameter B. As B increases, the system transitions from a periodic behaviour to a chaotic regime.In Figure 3, the transition of the system from its periodic operation to the chaotic state is clearly seen through the mechanisms of frequency doubling. The diagram includes a normal distribution as a reference, indicating the density of points through which the oscillator passes. This suggests that the chaotic behavior observed is not purely random but is governed by underlying structure.

In Figure 6 a Poincaré section is presented, this is a powerful tool used in the analysis of dynamical systems, particularly in studying complex, chaotic behavior. By taking a cross-section of the phase space, the Poincaré section captures the intersections of a trajectory with a lower-dimensional plane, allowing for a simplified yet insightful view of the system's dynamics.

The construction of the Poincare section is done using the same technique described for the bifurcation diagram. The simulation is repeated over a time interval corresponding to the period of the external signal from the source, $T = \frac{2\pi}{\omega}$. This process is conducted for $N = 10^7$ iterations. This number could be big enough, but in our computer (a laptop with an i7 processor), the simulation time was around one hour. This can be done with fewer iterations, but the quality of the image will not be as clear as in our result. After a period has passed, we keep the ($\dot{x}$ and x) values, which we use as initial values for the next simulation. This is repeated as many times as we have mentioned before. Once the data is collected, Figure 6, is constructed to visualize the results.

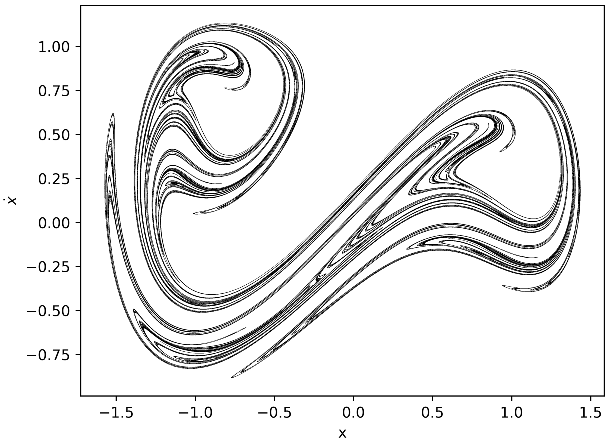

Fig. 6: Poincare section for Duffing's oscilator with parameters $\epsilon = 0.3$, $a = -1$, $b = 1$, $B = 0.4$ and $\omega = 1.2$

The detailed structure of a Poincaré section can be examined by increasing the size of a small part of the figure. By focusing on a small region of the strange attractor and further magnifying this area, one can observe that the intricate features at smaller scales closely resemble those observed at larger scales, revealing a fine, self-similar structure within the attractor. The recurrence of similar features across different regions of a figure and at varying scales is a hallmark of fractals. An attractor is considered fractal if it exhibits self-similarity, meaning that as the scale decreases, the pattern of points retains its structure.

4 Conclusion

Above, we have presented how we can construct the bifurcation diagram and the Poincare diagram for the case of Duffing's oscilator using Python. The above diagrams make it possible to present the complex dynamics that occur in nonlinear dynamic systems when they exhibit chaotic behavior. The proposed methodology can be used not only in the study of other non-dynamical systems but also in the context of a didactic approach in the framework of postgraduate studies in nonlinear systems or in the context of courses such as computational physics.

The transition of the system from periodic to chaotic mode can be used in the future in secure communication systems through the technique of adding the chaotic signal to the information signal at the receiver and removing it at the transmitter. Doing so would allow cryptography at the physical level, thus making such a future system more secure against attacks based on the use of decryption algorithms.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used Grammarly for language editing. After using this service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication

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