About the Tikhonov Regularization Method for the Solution of Incorrect Problems

V. M. RYABOV, I. G. BUROVA
Department of Computational Mathematics,
St. Petersburg State University,
7-9 Universitetskaya Embankment, St. Petersburg,
RUSSIA

Abstract: - From time to time, papers are published containing gross errors when solving integral equations of the first kind. This paper is devoted to the analysis of these errors. The paper considers Tikhonov's weak and operator regularization. To construct a solution to the integral equation, the local splines of the Lagrangian type of the second order of approximation, as well as the local splines of the Hermitian type of the fourth order of approximation of the first level, are used. The results of numerical experiments are presented.

Key-Words: integral equation of the first kind, regularization, local splines.

Received: October 8, 2022. Revised: May 17, 2023. Accepted: June 9, 2023. Published: June 28, 2023.

1 Introduction

From time to time, articles are published containing gross errors when solving integral equations of the first kind. This paper is devoted to the analysis of these errors. As is known, the Fredholm equation of the first kind

$$\int_a^b K(x,s)u(s)ds = f(x),\ c \le x \le d,$$

and the Volterra equation of the first kind

$$\int_a^x K(x,s)u(s)ds = f(x),\ c \le x \le d,$$

belong to ill-posed tasks. These problems are unstable on the right-hand side and, therefore, are ill-posed. In other words, there are arbitrarily small perturbations of the right-hand side, which correspond to large perturbations of the solution. Therefore, the solutions to these problems are unstable. In addition, in a number of cases, even if a solution exists for some right-hand side**s**, there are small changes in the right-hand side for which the solution does not exist.

Therefore, when solving the Fredholm or the Volterra integral equations of the first kind, one often has to resort to regularizing the solution. Two approaches to solving this problem should be noted. The first and easiest way is as follows. The integral equation is reduced to solving a system of linear algebraic equations. This system of equations usually has a large conditional number or can be degenerated. Therefore, it is necessary to carry out the regularization of the system of equations according to Tikhonov. Here we consider the second variant of regularization, namely, the operator regularization.

2 Operator Regularization

Given the equation

$$Az = f, \qquad (1)$$

where $A : Z \to F$. In the general case, the spaces Z and F are the Hilbert spaces.

2.1 The Case of Systems of Linear Algebraic Equations

The systems of linear algebraic equations (SLAEs) can be degenerate or ill-posed and we know approximations A_δ, u_δ such that

$$\| A_\delta - A \| \le \delta, \| f_\delta - f \| \le \delta.$$

Existence solutions to the original and perturbed SLAEs are not assumed. The concepts of pseudo-solutions and normal solutions are introduced, [1], [2], [3], [4].

To find a normal solution, we introduce the functional

$$M_\alpha(z, A_\delta, f_\delta) = \| A_\delta z - f_\delta \|^2 + \alpha\Omega(z),$$
$$\Omega(z) = \| z \|^2 . \qquad (2)$$

The minimum point of this functional is found as a solution to the equation Euler, [3]

$$(A_\delta^* A_\delta + \alpha E)z = A_\delta^* f_\delta. \qquad (3)$$

A solution to this equation exists and is unique due to the symmetry and positive definiteness of the matrix

$$A_\delta^* A_\delta + \alpha E \quad \text{for } \alpha > 0.$$

The set of solutions to the Euler equation for all $\alpha > 0$ is bounded, [3], and due to its compactness, one can choose a convergent sequence from it. In [3], a method was indicated for choosing the regularization parameter α which tends to be zero. This method also depends on the parameter δ which tends to be zero, at which the selected sequence converges to the normal solution of the original exact task. In the case of a symmetric positive definite matrix, the Euler equation (3) (i.e. regularization) should be carried out in the form [1], [2], [5], [6]

$$(A_\delta + \alpha E)z = f_\delta . \quad (4)$$

2.2 The Case of Fredholm Integral Equations of the First Kind

Equation (1) has the form

$$Az \equiv \int_0^1 \mu(x,s)z(s)ds = f(x), \ x \in [0,1], \quad (5)$$

where $\mu(x,s)$ is the kernel of the integral equation.

2.2.1 Case $\boldsymbol{Z = F = L_2[0,1]}$

Suppose $Z = F = L_2[0,1]$. The kernel of the operator A and the right side f are known with errors h and δ. In this case, the regularization is carried out in the form

$$(A_h^* A_h + \alpha E)z = A_h^* f_\delta . \quad (6)$$

and the convergence of solutions to Eq. (6) is guaranteed for a consistent tending to zero parameters h, δ, α to the exact solution of equation (5) (under the assumption of its existence) in the metric $L_2[0,1]$, [4].

2.2.2 Case $\boldsymbol{Z = C[0,1], F = L_2[0,1]}$

Suppose $Z = C[0,1], F = L_2[0,1]$. Therefore, $A : C[0,1] \to L_2[0,1]$. Errors are possible only in setting the right side of the equation. Let us assume that the desired solution belongs to some compact subset of the space of continuous functions. We use this as a priori of information about the solution in setting the stabilizing functional $\Omega(z)$ included in (2):

$$\Omega(z) = \int_0^1 \left(p(s)\big(z'(s)\big)^2 + r(s)z^2(s)\right)ds,$$

$$p(s), r(s) \geq 0.$$

The domain of the definition of this functional consists of functions uniformly bounded and equicontinuous on [0,1]. By the *Arzelà–Ascoli* theorem set

$$\{z : \Omega(z) < d\} \text{ for any } d > 0$$

is compact in space $C[0,1]$. Euler's equation for the functional

$$M_\alpha(z, A_\delta, f_\delta) = \| A_\delta z - f_\delta \|_{L_2}^2 + \alpha\Omega(z)$$

has the form [3],

$$(A_\delta^* A_\delta + \alpha L)z = A_\delta^* f_\delta. \quad (7)$$

where $Lz = -(p(s)\big(z'(s)\big))' + r(s)z(s)$. The operator $A_\delta^* A_\delta + \alpha L$ is symmetric and **a** positive definite, and the solutions of Eq. (7) belong to a compact set and converge to the normal solution of the integral equation as the parameters δ and α tend to zero, [3].

3 Problem Solution

It is stated in [7], that the "regularization" of Eq. (5) by the shift

$$\alpha z(x) + \int_0^1 \mu(x,s)z(s)ds = f(x), \ x \in [0,1], \quad (8)$$

and the simplest iterations as $\alpha \to 0$ lead to the solution of the original equation, and it is said that this was allegedly proven in the works of [3], [7], [8], which is completely wrong. In addition, paper, [7], gives examples of constructing "regularized" solutions in the form of a series, which are diverged, but they successfully sum up the row. Note that in this case no requirements are made regarding the kernel and the right side of the equation! The same error is repeated in the works of the authors of the paper, [9].

In the next section, for the Fredholm integral equation of the first kind, two methods of regularizations will be carried out. Weak regularization will be carried out first, i.e. the integral equation is reduced to solving a regularized system of linear algebraic equations (the Euler equation). Thus we obtain the regularized solution. While obtaining the system of equations, we used the spline approximation of the function $z(s)$. Further, in the second method, operator regularization will be carried out using local splines of the Hermitian type of the fourth order of approximation of the first level.

4 Numerical Experiments

Consider the Fredholm integral equations of the first kind

$$\int_0^1 K(x,t)\, u(t)dt = f(x),$$

Two approaches to the regularization of the solution were noted in section 2. The first and easiest way is as follows. The integral equation is reduced to solving a system of linear algebraic equations. This method belongs to weak regularization and is quite simple to implement. We use this method for solving the next integral equation.

Example. Consider the integral equation

WSEAS TRANSACTIONS on SYSTEMS
DOI: 10.37394/23202.2023.22.66

$\int_0^1 \sin(5+x-t)\, u(t)dt = f(x),\ x \in [0,1]$,
where the right part $f(x)$ is constructed using the exact solution $u(t) = (\sin(t)\sin(t-1))^3$.
So we have the expression

$$f(x) = \int_0^1 \sin(5+x-t)\,(\sin(t)\sin(t-1))^3\,dt =$$
$$-\frac{1}{20}\cos(2+x) - \frac{1}{140}\cos(8+x) - \frac{1}{4}\cos(4+x) - \frac{3}{20}\cos(6+x) + \frac{1}{20}\cos(7+x) + \frac{1}{140}\cos(1+x) + \frac{1}{4}\cos(5+x) + \frac{3}{20}\cos(3+x).$$

The plot of the function $u(t)$ is given in Fig. 1.

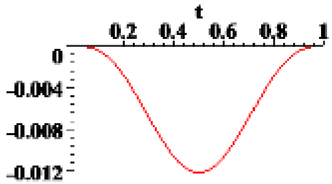

Fig. 1: The plot of the function $u(t)$

The plot of the function $f(x)$ is given in Fig. 2.

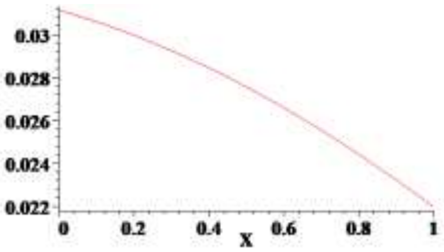

Fig. 2: The plot of the function $f(x)$.

First, we do regularization in the simplest way

4.1 The First Regularization Method

Let a grid of ordered nodes $\{x_j\}$ be constructed on the interval $[a,b]$. Let r, s, m, be integers, $r+s=m$, $r \geq 1, s \geq 0$, and the basis spline ω_k be such that $\operatorname{supp} \omega_k = [x_{k-s-1}, x_{k+r}]$. First, we approximate the unknown function $u(t)$ with the expression:

$$\tilde{u}(t) = \sum_{j=k-s}^{k+r} u_j \omega_j(t),\, t \in [x_k, x_{k+1}],$$

where u_j are the unknown coefficients. Let $s = 0, r = 1$. On the interval $[x_j, x_{j+1}]$ we approximate the function $u(t)$ by the following expression:

$$\tilde{u}(t) = u(x_j)\omega_j(t) + u(x_{j+1})\omega_{j+1}(t),$$
$$t \in [x_j, x_{j+1}],$$

where the basis splines are as follows:

$$\omega_j(t) = \frac{t - x_{j+1}}{x_j - x_{j+1}}, \qquad t \in [x_j, x_{j+1}],$$
$$\omega_{j+1}(t) = \frac{t - x_j}{x_{j+1} - x_j}, \quad t \in [x_j, x_{j+1}].$$

We call them splines of the second order of approximation (following Professor Mikhlin). So they are the local splines of the second order of approximation of the Lagrangian type.
We will use the norm of the vector of the form:

$$\| u \|_{[a,b]} = \max_{t \in [a,b]} | u(t) | .$$

Let $h = x_{j+1} - x_j$. The theorem of approximation of the function $u(x)$ with the splines is the following:

$$|u(t) - \tilde{u}(t)| \leq \frac{1}{8} h^2 \, \| u'' \|_{[x_j, x_{j+1}]} .$$

This splines were used, in papers, [5], [10].
Now we have the equation

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \sum_{j=k}^{k+1} u_j K(x,s)\omega_j(s)ds = f(x).$$

Next, we put $x = x_i$ and we can obtain the system of equations in the form

$$\sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \sum_{j=k}^{k+1} u_j K(x_i,s)\omega_j(s)ds = f(x_i),$$
$$i = 0, \dots n-1.$$

The integral $\int_{x_k}^{x_{k+1}} K(x_i,s)\omega_j(s)ds$ can be calculated exactly or by an appropriate quadrature formula. Now we solve the resulting system of linear algebraic equations. and after that we solve the regularization system of linear algebraic equations (the Euler equation) according to Tikhonov. And we obtain the regularized solution. Fig. 3 shows the exact solution and the solution obtained by solving the system of equations (without regularization). Similarly, Fig. 4 presents the plot of the error of the solution without regularization.

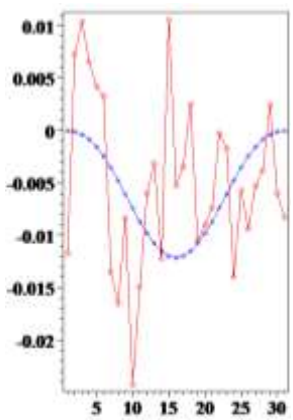

Fig. 3: The plot of the function $u(t)$ and the solution of the system of equations without regularization

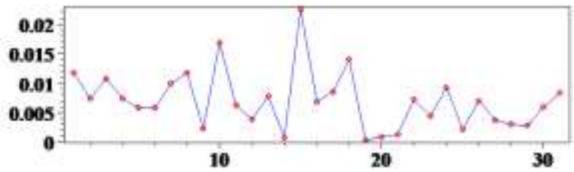

Fig. 4: The plot of the error of the solution without regularization

The next step is the following. We make a regularization according to Tikhonov. The results of the calculation are shown in Fig. 5 and Fig. 6. The plot of the function $u(t)$ and the solution after the regularization are given in Fig.5. The plot of the error of the solution after regularization is given in Fig.6. Note, that the numbers of the grid nodes are marked along the abscissa axis.

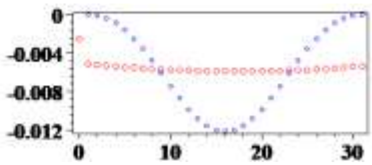

Fig. 5: The plot of the function $u(t)$ and the solution after the regularization

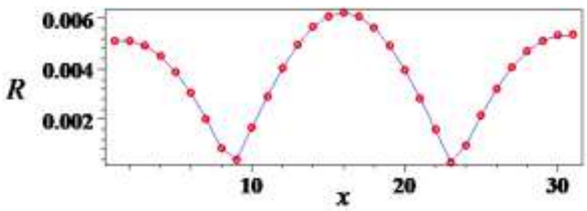

Fig. 6: The plot of the error of the solution after the regularization

4.2 The Second Regularization Method

Now consider the second regularization method. The second method of regularization refers to variational methods and is as follows. We assume that the kernel of the integral equation is continuous, and the homogeneous integral equation has only a zero solution. Let U, F be normed spaces. More specifically, let U be the Sobolev space W_2^n , and F be a Hilbert space. The essence of regularization according to Tikhonov is as follows. Instead of the equation $Au = f(x)$, where

$$Au \equiv \int_a^b K(x,s)u(s)ds,$$

we consider the modified problem:

$\int_c^d (Au - f)^2 dx + \alpha\Omega_n(u) = min,$

where the Tikhonov stabilizer $\Omega_n(u)$ has the form

$$\Omega_n(u) = \int_a^b \sum_{k=0}^{n} p_k\,(s)\left(\frac{d^k u(s)}{ds^k}\right)^2 ds,$$

p_k are continuous and $p_k \geq 0$, $\alpha > 0$. The Euler equation for this variational problem will be the following integro-differential equation:

$$\alpha \sum_{k=0}^{n} (-1)^k \frac{d^k}{ds^k}\left(p_k(s) u^{(k)}(s)\right) + \int_a^b Q(s,t)\,u(t)dt = P(s),$$

$a \leq s \leq b$, with the kernel $Q(s,t)$, and the right side $P(s)$ that are as follows:

$$Q(s,t) = \int_c^d K(x,s)K(x,t)dx,$$

$$P(s) = \int_c^d K(x,s)f(x)dx\,,$$

and with the boundary conditions:

$$q_r(u,s) = \sum_{k=r}^{n} (-1)^k \frac{d^{k-r}}{ds^{k-r}}\left(p_k(s)u^{(k)}(s)\right),$$

$$q_r(u,a) = q_r(u,b) = 0.$$

We consider a special case: $n = 1\,, r = 1$. The function $u(x)$ on each grid interval is replaced by the expression $\tilde{u}(x)$ using the basis splines of the fourth order of approximation and the first level. This approximation uses the values of the function and its first derivative at the grid nodes:

$$\tilde{u}(x) = u(x_j)w_{j,0}(x) + u(x_{j+1})w_{j+1,0}(x) + u'(x_j)w_{j,1}(x) + u'(x_{j+1})w_{j+1,1}(x),\ x \in [x_j, x_{j+1}],$$

where the basis splines have the form:

$$w_{j,0}(x) = \frac{(x - x_{j+1})^2}{(x_{j+1} - x_j)^2} + \frac{2(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^3},$$

$$w_{j+1,0}(x) = \frac{(x - x_j)^2}{(x_{j+1} - x_j)^2} + \frac{2(x_{j+1} - x)(x - x_j)^2}{(x_{j+1} - x_j)^3},$$

$$w_{j,1}(x) = \frac{(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^2},$$

$$w_{j+1,1}(x) = \frac{(x - x_{j+1})(x - x_j)^2}{(x_{j+1} - x_j)^2}.$$

The basis splines of the fourth order of approximation and the first level have the support: $\operatorname{supp} w_{j,0} = [x_{j-1}, x_{j+1}]$, $\operatorname{supp} w_{j,1} = [x_{j-1}, x_{j+1}]$. We call these splines the local splines of the Hermitian type of the fourth order of approximation of the first level.

We obtain these basis splines on the interval $[x_j, x_{j+1}]$ solving the system of identities:

$$\tilde{u}(x) = u(x), \qquad x = 1, x, x^2, x^3, x \in [x_j, x_{j+1}].$$

The approximation of the first derivative of a function with the splines of the fourth order of approximation is given with the relation:

$$\tilde{v}(x) = u(x_j)w'_{j,0}(x) + u(x_{j+1})w'_{j+1,0}(x) + u'(x_j)w'_{j,1}(x) + u'(x_{j+1})w'_{j+1,1}(x),\ x \in [x_j, x_{j+1}].$$

We can obtain the form of the basis splines on $[x_{j-1}, x_{j+1}]$ in the next way. We solve a similar system of equations on the interval $[x_{j-1}, x_j]$:

$$\bar{\bar{u}}(x) = u(x), \qquad x = 1, x, x^2, x^3, x \in [x_{j-1}, x_j].$$

Here we have the formula for the approximation the function $u(x)$, $x \in [x_{j-1}, x_j]$:

$$\bar{\bar{u}}(x) = u(x_j)w_{j,0}(x) + u(x_{j-1})w_{j-1,0}(x) + u'(x_j)w_{j,1}(x) + u'(x_{j-1})w_{j-1,1}(x),\ x \in [x_{j-1}, x_j].$$

Choosing the basis functions with common vertices, we obtain formulas for basis splines when $x \in [x_{j-1}, x_{j+1}]$.We have the next expressions:

$$w_{j,0}(x) = \begin{cases} \dfrac{(x-x_{j+1})^2}{(x_{j+1}-x_j)^2} + \dfrac{2(x-x_j)(x-x_{j+1})^2}{(x_{j+1}-x_j)^3}, x \in [x_j, x_{j+1}], \\ \dfrac{(x-x_{j-1})^2}{(x_{j-1}-x_j)^2} + \dfrac{2(x-x_j)(x-x_{j-1})^2}{(x_{j-1}-x_j)^3}, x \in [x_{j-1}, x_j], \end{cases}$$

$$w_{j,1}(x) = \begin{cases} \dfrac{(x-x_j)(x-x_{j+1})^2}{(x_{j+1}-x_j)^2}, x \in [x_j, x_{j+1}], \\ \dfrac{(x-x_j)(x-x_{j-1})^2}{(x_{j-1}-x_j)^2}, x \in [x_{j-1}, x_j]. \end{cases}$$

The plots of these basis splines when $j = 0, x_{j-1} = -1, x_{j+1} = 1$, are shown in Fig.7 and Fig. 8.

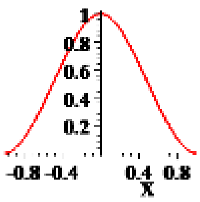

Fig. 7: The plot of the function $w_{j,0}(x)$

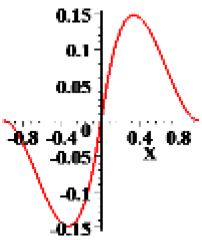

Fig. 8: The plot of the function $w_{j,1}(x)$

The plots of the first derivatives of these basis splines when $j = 0, x_{j-1} = -1, x_{j+1} = 1$, are shown in Fig. 9 and Fig. 10.

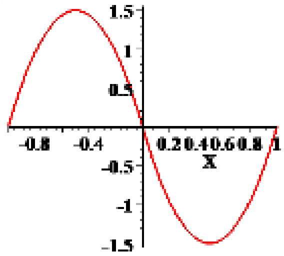

Fig. 9: The plot of the function $w'_{j,0}(x)$

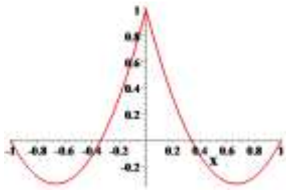

Fig. 10: The plot of the function $w'_{j,1}(x)$

Let $h = x_{j+1} - x_j$. The theorem of approximation of the function $u(x), x \in [x_j, x_{j+1}]$, is the following:

$$|u(x) - \tilde{u}(x)| \le Ch^4 \parallel u^{IV} \parallel_{[x_j, x_{j+1}]}, C = 0.0026.$$

The proof follows from the formula for the remainder term of the Hermite interpolation.

We have the equidistant set of nodes with the step $h = 0.1$ on the interval $[-1, 1]$. Now we can approximate the Runge function with the splines of the fourth order of approximation and the first level on the interval $[-1, 1]$. The plot of the error approximation of the Runge function $1/(1 + 25x^2)$ is given in Fig.11. The plot of the error approximation of the first derivative Runge function $1/(1 + 25x^2)$ is given in Fig.12.

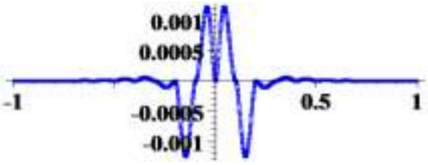

Fig. 11: The plot of the error approximation of the Runge function with the splines of the fourth order of approximation

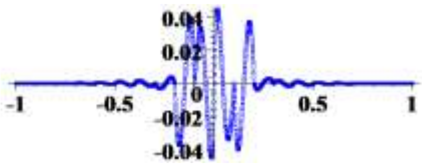

Fig. 12: The plot of the error approximation of the first derivative of the Runge function with the splines of the fourth order of approximation

Now we can use the approximate values of the first derivative. In the simplest way we can use the approximate expressions with the error of the second order. In this case, the derivative $u'(x_j)$ is replaced by the numerical differentiation formula:

$$u'_j = \frac{u_{j+1} - u_{j-1}}{2h}.$$

And, the derivative $u'(x_{j+1})$ is replaced by the numerical differentiation formula:

$$u'_{j+1} = \frac{3u_{j+1} - 4u_j + u_{j-1}}{2h}.$$

Now we have the expression:

$$\tilde{u}(x) \approx u_j w_{j,0}(x) + u_{j+1} w_{j+1,0}(x) + u'_j w_{j,1}(x) + u'_{j+1} w_{j+1,1}(x).$$

We apply the operator's regularization. According to Tikhonov's regularization, the new kernel has the form:

$$Q = \int_0^1 \sin(5 + x - t)\sin(5 + x - s)\,dx\,.$$

The new right side of the equation has the form

$$P = \int_0^1 \sin(5 + x - t) f(x) dx\,.$$

Now we solve the system of equations

$$\Omega + \sum_{r=1}^{n-1} W(x_j, r) - f(x_j) = 0\,,\quad j = 1, \dots n - 1.$$

where

$\Omega = 0.00000001(u_j + (u_{j+1} - 2u_j + u_{j-1})/h^2)$,

and

$$W(x,k) = u_k \int_{x_k}^{x_{k+1}} K(x,s)\, w_{k,0}(s) ds + u_{k+1} \int_{x_k}^{x_{k+1}} K(x,s)\, w_{k+1,0}(s) ds + u'_k \int_{x_k}^{x_{k+1}} K(x,s)\, w_{k,1}(s) ds + u'_{k+1} \int_{x_k}^{x_{k+1}} K(x,s)\, w_{k+1,1}(s) ds.$$

The graphs of the exact solution and its approximation after regularization are given in Fig. 13. The plot of the error of the solution after regularization is given in Fig. 14.

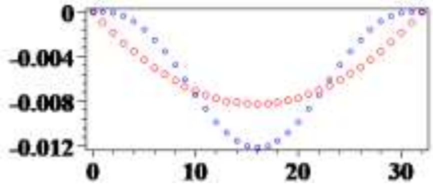

Fig. 13: The plot of the function $u(t)$ and the solution after regularization

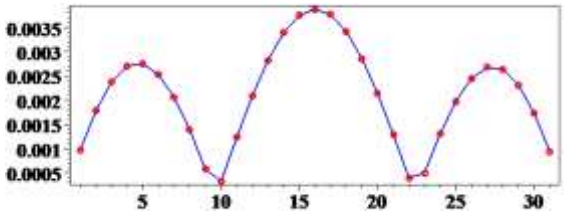

Fig. 14: The plot of the error of the solution after regularization

Now we also use the next stabilizer:

$$\Omega = 0.0000000001(x_j^2 u_j + 1000\left(\frac{1}{2} - x_j\right)^2 \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2})\,.$$

The graphs of the exact solution and its approximation after regularization are given in Fig. 15. The plot of the error of the solution after regularization is given in Fig. 16.

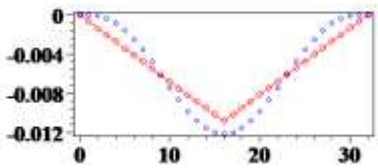

Fig. 15: The plot of the function $u(t)$ and the solution after regularization

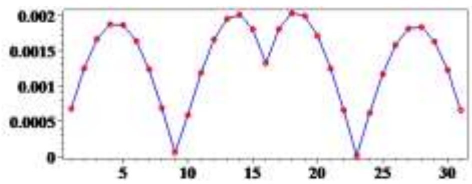

Fig. 16: The plot of the error of the solution after regularization

4.3 The Inversion of the Laplace Transform

Consider the problem of inverting the Laplace transform, i.e., finding the original $f(t)$ from its image $F(p), p > 0$, from the equation

$$\int_0^\infty exp(-pt) f(t) dt = F(p)\,.$$

Making a variable change $\exp(-t) = x, x \in [0,1], dt = -\frac{dx}{x}$, we obtain

$$\int_0^1 x^{p-1} f(-\ln x)dx = F(p)$$

Denote $\varphi(x) = f(-\ln x)$.
Thus, we get an integral equation of the first kind:

$$\int_0^1 x^{p-1} \varphi(x)dx = F(p).$$

This equation can be solved by the methods considered in this work.

5 Conclusion

The solution of integral equations of the first kind requires great care. Without a deep knowledge of the theory of operator regularization, it is recommended to reduce the solution of the integral equation to the solution of a system of linear algebraic equations. Next, regularize the solution of the system of equations using A.N.Tikhonov's theory. In the following works, we will consider the problem of numerically solving the Laplace transform by solving integral equations of the first kind in more detail.

References:

[1] A. V. Lebedeva, V.M. Ryabov, On the numerical solution of systems of linear algebraic equations with ill-conditioned matrices, *Vestnik St. Petersburg University, Mathematics*, 2019, Vol. 52, No. 4, 2019, pp. 388–393.

[2] V. M. Ryabov and others, Inverse and ill-posed problems, http://hdl.handle.net/11701/36400

[3] A. N. Tikhonov, V.Ya.Arsenin, *Solutions of Ill-Posed Problems*, Winston, 1977. 258 p.

[4] A. B. Bakushinsky, A.V. Goncharsky, *Incorrect tasks. Numerical methods and applications*. Moscow University Press. 1989.

[5] I. G. Burova, V. M. Ryabov, On the solution of Fredholm integral equations of the first kind, *WSEAS Transactions on Mathematics*, Vol.19, 2020, pp. 699-708.

[6] I. G. Burova, M.A. Kalnitskaia, A.V. Malevich, On the numerical solution of system of linear algebraic equations with positive definite symmetric ill-posed matrices, *WSEAS Transactions on Mathematics*, Vol.17, 2018, pp. 13-19.

[7] Abdul-Majid Wazwaz, The regularization method for Fredholm integral equations of the first kind, *Computers and Mathematics with Applications*, Vol. 61, 2011, pp. 2981–2986.

[8] D. L. Phillips, A technique for the numerical solution of certain integrals equations of the first kind, *J. Ass. Comput. Mach.*, Vol. 9, 1962, pp. 84–96.

[9] E. K. Kulikov, A.A. Makarov., A method for solving the Fredholm integral equation of the first kind, *Journal of Mathematical Sciences*, 2023, Vol. 272, No. 4, May, 2023, pp.558-565, https://doi.org/10.1007/s10958-023-06449-3

[10] I. G. Burova, G.O. Alcybeev, Application of Splines of the second order approximation to Volterra integral equations of the second kind. Applications in Systems, Theory and Dynamical Systems, *International Journal of Circuits, Systems and Signal Processing*, Vol. 15, 2021, pp. 63-71.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

I. G. Burova was responsible for the theory.
V. M. Ryabov was responsible for the theory.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The authors are gratefully indebted to St. Petersburg University for their financial support in the preparation of this paper (Pure ID 94029567, ID 104210003, 104625746), as well as to a resource center for providing the package Maple.

Conflict of Interest

The authors have no conflict of interest to declare.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US