Asymptotic Properties of Zero Dynamics of Multivariable Sampled-Data models with Time Delay

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Abstract: It is well-known that the existence of unstable zero dynamics is recognized as a major barrier in many control systems, and deeply limit the achievable control performance when controlling a system. When a continuous-time multivariable system with at least one of relative degrees greater than two is discretized in the case of the time delay and a zero-order hold (ZOH) assumption, at least one zero dynamics of the resulting sampled-data model is obviously unstable for sufficiently small sampling periods. Thus, attention is here focused on continuous-time multivariable systems with time delay, and some of the relative degrees are one and the rest are two. This paper investigates the asymptotic properties of zero dynamics for the sampled-data models corresponding to the continuous-time multivariable systems mentioned above, and further derives an approximate expression of the zero dynamics in the form of a power series expansion up to the first order term of sampling period. Meanwhile, the condition for obtaining stable zero dynamics for sufficiently small sampling periods is also presented. The ideas presented here generalize well-known results from the delay-free multivariable control system to the time-delay case.

Key–Words: Discrete-time model; Zero dynamics; Zero-order hold; Time discretization; Taylor method; Time delay; Asymptotic properties; Stability conditions.

1 Introduction

Zero dynamics play an important role in control system design. It is well-known that unstable zero dynamics limit the control performance that can be achieved when controlling a system. Further, many techniques for design of adaptive control systems are based on the cancellation of the zero dynamics [1, 2]. Such methods will not work when a plant has unstable zero dynamics [3, 4]. When a continuous-time plant is discretized using a sampler and a zero-order hold (ZOH), poles p_i are transformed as $p_i \to exp(p_iT)$, where T is a sampling period. The transformations of zero dynamics, however, are much more complicated and the stability of zero dynamics is not necessarily preserved in the discretization process except in special cases [5, 6]. In the past years, for this reason the asymptotic properties and stability conditions of zero dynamics are studied by many researchers [3,4,7–21]. Because of its simplicity, most digital control systems use a ZOH and the studies on properties of zero dynamics of the discretized system are mainly limited in the ZOH case [3, 7, 8, 10–14, 18, 21].

Roughly speaking, the zero dynamics of a discrete-time system are classified into two categories [18]. The former ones have counterpart in the underlying continuous-time system and they are called intrinsic zero dynamics. The latter ones are generated in the sampling process and they are called sampling zero dynamics. The intrinsic zero dynamics are located inside (outside) the unit circle for sufficiently small sampling periods when the corresponding continuous-time zero dynamics lie strictly in the left (right) half plane [22]. For sampling zero dynamics of single-input single-output (SISO) systems, at least one of the sampling zero dynamics lies strictly outside the unit circle if the relative degree of a continuous-time transfer function is greater than two [3]. This fact indicates that even though all the zero dynamics of such a continuous-time system are stable, the corresponding discrete-time model has at least one unstable sampling zero dynamics. The same property also holds for multivariable systems, i.e., when at least one of the relative degrees of a decouplable multivariable system is greater than two, the sampling zero dynamics of the corresponding discrete-time model stay outside the unit circle. Therefore, attention is here focused on continuous-time multivariable systems with all relative degrees less than three from the viewpoint of stability of zero dynamics of the corresponding sampled-data plants.

However, these well-known results proposed [3, 4, 7–18, 21] have presented the asymptotic characterization of zero dynamics for the discretized systems without time delay. In practice, time delay of controlled systems inherently exists in many mechanical engineering applications and therefore must be integrated into system models. It is inevitable that the influence of time delay on digital control system must be considered because it also occurs frequently in information transmission between elements or systems, transport of controls and sensors, data computation, etc. For the time-delay system, perhaps the first attempt to study discrete system zero dynamics with time delay was given by Åström and coworkers [3], who have considered a continuous-time system with time delay and shown that it can locate the zero dynamics of the discretized plant inside the unit circle in some cases, through examples only. In the very motivating work by Hara [23], the asymptotic behavior of the discretized system zero dynamics with time delay have been presented for fast sampling rate as the original continuous-time plant is discretized with ZOH. Further results on the asymptotic behaviors and stability conditions of the time-delay plants have been reported [24–28]. So far, the properties of zero dynamics of the discrete-time systems with time delay are discussed mainly on SISO systems. Hence, it is important to investigate the asymptotic properties of zero dynamics in the sampled-data models corresponding to the continuous-time multivariable plants with time delay for the digital control system design.

The purpose of this paper is to deal with the asymptotic properties of the discrete zero dynamics of sampled-data models corresponding to continuous-time multivariable plants with time delay when ZOH is used for signal reconstruction. It is obvious that the asymptotic behavior of the zero dynamics is an interesting issue because it is stable for sufficiently small sampling periods if it approaches the unit circle from inside as the sampling period tends to zero. Also, when on the basis of the normal form representation of the multivariable continuous-time systems is discretized by Taylor method, the approximate asymptotic expressions of zero dynamics for the corresponding discrete-time models are explicitly given as power series expansions with respect to a sampling period up to the first order term. A condition that assures stability of all zero dynamics for the discretized system is also presented for sufficiently small sampling periods. An insightful interpretation of the results obtained can be made in terms of the explicit characterization of discrete zero dynamics with multivariable time-delay systems. Moreover, the current paper extends the well-known ideas with time delay from the SISO case to multivariable system.

The layout of the paper is structured as follows. In Section 2, we describe the results that are multivariable system description, and the ZOH assumption. Section 3 presents the main result of this paper, namely, the approximate asymptotic expressions of zero dynamics for the multivariable sampled-data models in the case of the time delay. A numerical simulation is represented in Section 4. Finally, conclusions are presented in Section 5.

2 Preliminaries

Consider square time-invariant multivariable system expressed by

$$S_C: \quad \begin{cases} \dot{\overline{x}} = A_0\overline{x} + B_0 u(t-\tau) \\ y = C_0\overline{x}, \quad C_0 = [c_1, c_2, \cdots, c_m]^T \end{cases} \tag{1}$$

with a state vector $\overline{x} \in R^n$, an input vector $u \in R^m$, an output vector $y \in R^m$ and time delay τ . It is assumed in this paper that the system S_C is minimal and decouplable by static state feedback, and that the transfer function matrix $G_s = C_0(sI - A_0)^{-1}B_0$ has full rank.

Zero dynamics of multivariable systems are defined in several ways. Multivariable zero dynamics can be termed system zero dynamics, invariant zero dynamics, transmission zero dynamics and so on. In spite of many differences and ambiguities, all those definitions of multivariable zero dynamics refer or claim to be extensions to those for SISO system. Then, the definition of invariant zero dynamics, transmission zero dynamics and system zero dynamics for the system all coincide and some of the properties of zero dynamics in SISO systems are inherited in the discretization process [29]. Thus, these zero dynamics are simply called the zero dynamics throughout this article. We denote the relative degree corresponding to the ith output y_i by r_i, i.e., $c_i^T A_0^{r_i-1} B_0 \neq \mathbf{0^T}$ and $c_i^T A_0^j B_0 = \mathbf{0^T}, j = 0, 1, \cdots, r_i - 2$. The decoupling matrix D is then represented as

$$D = \begin{bmatrix} c_1^T A_0^{r_1-1} B_0 \\ c_2^T A_0^{r_2-1} B_0 \\ \vdots \\ c_m^T A_0^{r_m-1} B_0 \end{bmatrix} \quad (2)$$

and nonsingular from the assumption.

We are interested in the sampled-data model for the linear systems when the input is a piecewise constant signal generated by a ZOH. Thus, for a sampling period T,

$$u(t) = u(kT) \equiv u_k, \quad kT \leq t < kT + T \quad (3)$$

In the case of time delay and ZOH, the sampled-data models consist of the sampled counterpart of the continuous-time zero dynamics and the additional zero dynamics produced by the sampling process. The former ones have counterpart in the underlying continuous-time system and they are also called intrinsic zero dynamics. The latter ones are newly generated in the sampling process and they are also called sampling zero dynamics. In particular, for the SISO systems, the pulse transfer function with time delay $H_\Delta(z)$ approaches a scalar multiple of the pulse transfer function corresponding to $\frac{1}{s^m}$ when the sampling period tends to zero [23]. Of particular contribution is that the sampling zero dynamics of $H_\Delta(z)$ contain unstable ones for any $\Delta \in [0, 1]$ if $m > 2$.

The pulse transfer function $H_\Delta(z)$ for $G(s) = \frac{1}{s^m}$ is given by

$$H_\Delta(z) = \frac{T^m B_m(z, \Delta)}{m!(z-1)^m z} \quad (4)$$

where

$$\begin{aligned} B_m(z, \Delta) &= b_0^m(\Delta) z^m + b_1^m(\Delta) z^{m-1} + \cdots + b_m^m(\Delta) \\ b_k^m(\Delta) &= \sum_{\ell=0}^{m-k} (-1)^{m-k-\ell} (\ell + \Delta)^m \begin{pmatrix} m+1 \\ m-k-\ell \end{pmatrix} \\ & k = 0, 1, \cdots, m \end{aligned}$$

and $\begin{pmatrix} m \\ k \end{pmatrix}$ indicates $\frac{m!}{(m-k)!k!}$. The polynomials $B_m(z, \Delta)$ are listed below for $m = 1, 2, 3$.

$$\begin{aligned} B_1(z, \Delta) &= (1 - \Delta) z + \Delta && (5) \\ B_2(z, \Delta) &= (1 - 2\Delta + \Delta^2) z^2 + (1 + 2\Delta \\ & \quad -2\Delta^2) z + \Delta^2 && (6) \\ B_3(z, \Delta) &= (1 - 3\Delta + 3\Delta^2 - \Delta^3) z^3 + (4 - 6\Delta^2 \\ & \quad +3\Delta^3) z^2 + (1 - 3\Delta + 3\Delta^2 - 3\Delta^3) z \\ & \quad +\Delta^3 && (7) \end{aligned}$$

3 Main Results

This paper is related to zero dynamics of the discrete-time model for sufficiently small sampling periods. Through above mentioned analyzing, attention then is restricted to the case when all the relative degrees of continuous-time system with time delay (1) are less than three from the viewpoint of stability of the zero dynamics of the corresponding discrete-time model. Without loss of generality, we mainly consider here that some of the relative degrees are one and the rest are two (see also Remark 1).

Let S_D be a sampled-data system of a series connection of a ZOH, the continuous-time system S_C and a sampler with a sampling period T. In the remainder of this section, this manuscript derives the approximate model of the discrete-time system S_D on the basis of the normal form representation of the continuous-time system S_C, and gives the approximate asymptotic expressions of the zero dynamics of S_D as power series expansions with respect to a sampling period T up to the first order term.

Remark 1 *When all the relative degrees of the multivariable continuous-time systems with time delay are less than three, the stability of all zero dynamics of corresponding discrete-time models could remain stable. At this point, three cases, i.e., 1) some of the relative degrees are one and the rest are two, 2) all the relative degrees are one, and 3) all the relative degrees are two, should be considered. However, in practice, all the relative degrees of many intelligent mechanical systems, irrespectively of whether they involve time delay or not, in the practical field are mainly consist of one and two, both linear and nonlinear plants.*

It is assumed that the relative degrees $r_1, \cdots, r_p, 0 < p < m$ for the outputs $y_1, \cdots, y_p$ are one and the rest $r_{p+1}, \cdots, r_m$ for the outputs $y_{p+1}, \cdots, y_m$ are two; i.e., $r_1 = \cdots = r_p = 1$, $r_{p+1} = \cdots = r_m = 2$, without loss of generality.

Then, the system matrix C_0 and the decoupling matrix have the following form.

$$C_0 = \begin{bmatrix} C_{10} \\ C_{20} \end{bmatrix}, D = \begin{bmatrix} C_{10}B_0 \\ C_{20}A_0B_0 \end{bmatrix} \neq 0, \\ C_{20}B_0 = O \tag{8}$$

where C_{10} and C_{20} are $p\times n$ and $(m-p)\times n$ matrices, respectively.

First, a sampled-data model is derived from the normal form of S_C. The normal form of (1) with $r_1 = \cdots = r_p = 1$ and $r_{p+1} = \cdots = r_m = 2$ is expressed by

$$\begin{cases} \dot{x} = Ax + Bu(t-\tau) \\ y = Cx \end{cases} \tag{9}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ O & O & I_{m-p} & O \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ O \\ B_2 \\ O \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} I_p & O & O & O \\ O & I_{m-p} & O & O \end{bmatrix}$$

where

$$x = \left[\mathbf{y_1^T}, \mathbf{y_2^T}, \mathbf{y_3^T}, \upsilon^{\mathbf{T}}\right]^T, \mathbf{y_1} = [y_1, \cdots, y_p]^T,$$
$$\mathbf{y_2} = [y_{p+1}, \cdots, y_m]^T, \mathbf{y_3} = \dot{\mathbf{y}}_\mathbf{2} = [\dot{y}_{p+1}, \cdots, \dot{y}_m]^T,$$
$$\upsilon = [x_{2m-p+1}, \cdots, x_n]^T, x = S_b\overline{x},$$
$$S_b = \begin{bmatrix} C_{10}^T & C_{20}^T & (C_{20}A_0)^T & W^T \end{bmatrix}^T,$$
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \end{bmatrix} = C_{10}A_0S_b^{-1},$$
$$\begin{bmatrix} A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix} = C_{20}A_0^2S_b^{-1},$$
$$A = S_bA_0S_b^{-1}, B = S_bB_0, C = C_0S_b^{-1}$$

and matrices $W, A_{31}, A_{32}, A_{33}, A_{34}$ are necessarily obtained by $WB_0 = 0$ and $WA_0 = A_{31}C_{10} + A_{32}C_{20} + A_{33}C_{20}A_0 + A_{34}W$. Further, the roots of the equation $|sI - A_{34}| = 0$ are zero dynamics of S_C and the following relations hold.

$$B_1 = C_1B = C_{10}B_0, C_2B = C_{20}B = O, \\ B_2 = C_2AB = C_{20}A_0B_0 \tag{10}$$

The main result of this paper is given by the following Theorem 2, which mainly describes the approximate asymptotic expressions of zero dynamics of the discrete-time model for the corresponding multivariable continuous-time system with time delay (1). More importantly, the explicit characterization of these significant results is also presented as power series expansions with respect to a sampling period up to the first order term.

Theorem 2 *The zero dynamics of a discrete-time model corresponding to a continuous-time system with time delay (1), with $r_1 = \cdots = r_p = 1$ and $r_{p+1} = \cdots = r_m = 2$, are given approximately for sufficiently small sampling periods T by the roots of*

$$|F_I(z)F_S(z)| = 0 \tag{11}$$

where

$$\begin{aligned} F_I(z) &= (z-1)I - TA_{34} \\ F_S(z) &= (z+1)I + C_2ABD_R \\ &\quad + \left(2A_{23} + \frac{1}{3}C_{20}A_0^2B_0D_R\right)T \end{aligned}$$

where D_L and D_R are a $m \times p$ matrix and a $m \times (m-p)$ matrix, respectively, which are defined by $\begin{bmatrix} D_L & D_R \end{bmatrix} = D^{-1}$.

Proof: When a ZOH is used, and notice that the relations $\dot{u} = \ddot{u} = \cdots = 0$ are taken into account, (9) leads to the derivatives of the output vector y and the vector υ represented by x and $\overline{u}(= u(t-\tau))$ as

$$\dot{\mathbf{y}}_\mathbf{1} = C_1Ax + C_1B\overline{u} \tag{12}$$
$$\ddot{\mathbf{y}}_\mathbf{1} = C_1A^2x + C_1AB\overline{u} \tag{13}$$
$$\vdots$$
$$\dot{\mathbf{y}}_\mathbf{2} = C_2Ax + C_2B\overline{u} \tag{14}$$
$$\ddot{\mathbf{y}}_\mathbf{2} = C_2A^2x + C_2AB\overline{u} \tag{15}$$
$$\mathbf{y_2^{(3)}} = C_2A^3x + C_2A^2B\overline{u} \tag{16}$$
$$\vdots$$
$$\dot{\upsilon} = \begin{bmatrix} A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} x \tag{17}$$
$$\vdots$$

In addition, a sampled-data state vector is defined by $x_k^T = [\mathbf{y_{1,k}^T}, \mathbf{y_{2,k}^T}, \mathbf{y_{3,k}^T}, \upsilon_\mathbf{k}^\mathbf{T}]^\mathbf{T}$, where the subscript k denotes the sampling instant $t = kT, k = 0, 1, 2, \cdots$.

As it can be easily inferred from (9), the input variable $u(t)$ remains constant within the two subintervals: $[kT, kT+\tau)$, $[kT+\tau, kT+T)$. One readily obtains

$$\begin{cases} y_{j,k+\tau} = \sum\limits_{i=0}^{4} \frac{T^i}{i!} y_{j,k}^{(i)} + O(T^5) \\ \dot{y}_{j,k+\tau} = \sum\limits_{i=0}^{3} \frac{T^i}{i!} y_{j,k}^{(i+1)} + O(T^5) \\ \ddot{y}_{j,k+\tau} = \sum\limits_{i=0}^{2} \frac{T^i}{i!} y_{j,k}^{(i+2)} + O(T^5) \\ y_{j,k+\tau}^{(3)} = \sum\limits_{i=0}^{1} \frac{T^i}{i!} y_{j,k}^{(i+3)} + O(T^5), \quad j = 1, 2 \end{cases} \tag{18}$$

and

$$\begin{cases} y_{j,k+1} = \sum\limits_{i=0}^{4} \frac{T^i}{i!} y_{j,k+\tau}^{(i)} + O(T^5) \\ \dot{y}_{j,k+1} = \sum\limits_{i=0}^{3} \frac{T^i}{i!} y_{j,k+\tau}^{(i+1)} + O(T^5) \\ \ddot{y}_{j,k+1} = \sum\limits_{i=0}^{2} \frac{T^i}{i!} y_{j,k+\tau}^{(i+2)} + O(T^5) \\ y_{j,k+1}^{(3)} = \sum\limits_{i=0}^{1} \frac{T^i}{i!} y_{j,k+\tau}^{(i+3)} + O(T^5), \quad j = 1,2 \end{cases} \tag{19}$$

from which representing the discrete-time model of the original continuous-time system (1) with time delay τ, namely, the value of the output vector at $(k+1)T$ is a linear combination of the output evaluated at kT and the past value of the input variable u. In addition, note that

$$\upsilon_{k+1} = \sum_{i=0}^{\infty} \frac{T^i}{i!} \upsilon_k^i \tag{20}$$

In the following, the approximate model of the above sampled-data time-delay system is considered by neglecting the higher order terms, and the approximate expression of the zero dynamics is also calculated. Here, the following approximate expressions of multivariable discrete-time models in the case of a time delay are treated.

$$\begin{aligned} y_{1,k+1} =& y_{1,k} + 2T(C_1Ax_k + C_1B\overline{u}_k) \\ &+ 2T^2(C_1A^2x_k + C_1AB\overline{u}_k) + O(T^3) \end{aligned} \tag{21}$$

$$\begin{aligned} y_{2,k+1} =& y_{2,k} + 2T\dot{y}_{2,k} + 2T^2(C_2A^2x_k + C_2AB\overline{u}_k) \\ &+ \frac{4T^3}{3}(C_2A^3x_k + C_2A^2B\overline{u}_k) + O(T^4) \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{y}_{2,k+1} =& \dot{y}_{2,k} + 2T(C_2A^2x_k + C_2AB\overline{u}_k) \\ &+ 2T^2(C_2A^3x_k + C_2A^2B\overline{u}_k) + O(T^3) \end{aligned} \tag{23}$$

$$\upsilon_{k+1} = \upsilon_k + T\left[A_{31}\ A_{32}\ A_{33}\ A_{34}\right]x_k + O(T^2) \tag{24}$$

Setting $y_{1,k} = y_{1,k+1} = y_{2,k} = y_{2,k+1} = 0$ yields the zero dynamics of the sampled-data system such that

$$M\left[\ Z[\dot{y}_{2,k}] \quad Z[\overline{u}_k] \quad Z[\upsilon_k]\ \right]^T = 0 \tag{25}$$

where $Z[.]$ denotes the z-transform and

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \\ M_{41} & M_{42} & M_{43} \end{bmatrix} \tag{26}$$

where all the elements of the matrix M are approximately represented as

$$\begin{aligned} &M_{11} = (-z+1)I + 2TA_{23}, M_{12} = 2TC_2AB, \\ &\quad +2T^2C_2A^2B, M_{13} = 2TA_{24}, M_{21} = 2A_{13}, \\ &M_{22} = 2C_1B + 2TC_1AB, M_{23} = 2A_{14}, \\ &M_{31} = I + 2TA_{23}, M_{32} = 2TC_2AB + \frac{4T^2}{3}C_2A^2B, \\ &M_{033} = 2TA_{24}, M_{41} = TA_{33}, M_{42} = O, \\ &M_{43} = (-z+1)I + TA_{34} \end{aligned}$$

Hence, the zero dynamics of multivariable sampled-data system with time delay is obtained by $det(M) = 0$.

From the relationship

$$\overline{M} = \begin{bmatrix} I_{m-p} & O & -2I_{m-p} & O \\ O & I_m & O & O \\ O & O & I_{m-p} & O \\ O & O & O & I_{n-2m+p} \end{bmatrix} M \tag{27}$$

it is obvious that the condition $det(\overline{M}) = 0$ is equivalent to $det(M) = 0$. The matrix multiplication (27) gives

$$\overline{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{28}$$

where

$$M_{11} \approx \begin{bmatrix} -(z+1)I - 2TA_{23} & 2TC_2AB + \frac{2T^2}{3}C_2A^2B \\ 2A_{13} & 2C_1B + 2TC_1AB \\ I + 2TA_{23} & 2TC_2AB + \frac{4T^2}{3}C_2A^2B \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} -2TA_{24} & 2A_{14} & 2TA_{24} \end{bmatrix}^T$$

$$M_{21} = \begin{bmatrix} TA_{33} & O \end{bmatrix} \quad M_{22} = (-z+1)I + TA_{34}$$

Simple calculation yields

$$|\overline{M}| = |M_{22}||M_{11} - M_{12}M_{22}^{-1}M_{21}| \tag{29}$$

and

$$\begin{aligned} M_{12}M_{22}^{-1}M_{21} \approx& \begin{bmatrix} -2TA_{24} \\ 2A_{14} \\ 2TA_{24} \end{bmatrix} \frac{1}{(1-z)} \\ &\times\left(I - \frac{1}{(1-z)}TA_{34}\right)\left[TA_{33} \quad O\right] \\ =& \begin{bmatrix} O(T^2) & \\ O(T) & O \\ O(T^2) & \end{bmatrix} \end{aligned} \tag{30}$$

Note here that the order of each block matrix of the left hand side in M_{11} is lower than that in $M_{12}M_{22}^{-1}M_{21}$, then we have

$$|M_{11} - M_{12}M_{22}^{-1}M_{21}| \approx |M_{11}|$$

Next, determinant calculation leads to

$$\begin{aligned} |M_{11}| =& \begin{vmatrix} 2C_1B + 2TC_1AB \\ 2TC_2AB + \frac{4T^2}{3}C_2A^2B \end{vmatrix} |[-(z+1)I \\ &-2TA_{23}] - \left(2TC_2AB + \frac{2T^2}{3}C_2A^2B\right) \end{aligned}$$

$$\times\left[\begin{array}{c} 2C_1B+2TC_1AB \\ 2TC_2AB+\frac{4T^2}{3}C_2A^2B \end{array}\right]^{-1}$$
$$\times\left[\begin{array}{c} 2A_{13} \\ I+2TA_{23} \end{array}\right]\Bigg|$$

from which we have

$$\left[\begin{array}{c} 2C_1B+2TC_1AB \\ 2TC_2AB+\frac{4T^2}{3}C_2A^2B \end{array}\right]^{-1}\left[\begin{array}{c} 2A_{13} \\ I+2TA_{23} \end{array}\right]$$
$$=\left\{\left[\begin{array}{cc} I & O \\ O & 2TI \end{array}\right]D\left(I+D^{-1}\left[\begin{array}{c} 2C_1AB \\ \frac{2}{3}C_2A^2B \end{array}\right]T\right)\right\}^{-1}$$
$$\times\left[\begin{array}{c} 2A_{13} \\ I+2TA_{23} \end{array}\right]$$
$$\approx\left(I-D^{-1}\left[\begin{array}{c} 2C_1AB \\ \frac{2}{3}C_2A^2B \end{array}\right]T\right)D^{-1}\left[\begin{array}{cc} I & O \\ O & \frac{I}{2T} \end{array}\right]$$
$$\times\left[\begin{array}{c} 2A_{13} \\ I+2TA_{23} \end{array}\right]$$
$$\approx\left(I-D^{-1}\left[\begin{array}{c} 2C_1AB \\ \frac{2}{3}C_2A^2B \end{array}\right]T\right)D^{-1}\left[\begin{array}{c} 2A_{13} \\ \frac{I}{2T} \end{array}\right]$$
$$\approx\left[\begin{array}{cc} D_L & D_R \end{array}\right]\left[\begin{array}{c} 2A_{13} \\ \frac{I}{2T} \end{array}\right]$$
$$\approx\frac{1}{2T}D_R$$

Further

$$(z+1)I+2TA_{23}+\left(2TC_2AB+\frac{2T^2}{3}C_2A^2B\right)\frac{D_R}{2T}$$
$$\approx(z+1)I+C_2ABD_R+\left(2A_{23}+\frac{1}{3}C_2A^2BD_R\right)T$$

As a result, it holds with a constant K_0 that

$$|M|\approx K_0\,|(z-1)I-TA_{34}|\,\Big|(z+1)I+C_2ABD_R + \left(2A_{23}+\frac{1}{3}C_2A^2BD_R\right)T\Big|$$
$$=K_0|F_I(z)F_S(z)| \tag{31}$$

The proof is completed. □

Remark 3 *It is found from Theorem 2 that an approximate value of the sampling zero dynamics for multivariable time-delay system is expressed as*

$$\left|(z+1)I+C_2ABD_R+\left(2A_{23}+\frac{1}{3}C_2A^2BD_R\right)T\right| = 0 \tag{32}$$

and the approximate values of the intrinsic zero dynamics with multivariable time-delay plant are derived from

$$|(z-1)I-TA_{34}|=0 \tag{33}$$

Remark 4 *The way of constructing polynomial matrix, which has been used by many other authors [16, 30–33], serves as the main tool in the zero dynamics of multivariable sampled-data models. In this method, the properties of the zero dynamics for multivariable systems are mainly characterized by the degrees of the infinite elementary divisors [34] of a system matrix. Moreover, the degrees of the infinite elementary divisors is closely related to the relative degrees mentioned above which also determine the behavior of the discrete system zero dynamics [35]. However, this approach, i.e., constructing polynomial matrix, could not be applied to the case that at least one of the degrees of the infinite elementary divisors or the relative degrees is very high, or time-delay case. On the basis of the normal form representation used in the proof of Theorem 2 is a very effective tool for the zero dynamics of discrete multivariable system with relative degrees being high or time-delay multivariable systems. Roughly speaking, one may first establish the normal form representation of the continuous-time multivariable systems with time delay, then propose the corresponding sampled-data models using the Taylor series expression and Z-transform, and consequently investigate the zero dynamics of resulting multivariable discrete-time plants in the case of time delay. This is two kinds of completely different methods, which could commonly use to the multivariable discrete-time systems and their zero dynamics.*

The following corollary shows stability conditions of all the zero dynamics, including intrinsic zero dynamics and sampling zero dynamics, of discrete-time models for continuous-time multivariable systems with time delay.

Corollary 5 *Assume that a continuous-time multivariable system with time delay has relative degrees being one and two, and the matrix A_{23} and A_{34} are nonsingular and have no pure imaginary roots. If all the eigenvalues of the matrix $C_{20}A_0B_0D_R$, A_{23} and $C_{20}A_0^2B_0D_R$ have negative real parts, then the sampling zero dynamics are stable for sufficiently small sampling periods T. Similarly, when all the eigenvalues of the matrix A_{34} have negative real parts, the intrinsic zero dynamics are located strictly inside the unit circle when the sampling periods tend to zero.*

Remark 6 *When at least one of the eigenvalues of the matrix A_{34} ($C_{20}A_0B_0D_R$ or A_{23} or $C_{20}A_0^2B_0D_R$) has a positive real part, the intrinsic zero dynamics (sampling zero dynamics) stay strictly outside the unit circle for sufficiently small sampling periods. In particular, when these matrix are singular, or at least* *one pair of the eigenvalues of these matrix are a pure imaginary number and the rest are stable, we need to derive the more accuracy asymptotic expression than that of Theorem 2 in order to explore the stability of zero dynamics of discrete-time models for the continuous-time multivariable systems.*

Remark 7 *It is trivial to obtain the Corollary 5 from the Theorem 2 because it is easy found from the equation (31) that intrinsic zero dynamics and sampling zero dynamics for a sufficiently small sampling period T are respectively expressed approximately by the eigenvalues of the matrix A_{34}, and the linear combination of $C_{20}A_0B_0D_R$, A_{23} and $C_{20}A_0^2B_0D_R$.*

Remark 8 *If the continuous-time input $u(t)$ is generated by a different hold device, for example, a first-order hold (FOH), this information can be used to include more complicated steps in the proof of Theorem 2. This, of course, would lead us to a different approximate discrete-time model and asymptotic expression of corresponding zero dynamics in Theorem 2, with different stability conditions of discretized zero dynamics in Corollary 5.*

Remark 9 *It is difficult to derive higher approximate expression in the previous case that some of the relative degrees of S_C are one and the rest are two. However, the decent results about sampled-data models could be presented in the cases that all the relative degrees are one or two, and further obtain the corresponding approximate asymptotic properties and stability conditions.*

4 NUMERICAL SIMULATION

This section presents an interesting example to show the asymptotic properties of zero dynamics of a sampled-data model for a multivariable continuous-time system in the case of a ZOH and time delay when some of the relative degrees are one and the rest are two. Furthermore, in this case, the conditions that assure stability of all zero dynamics for sufficiently small sampling periods are also derived. All of the corresponding zero dynamics are calculated by use of MATLAB, and in the simulation figures, the solid line and dotted line indicate the approximate values and exact values, respectively.

Consider the following abstracted longitudinal control design with time delay example for a open-loop CH-47 tandem rotor helicopter model [36]. This multivariable continuous-time system S_C has a zero dynamics at -0.018. A nominal model for the dynamics relating these variables at certain knot airspeed and the corresponding state space matrices are:

$$\begin{cases} \dot{x} = A_0x + B_0u(t-\tau) \\ y = C_0x \end{cases} \tag{34}$$

where

$$A_0 = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & 1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ -0.35 & 0.009 \\ 0 & 0 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is obvious to see that the above continuous-time system has relative degrees being one and two. More specifically, the relative degrees of this system are given by $r_1 = 1$ and $r_2 = 2$ for simplicity description, and first the corresponding decoupling matrix is given by

$$D = \begin{bmatrix} C_{10}B_0 \\ C_{20}A_0B_0 \end{bmatrix} = \begin{bmatrix} 0.36 & -8.6 \\ -0.35 & 0.009 \end{bmatrix}$$

Next, it follows from Theorem 2 that the sampling zero dynamics are located strictly inside the unit circle for sufficiently small sampling periods since $C_{20}A_0B_0D_R = -0.998 < 0$, $|A_{23}| = 0$ and $C_{20}A_0^2B_0D_R = -1.597 < 0$ have negative real parts, and the intrinsic zero dynamics are also stable owing to the continuous-time zero dynamics at -0.018. Therefore, the intrinsic zero dynamics z_1 and the sampling zero dynamics z_2 are expressed approximately as respectively

$$z_1 \approx 1 - 0.018T, \quad z_2 \approx -1 + W \tag{35}$$

where $W = 0.998 + \frac{1.597}{3}T$. The approximate values and the exact values of the multivariable discrete-time zero dynamics with time delay are shown in Table I and Figure I-II, where Figure I and Figure II are respectively represented the values of sampling zero dynamics and intrinsic zero dynamics.

Table I Zero dynamics of the multivariable discrete-time system with time delay and relative degrees being one and two

T	Approximate values	Exact values
0.01	0.999820000, 0.003320000	0.999820115, 0.003310145
0.02	0.999640000, 0.008640000	0.999640263, 0.008638424
0.05	0.999100000, 0.024600000	0.999100901, 0.024304410
0.1	0.998200000, 0.051200000	0.998202612, 0.049908512
0.2	0.996400000, 0.104400000	0.996408465, 0.092006592

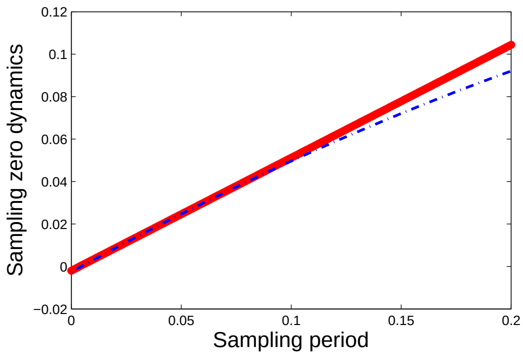

Figure I Sampling zero dynamics of multivariable sampled-data model with time delay and relative degrees being one and two

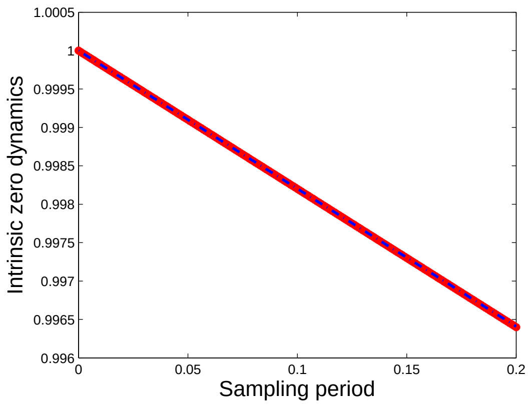

Figure II Intrinsic zero dynamics of multivariable sampled-data model with time delay and relative degrees being one and two

5 Conclusion

This paper analyzes the asymptotic properties of zero dynamics for the discrete-time system when the continuous-time multivariable system with time delay is explicitly discretized in the case of a ZOH, and all the relative degrees are one and two. We also give an approximate asymptotic expression of the zero dynamics for the discrete systems with time delay as power series expansions up to first order term with respect to sampling periods. The stability of discrete zero dynamics is further discussed for sufficiently small sampling periods in this case and a corresponding stability condition is also derived. Moreover, an insightful interpretation is given in terms of an explicit characterization of the sampling zero dynamics of the obtained model in the case of time delay. Our contributions in this paper are a further extension of well-known results from the delay-free cases to the time-delay systems. The idea of this paper is believed to give important insights which are relevant to many aspects of linear multivariable discrete-time mechanical control theory in the case of the time-delay.

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