Robust Recursive Least-Squares Finite Impulse Response Filter in Linear Continuous-Time Stochastic Systems with Uncertainties

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Abstract: - The current research designs an original robust recursive least-squares (RLS) finite impulse response (FIR) filter for linear continuous-time systems with uncertainties in both the system and observation matrices. These uncertainties in the state-space model generate the degraded signal and observed value. The robust RLS FIR filter does not account for the norm-bounded uncertainties in the system and observation matrices. This study uses an observable companion form to represent the degraded signal state-space model. The system and observation matrices are estimated based on the author's previous computational methods. The robust RLS FIR filtering problem aims to minimize the mean-square errors in FIR filtering for the system state. The robust FIR filtering estimate is formulated as an integral transformation of the degraded observations using an impulse response function. Section 3 obtains the integral equation satisfied by the optimal impulse response function. Theorem 1 presents the robust RLS FIR filtering algorithms for the signal and the system state. This integral equation derives the robust RLS-FIR filtering algorithms. Numerical simulation examples show the validity of the proposed robust RLS FIR filter.

Key-Words: - Robust RLS FIR filter, degraded signal, stochastic systems with uncertainties, observable companion form, continuous-time stochastic systems, covariance information.

Received: March 27, 2024. Revised: October 11, 2024. Accepted: November 15, 2024. Published: December 27, 2024.

1 Introduction

Kalman filter and the finite impulse response (FIR) filter for signal and state estimation are widely used in the application area of navigation, communication systems, and signal processing in stochastic systems [1], [2]. In [3], a one-step H_2 optimal FIR predictor is designed and applied to the robot predictive tracking problem. In [4], an unbiased FIR filter estimates the clock state by measuring the time interval error based on an interval of finite most recent past points. Researchers have studied finite impulse response (FIR) estimation techniques in discrete-time and continuous-time stochastic systems, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34]. In the book [35], there is a thorough discussion of FIR estimation techniques. For state-space models with uncertainties, FIR estimators perform better than conventional recursive estimators in linear discrete-time stochastic systems, [5]. Below is a classification of FIR estimation techniques.

(1) Some references to continuous-time FIR estimators are as follows. FIR filter and FIR smoother in linear stochastic systems, [6]. Robust FIR filter in linear stochastic systems with bounded uncertainties, [7]. FIR filter for input-delayed stochastic systems, [8]. Recursive least-squares (RLS) FIR filter using covariance information in linear continuous-time stochastic systems, [9].
(2) Receding horizon FIR filter in linear discrete-time stochastic systems, [10].
(3) Iterative FIR filter in linear discrete-time stochastic systems, e.g. [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].
(4) Strictly passive FIR filter in linear discrete-time stochastic systems, [22].
(5) Fixed-lag FIR smoother in linear discrete-time stochastic systems, [23].
(6) Robust RLS Wiener FIR filter in linear discrete-time stochastic systems, [24].
(7) FIR filter in nonlinear discrete-time stochastic systems, [25].
(8) Confidence set-membership FIR filter in linear discrete time-variant stochastic systems, [26].
(9) Unified FIR filter and smoother in linear discrete-time stochastic systems, [27].

(10) FIR filter for systems with delays and missing observations in linear discrete-time stochastic systems, [28], [29], [30], [31].
(11) Systematical analysis of batch FIR filtering algorithms, [32].
(12) Backward FIR filter in linear discrete-time variant stochastic systems, [33].
(13) FIR smoother estimating signal at the starting time of fixed-interval based on algebraic calculations by the Levinson-Durbin algorithm, [34].

The RLS Wiener filter is designed for linear continuous-time stochastic systems with uncertainties, as described in [36]. This paper extends the robust RLS Wiener filter to the robust RLS FIR filter by utilizing covariance information in linear continuous-time stochastic systems with uncertainties. The robust RLS FIR filter uses the cross-covariance function of the system state with the degraded observed value and the auto-covariance function of the degraded state.

The organization of this paper is as follows: Section 2 presents he estimation method for the system and observation matrices, [36]. As explained in [36], the observable companion form expresses the differential equations for uncertain states. For robust filtering problems in linear continuous-time stochastic systems with uncertainties, this paper uses the state-space model of the observable companion form for the degraded signal. Section 3 introduces the least-squares FIR filtering problem. In Section 4, Theorem 1 presents robust RLS FIR filtering algorithms for both the signal and the system state. Section 5 demonstrates two numerical simulation examples for the robust RLS FIR filter. For finite observation intervals, we compare the estimation accuracy of the robust RLS FIR filter. We also compare the estimation accuracy of the robust RLS FIR filter in Theorem 1 with that of the robust RLS filter in Theorem 1.

2 Nominal and Degraded State-Space Models and Degraded System Realization

Let (1) be a nominal state-space model in linear continuous-time stochastic systems.

$$\begin{aligned}
&y(t)=z(t)+v(t), z(t)=Hx(t),\\
&\frac{dx(t)}{dt}=Ax(t)+\Gamma w(t), x(0)=c,\\
&E[v(k)v^T(s)]=R\delta(t-s), R>0\\
&E[w(t)w^T(s)]=Q\delta(t-s), Q>0\\
&E[v(t)w^T(s)]=0, E[x(0)w^T(t)]=0.
\end{aligned} \quad (1)$$

Here, $x(t)\in R^n$ is the state vector, while $z(t)\in R^m$ is the signal vector. Input noise $w(t)\in R^l$ and observation noise $v(t)$ are independent, zero mean, white Gaussian noises. Γ is the $n\times l$ input matrix, and H is the $m\times n$ observation matrix. The auto-covariance functions for the input noise $w(t)$ and the observation noise $v(t)$ are given by (1), respectively. Let the state and observation equations with uncertain parameters be given by (2).

$$\begin{aligned}
&\breve{y}(t)=\breve{z}(t)+v(t),\\
&\breve{z}(t)=\overleftrightarrow{H}(t)\overleftrightarrow{x}(t), \overleftrightarrow{H}(t)=H+\Delta H(t),\\
&\frac{d\overleftrightarrow{x}(t)}{dt}=\overleftrightarrow{A}(t)\overleftrightarrow{x}(t)+\Gamma w(t),\\
&\overleftrightarrow{A}(t)=A+\Delta\mathrm{A}(t), \overleftrightarrow{x}(0)=\overleftrightarrow{c},\\
&\mathrm{E}[\mathrm{v(t)}w^T(\mathrm{s})]=0, \mathrm{E}[\Delta\mathrm{A(t)}w^T(\mathrm{s})]=0,\\
&E[\Delta\mathrm{C}(t)v^T(s)]=0, E[\overleftrightarrow{x}(0)w^T(t)]=0,\\
&E[\overleftrightarrow{x}(0)v^T(t)]=0
\end{aligned} \quad (2)$$

In (2), $\overleftrightarrow{A}(t)$ and $\overleftrightarrow{C}(t)$ denote the degraded system matrix and the degraded observation matrix, respectively. In (2), $\Delta\mathrm{A}(t)$ and $\Delta\mathrm{C}(t)$ are uncertain matrices. The initial state of the system, $\overleftrightarrow{x}(0)$, is a random vector uncorrelated with both the system input noise $w(t)$ and the measurement noise $v(t)$.

Suppose that the degraded signal is represented as $\breve{z}(t)=\breve{H}\breve{x}(t)$ using the degraded state vector $\breve{x}(t)$, where $\breve{x}(t)$ assumes n components.

$$\begin{aligned}
&\breve{z}(t)=\breve{H}\breve{x}(t), \breve{z}(t)=\breve{x}_1(t),\\
&\breve{H}=[I_{m\times m}\quad 0\quad 0\quad \cdots\quad 0],\\
&\breve{x}(t)=\begin{bmatrix}\breve{x}_1(t)\\ \breve{x}_2(t)\\ \vdots\\ \breve{x}_{n-1}(t)\\ \breve{x}_n(t)\end{bmatrix}
\end{aligned} \quad (3)$$

Let $\breve{x}_1(t)$ satisfies a differential equation

$$\begin{aligned}
\frac{d\breve{x}_1^n(t)}{dt^n}=&-\breve{a}_1\frac{d\breve{x}_1^{n-1}(t)}{dt^n}-\breve{a}_2\frac{d\breve{x}_1^{n-2}(t)}{dt^{n-2}}\cdots\\
&-\breve{a}_{n-1}\frac{d\breve{x}_1(t)}{dt}-\breve{a}_n\breve{x}_1(t)\\
&+\xi(t).
\end{aligned} \quad (4)$$

(4) is transformed into the observable companion form of the state differential equations:

$$\begin{aligned}
&\frac{d\breve{x}(t)}{dt}=\breve{A}\breve{x}(t)+\breve{\Gamma}\xi(t),\\
&E[\xi(t)\xi^T(s)]=\breve{Q}\delta(t-s),\\
&\breve{A}=\begin{bmatrix}0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\breve{a}_n & -\breve{a}_{n-1} & -\breve{a}_{n-2} & \cdots & -\breve{a}_1\end{bmatrix},\\
&\breve{\Gamma}=[0\quad 0\quad \cdots\quad 0\quad I_{I_{m\times m}}]^T.
\end{aligned} \quad (5)$$

In (4), $\xi(t)$ denotes the residual in approximating the degraded signal $\breve{z}(t)$. The degraded system matrix $\breve{A}$ is estimated by (6), [36].

$$\breve{A} = E\left[\frac{d\breve{x}(t)}{dt}\breve{x}^T(s)\right]E[\breve{x}(s)\breve{x}^T(s)]^{-1}, \quad 0 \le s < t,$$
$$\breve{x}^T(t) = [\breve{x}_1(t) \quad \breve{x}_2(t) \quad \cdots \quad \breve{x}_{n-1}(t) \quad \breve{x}_n(t)],$$
$$\breve{x}_1(t) = \breve{z}(t), \breve{x}_2(t) = \frac{d\breve{z}(t)}{dt}, \cdots, \quad (6)$$
$$\breve{x}_{n-1}(t) = \frac{d^{n-2}\breve{z}(t)}{dt^{n-2}}, \breve{x}_n(t) = \frac{d^{n-1}\breve{z}(t)}{dt^{n-1}}$$

Also, $\breve{H}$ is estimated by

$$\breve{H} = E[\breve{y}(t)\breve{x}^T(t)]E[\breve{x}(t)\breve{x}^T(t)]^{-1} \quad (7)$$

[36].

3 Robust Finite Impulse Response Filtering Problem

Let the FIR filtering estimate $\hat{x}(t, t+T)$ of $x(t+T)$ be given by

$$\hat{x}(t, t+T) = \int_t^{t+T} h(t+T, s)\breve{y}(s)ds \quad (8)$$

as a linear transformation of the degraded observed value $\breve{y}(s)$, $t \le s \le t+T$. Here, $h(t+T, s)$ represents an impulse response function. Let us consider minimizing the mean-square value:

$$J = E[(x(t+T) - \hat{x}(t, t+T))^T(x(t+T) - \hat{x}(t, t+T))] \quad (9)$$

of the FIR filtering error $x(t+T) - \hat{x}(t, t+T)$. The filtering estimate $\hat{x}(t, t+T)$ to minimize the cost function J satisfies the relationship:

$$x(t+T) - \hat{x}(t, t+T) \perp \breve{y}(s), \quad t < s < t+T, \quad (10)$$

from the orthogonal projection lemma [37], [38], [39], [40]. Hence, the optimal impulse response function satisfies the Wiener-Hopf integral equation:

$$E[x(t+T)\breve{y}^T(s)] = \int_t^{t+T} h(t+T, \tau)E[\breve{y}(\tau)\breve{y}^T(s)]d\tau. \quad (11)$$

Substituting the degraded observation equation in (2) into (11), (11) is transformed into:

$$h(t+T, s)R = K_{x\breve{y}}(t, s) - \int_t^{t+T} h(t+T, \tau)\breve{H}K_{\breve{x}}(\tau, s)\breve{H}^T d\tau,$$
$$K_{x\breve{y}}(t, s) = E[x(t)\breve{y}^T(s)], \quad (12)$$
$$K_{\breve{x}}(\tau, s) = E[\breve{x}(\tau)\breve{x}^T(s)].$$

Starting from (12), the robust RLS FIR filtering algorithms for the signal and the system state are derived. Consider the cross-covariance function $K_{x\breve{y}}(t, s)$ of $x(t)$ with $\breve{y}(s)$, expressed as:

$$K_{x\breve{y}}(t, s) = \begin{cases} \alpha(t)\beta^T(s), 0 \le s \le t, \\ \gamma(t)\delta^T(s), 0 \le t \le s, \end{cases}$$
$$\alpha(t) = e^{At}, \quad \beta^T(s) = e^{-As}K_{x\breve{y}}(s, s). \quad (13)$$

Let $K_{\breve{x}}(t, s)$ be the covariance function of $\breve{x}(t)$, expressed as:

$$K_{\breve{x}}(t, s) = \begin{cases} \breve{C}(t)\breve{D}^T(s), 0 \le s \le t, \\ \breve{D}(t)\breve{C}^T(s), 0 \le t \le s, \end{cases} \quad (14)$$
$$\breve{C}(t) = e^{\breve{A}t}, \breve{D}^T(s) = e^{-\breve{A}s}K_{\breve{x}}(s, s).$$

The use of covariance information $\alpha(t)$, $\beta(t)$, $\breve{C}(t)$, and $\breve{D}(t)$ characterizes the current robust RLS FIR filter in Theorem 1.

4 Robust RLS FIR Filtering Algorithms

Theorem 1 presents the robust RLS FIR filtering algorithms for $\hat{z}(t, t+T)$ of the signal $z(t+T)$ and $\hat{x}(t, t+T)$ of the system state $x(t+T)$.

Theorem 1 Let the state-space model for the signal $z(t)$ be given by (1). Let the state-space model for the degraded signal $\breve{z}(t)$ be given by (2). Let the cross-covariance function $K_{x\breve{y}}(t, s)$ of the state $x(t)$ with the degraded observed value $\breve{y}(s)$ be given by (13). Let the autocovariance function $K_{\breve{x}}(t, s)$ of the degraded state $\breve{x}(t)$ be given by (14). Then robust RLS FIR filtering algorithms for the signal $z(t+T)$ and the state $x(t+T)$ using the information on the degraded observations $\breve{y}(s), t \le s \le t+T$, and the covariances consist of the following equations (15)-(43).

FIR filtering estimate of the signal $z(t+T)$: $\hat{z}(t, t+T)$

$$\hat{z}(t, t+T) = H\hat{x}(t, t+T) \quad (15)$$

FIR filtering estimate of the state $x(t+T)$: $\hat{x}(t, t+T)$

$$\hat{x}(t, t+T) = \alpha(t+T)e(t, t+T) \quad (16)$$

$$J(t+T, t+T) = (\beta^T(t+T) - r(t, t+T)\breve{C}^T(t+T)\breve{H}^T)R^{-1} \quad (17)$$

$$J(t+T, t) = (\beta^T(t) - \breve{r}(t, t+T)\breve{C}^T(t)\breve{H}^T)R^{-1} \quad (18)$$

$$\breve{J}(t+T, t+T) = (\breve{C}^T(t+T)\breve{H}^T - \overleftarrow{r}(t, t+T)\breve{C}^T(t+T)\breve{H}^T)R^{-1} \quad (19)$$

$$\begin{aligned}&\breve{J}(t+T,t)=(\breve{C}^T(t)\breve{H}^T\\&-\overleftrightarrow{\Gamma}(t,t+T)\breve{D}^T(t)\breve{H}^T)R^{-1}\end{aligned}\tag{20}$$

$$\begin{aligned}&L(t+T,t+T)=(\breve{D}^T(t+T)\breve{H}^T\\&-S(t,t+T)\breve{C}^T(t+T)\breve{H}^T)R^{-1}\end{aligned}\tag{21}$$

$$\begin{aligned}&L(t+T,t)=(\breve{D}^T(t)\breve{H}^T\\&-\breve{p}(t,t+T)\breve{D}^T(t)\breve{H}^T)R^{-1}\end{aligned}\tag{22}$$

$$\begin{aligned}&\frac{de(t,t+T)}{dt}=J(t+T,t+T)(\breve{y}(t+T)\\&-\breve{H}\breve{C}(t+T)f(t,t+T))-J(t+T,t)\\&\times(\breve{y}(t)-\breve{H}\breve{D}(t)\breve{e}(t,t+T))\end{aligned}\tag{23}$$

$$\begin{aligned}&\frac{d\breve{e}(t,t+T)}{dt}=\breve{J}(t+T,t+T)(\breve{y}(t+T)\\&-\breve{H}\breve{C}(t+T)f(t,t+T))\\&-\breve{J}(t+T,t)(\breve{y}(t)-\breve{H}\breve{D}(t)\breve{e}(t,t+T))\end{aligned}\tag{24}$$

$$\begin{aligned}&\frac{df(t,t+T)}{dt}=L(t+T,t+T)(\breve{y}(t+T)\\&-\breve{H}\breve{C}(t+T)f(t,t+T))\\&-L(t+T,t)(\breve{y}(t)-\breve{H}\breve{D}(t)\breve{e}(t,t+T))\end{aligned}\tag{25}$$

$$\begin{aligned}&\frac{dr(t,t+T)}{dt}=J(t+T,t+T)(\breve{H}\breve{D}(t+T)\\&-\breve{H}\breve{C}(t+T)S(t,t+T))\\&-J(t+T,t)(\breve{H}\breve{D}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T))\end{aligned}\tag{26}$$

$$\begin{aligned}&\frac{dS(t,t+T)}{dt}=L(t+T,t+T)(\breve{H}\breve{D}(t+T)\\&-\breve{H}\breve{C}(t+T)S(t,t+T))\\&-L(t+T,t)(\breve{H}\breve{D}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T))\end{aligned}\tag{27}$$

$$\begin{aligned}&\frac{d\breve{r}(t,t+T)}{dt}=J(t+T,t+T)(\breve{H}\breve{C}(t+T)\\&-\breve{H}\breve{C}(t+T)\breve{p}(t,t+T))\\&-J(t+T,t)(\breve{H}\breve{C}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T))\end{aligned}\tag{28}$$

$$\begin{aligned}&\frac{d\overleftrightarrow{r}(t,t+T)}{dt}=\breve{J}(t+T,t+T)(\breve{H}\breve{D}(t+T)\\&-\breve{H}\breve{C}(t+T)S(t,t+T))\\&-\breve{J}(t+T,t)(\breve{H}\breve{D}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T))\end{aligned}\tag{29}$$

$$\begin{aligned}&\frac{d\breve{p}(t,t+T)}{dt}=L(t+T,t+T)(\breve{H}\breve{C}(t+T)\\&-\breve{H}\breve{C}(t+T)\breve{p}(t,t+T))\\&-L(t+T,t)(\breve{H}\breve{C}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T))\end{aligned}\tag{30}$$

$$\begin{aligned}&\frac{d\overleftrightarrow{\Gamma}(t,t+T)}{dt}=\breve{J}(t+T,t+T)(\breve{H}\breve{C}(t+T)\\&-\breve{H}\breve{C}(t+T)\breve{p}(t,t+T))\\&-\breve{J}(t+T,t)(\breve{H}\breve{C}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T))\end{aligned}\tag{31}$$

$$\begin{aligned}&J(T,T)\\&=(\beta^T(T)-r(0,T)\breve{C}^T(T)\breve{H}^T)R^{-1}.\end{aligned}\tag{32}$$

$$\begin{aligned}&\breve{J}(T,T)=(\breve{C}^T(T)\breve{H}^T\\&\qquad-\overleftrightarrow{r}(0,T)\breve{C}^T(T)\breve{H}^T)R^{-1}\end{aligned}\tag{33}$$

$$\begin{aligned}&L(T,T)\\&=(\breve{D}^T(T)\breve{H}^T-S(0,T)\breve{C}^T(T)\breve{H}^T)R^{-1}.\end{aligned}\tag{34}$$

Initial condition of $e(t,t+T)$ at $t=0$: $e(0,T)$

$$\begin{aligned}&\frac{de(0,T)}{dT}=J(T,T)(\breve{y}(T)-\breve{H}\breve{C}(T)f(0,T)),\\&e(0,0)=0\end{aligned}\tag{35}$$

Initial condition of $\breve{e}(t,t+T)$ at $t=0$: $\breve{e}(0,T)$

$$\begin{aligned}&\frac{d\breve{e}(0,T)}{dT}=\breve{J}(T,T)(\breve{y}(T)-\breve{H}\breve{C}(T)f(0,T)),\\&\breve{e}(0,0)=0\end{aligned}\tag{36}$$

Initial condition of $f(t,t+T)$ at $t=0$: $f(0,T)$

$$\begin{aligned}&\frac{df(0,T)}{dT}=L(T,T)(\breve{y}(T)-\breve{H}\breve{C}(T)f(0,T)),\\&f(0,0)=0\end{aligned}\tag{37}$$

Initial condition of $r(t,t+T)$ at $t=0$: $r(0,T)$

$$\begin{aligned}&\frac{dr(0,T)}{dT}=J(T,T)(\breve{H}\breve{D}(T)\\&\qquad-\breve{H}\breve{C}(T)S(0,T)),\\&r(0,0)=0\end{aligned}\tag{38}$$

Initial condition of $S(t,t+T)$ at $t=0$: $S(0,T)$

$$\begin{aligned}&\frac{dS(0,T)}{dT}=L(T,T)(\breve{H}\breve{D}(T)\\&\qquad-\breve{H}\breve{C}(T)S(0,T)),\\&S(0,0)=0\end{aligned}\tag{39}$$

Initial condition of $\breve{r}(t,t+T)$ at $t=0$: $\breve{r}(0,T)$

$$\begin{aligned}&\frac{d\breve{r}(0,T)}{dT}=J(T,T)(\breve{H}\breve{C}(T)\\&\qquad-\breve{H}\breve{C}(T)\breve{p}(0,T)),\\&\breve{r}(0,0)=0\end{aligned}\tag{40}$$

Initial condition of $\overleftrightarrow{r}(t,t+T)$ at $t=0$: $\overleftrightarrow{r}(0,T)$

$$\begin{aligned}&\frac{d\overleftrightarrow{r}(0,T)}{dT}=\breve{J}(T,T)(\breve{H}\breve{D}(T)\\&\qquad-\breve{H}\breve{C}(T)S(0,T)),\\&\overleftrightarrow{r}(0,0)=0\end{aligned}\tag{41}$$

Initial condition of $\breve{p}(t,t+T)$ at $t=0$: $\breve{p}(0,T)$

$$\frac{d\breve{p}(0,T)}{dT} = L(T,T)(\breve{H}\breve{C}(T) - \breve{H}\breve{C}(T)\breve{p}(0,T)), \quad \breve{p}(0,0) = 0 \tag{42}$$

Initial condition of $\overleftrightarrow{\Gamma}(t,t+T)$ at $t=0$: $\overleftrightarrow{\Gamma}(0,T)$

$$\frac{d\overleftrightarrow{\Gamma}(0,T)}{dT} = \breve{J}(T,T)(\breve{H}\breve{C}(T) - \breve{H}\breve{C}(T)\breve{p}(0,T)), \quad \overleftrightarrow{\Gamma}(0,0) = 0 \tag{43}$$

Initial condition $e(0,T)$ of the differential equation (23) for $e(t,t+T)$ at $t=0$ is calculated by (35), starting with $e(0,0)=0$. Initial condition $\breve{e}(0,T)$ of the differential equation (24) for $\breve{e}(t,t+T)$ at $t=0$ is calculated by (36), starting with $\breve{e}(0,0)=0$. Initial condition $f(0,T)$ of the differential equation (25) for $f(t,t+T)$ at $t=0$ is calculated by (37), starting with $f(0,0)=0$. Initial condition $r(0,T)$ of the differential equation (26) for $r(t,t+T)$ at $t=0$ is calculated by (38), starting with $r(0,0)=0$. Initial condition $S(0,T)$ of the differential equation (27) for $S(t,t+T)$ at $t=0$ is calculated by (39), starting with $S(0,0)=0$. Initial condition $\breve{r}(0,T)$ of the differential equation (28) for $\breve{r}(t,t+T)$ at $t=0$ is calculated by (40), starting with $\breve{r}(0,0)=0$. Initial condition $\overleftrightarrow{r}(0,T)$ of the differential equation (29) for $\overleftrightarrow{r}(t,t+T)$ at $t=0$ is calculated by (41), starting with $\overleftrightarrow{r}(0,0)=0$. Initial condition $\breve{p}(0,T)$ of the differential equation (30) for $\breve{p}(t,t+T)$ at $t=0$ is calculated by (42), starting with $\breve{p}(0,0)=0$. Initial condition $\overleftrightarrow{\Gamma}(0,T)$ of the differential equation (31) for $\overleftrightarrow{\Gamma}(t,t+T)$ at $t=0$ is calculated by (43), starting with $\overleftrightarrow{\Gamma}(0,0)=0$.

From (15), the robust RLS filtering estimate $\hat{z}(0,T)$ of the signal $z(T)$ is calculated as $\hat{z}(0,T) = H\hat{x}(0,T)$. From (16), the robust RLS filtering estimate $\hat{x}(0,T)$ of the system state $x(T)$ is calculated as $\hat{x}(0,T) = \alpha(T)e(0,T)$. (32), (34), (35), (37), (38) and (39) compute $e(0,T)$ recursively. Section 5 provides a numerical comparison of the estimation accuracy between the robust FIR and robust RLS filters.

See the Appendix for proving Theorem 1.

5 A Numerical Simulation Example

Example 1

Let (44) give the state-space model for the observed value $y(t)$ and the nominal system state $x(t)$.

$$\begin{aligned}
&y(t) = z(t) + v(t), z(t) = Hx(t),\\
&H = [1 \quad 0],\\
&\frac{dx(t)}{dt} = Ax(t) + \Gamma w(t),\\
&x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix},\\
&A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, a_1 = 4, a_2 = 3,\\
&\Gamma = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\\
&E[v(k)v(s)] = R\delta(t-s),\\
&E[w(t)w(s)] = Q\delta(t-s), Q = 1,\\
&E[v(t)w(s)] = 0
\end{aligned} \tag{44}$$

Let the state-space model for the degraded observed value $\breve{y}(t)$ and the degraded state $\overleftrightarrow{x}(t)$ be given by:

$$\begin{aligned}
&\breve{y}(t) = \breve{z}(t) + v(t), \breve{z}(t) = \overleftrightarrow{H}(t)\overleftrightarrow{x}(t),\\
&\frac{d\overleftrightarrow{x}(t)}{dt} = \overleftrightarrow{A}(t)\overleftrightarrow{x}(t) + \Gamma w(t),\\
&\overleftrightarrow{A}(t) = A + \Delta\mathrm{A}(t), \overleftrightarrow{H}(t) = H + \Delta\mathrm{H}(t),\\
&\Delta\mathrm{A}(t) = \begin{bmatrix} 0 & 0 \\ -0.1 * rand & -0.1 * rand \end{bmatrix},\\
&\Delta\mathrm{H}(t) = [0.1 \quad 0],\\
&E[\overleftrightarrow{x}(0)w(t)] = 0.
\end{aligned} \tag{45}$$

Here, the degraded signal $\breve{z}(t)$ is observed with additive white Gaussian noise $v(t)$. $\Delta A(t)$ denotes an uncertain matrix additional to the system matrix A. "$rand$" denotes a scalar random number that follows a uniform distribution in the interval (0,1). Along with the state-space model in the observable companion form of (3) and (5), the observation equation for the degraded signal $\breve{z}(t)$ and the state differential equations for the degraded state $\breve{x}(t)$ is represented by:

$$\begin{aligned}
&\breve{y}(t) = \breve{z}(t) + v(t), \breve{z}(t) = \breve{H}\breve{x}(t),\\
&\breve{z}(t) = \breve{x}_1(t),\\
&\breve{x}(t) = \begin{bmatrix} \breve{x}_1(t) \\ \breve{x}_2(t) \end{bmatrix},\\
&\frac{d\breve{x}(t)}{dt} = \breve{A}\breve{x}(t) + \breve{\Gamma}\xi(t), \breve{\Gamma} = [0 \quad 1]^T,\\
&E[\xi(t)\xi(s)] = \breve{Q}\delta(t-s).
\end{aligned} \tag{46}$$

The system matrix $\breve{A}$ for the degraded state-space model (46) is calculated by:

$$\breve{A}=\begin{bmatrix} E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(s)\right] & E\left[\frac{d\breve{z}(t)}{dt}\frac{d\breve{z}(s)}{dt}\right] \\ E\left[\frac{d^2\breve{z}(t)}{dt^2}\breve{z}(s)\right] & E\left[\frac{d^2\breve{z}(t)}{dt^2}\frac{d\breve{z}(s)}{dt}\right]\end{bmatrix} \times \begin{bmatrix} E[\breve{z}(t)\breve{z}(s)] & E\left[\breve{z}(t)\frac{d\breve{z}(s)}{dt}\right] \\ E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(s)\right] & E\left[\frac{d\breve{z}(t)}{dt}\frac{d\breve{z}(s)}{dt}\right]\end{bmatrix}^{-1}, \quad (47)$$
$s=t-h$, $h=0.001$.

The estimate of $\breve{A}$ by (47) for $n=2$ is based on the relationship $\frac{\partial K_{\breve{x}}(t,s)}{\partial t}=\breve{A}K_{\breve{x}}(t,s)$, $0\le s<t$, [36].

To approximate the derivatives in (47) for the data sampling interval of $h=0.001$, a four-point forward difference formula with a truncation error of $O(h^2)$ is employed in the numerical differentiation. In the calculation of the expectation in (47), for example, $E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(t)\right]$ is evaluated by, $\frac{1}{T}\int_0^T\frac{d\breve{z}(t)}{dt}\breve{z}(t)dt$, $T=2.0$. Simpson's $\frac{1}{3}$ rule computes the numerical integration with an integration step size of $h=0.001$. For the degraded state $\breve{x}(t)$ in (46), the estimate of the system matrix $\breve{A}$ results in:

$$\begin{bmatrix}-1.636926\times10^{-17} & 1.000000\\ -3.217657 & -4.216199\end{bmatrix}. \quad (48)$$

Table 1. Estimates of $\breve{H}$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$

White Gaussian observation noise	Estimates of $\breve{H}$
$N(0,0.3^2)$	$[1 \quad -1.725904\times10^{-16}]$
$N(0,0.4^2)$	$[1 \quad -3.363866\times10^{-17}]$
$N(0,0.5^2)$	$[1 \quad 3.693192\times10^{-16}]$

Table 1 shows the estimates of $\breve{H}$ by (7) for the white Gaussian observation noises $N(0,0.4^2)$, and $N(0,0.5^2)$. $N(0,0.4^2)$, and $N(0,0.5^2)$. The estimate of $\breve{H}$ is precisely close to $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$.

Substituting the covariance information $\alpha(t)$, $\beta(t)$, $\breve{C}(t)$, and $\breve{D}(t)$ into the robust RLS FIR filtering algorithm of Theorem 1, the FIR filtering estimate $\hat{x}(t,t+T)$ of the system state $x(t+T)$ is recursively computed. $K_{x\breve{y}}(s,s)=E[x(s)\breve{y}(s)]$ is evaluated by $\frac{1}{T}\int_0^T x(t)\breve{y}(t)dt$, $T=2.0$. Table 2 shows the estimates of $K_{x\breve{y}}(s,s)$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$.

Table 2. Estimates of $K_{x\breve{y}}(s,s)$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$

White Gaussian obser-vation noise	Estimates of $K_{x\breve{y}}(s,s)$
$N(0,0.3^2)$	$[2.404773\times10^{-1} \quad -8.877329\times10^{-2}]$
$N(0,0.4^2)$	$[2.395386\times10^{-1} \quad -8.877390\times10^{-2}]$
$N(0,0.5^2)$	$[2.383359\times10^{-1} \quad -8.857582\times10^{-2}]$

Figure 1 illustrates the FIR filtering estimate $\hat{x}_1(t,t+T)$, $T=1$, of the state variable $x_1(t+T)$ vs. t, $0\le t\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$. Figure 1 shows that $\hat{x}_1(t,t+T)$ converges to $x_1(t+T)$ as t increases. Figure 2 illustrates the FIR filtering estimate $\hat{x}_2(t,t+T)$, $T=1$, of the state variable $x_2(t+T)$ vs. t, $0\le t\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$. Figure 2 shows that $\hat{x}_2(t,t+T)$ gradually converges to $x_2(t+T)$ as t increases.

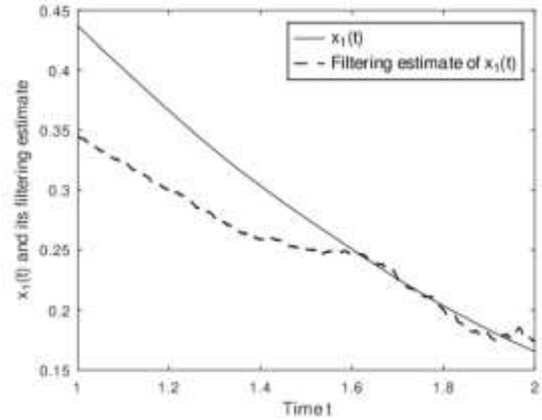

Fig. 1: FIR filtering estimate $\hat{x}_1(t,t+T)$, $T=1$, of the state variable $x_1(t+T)$ vs. t, $0\le t\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$

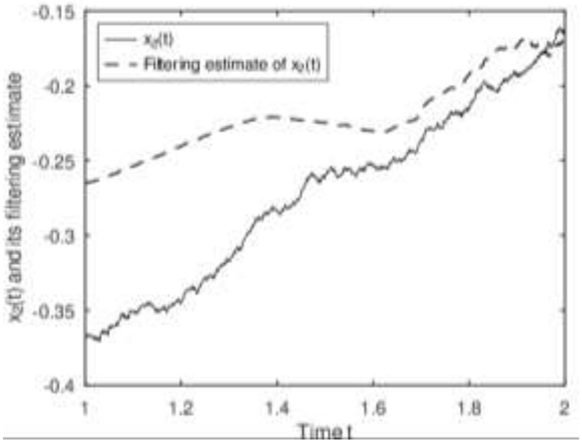

Fig. 2: FIR filtering estimate $\hat{x}_2(t,t+T)$, $T=1$, of the state variable $x_2(t+T)$ vs. t, $0\le t\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$

Table 3 shows the mean-square values (MSVs) of the filtering errors $x_1(t+T)-\hat{x}_1(t,t+T)$ and $x_2(t+T)-\hat{x}_2(t,t+T)$ in the case of $T=0.5$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i\cdot h+T)-\hat{x}_1(i\cdot h,i\cdot h+T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i\cdot h+T,)-\hat{x}_2(i\cdot h,i\cdot h+T)\right)^2$, $h=0.001$.

Table 3. Mean-square values of the FIR filtering errors $x_1(t)-\hat{x}_1(t,t+T)$ and $x_2(t)-\hat{x}_2(t,t+T)$, $T=0.5$, for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$

White Gaussian observation noise	MSV of $x_1(t)-\hat{x}_1(t,t+T)$	MSV of $x_2(t)-\hat{x}_2(t,t+T)$
$N(0,0.3^2)$	1.17634×10^{-1}	6.33826×10^{-2}
$N(0,0.4^2)$	1.74127×10^{-1}	7.58909×10^{-2}
$N(0,0.5^2)$	2.39987×10^{-1}	1.05730×10^{-1}

Table 4. Mean-square values of the FIR filtering errors $x_1(t)-\hat{x}_1(t,t+T)$ and $x_2(t)-\hat{x}_2(t,t+T)$, $T=1$, for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$

White Gaussian observation noise	MSV of $x_1(t)-\hat{x}_1(t,t+T)$	MSV of $x_2(t)-\hat{x}_2(t,t+T)$
$N(0,0.3^2)$	1.13525×10^{-1}	6.19606×10^{-2}
$N(0,0.4^2)$	1.66809×10^{-1}	7.37079×10^{-2}
$N(0,0.5^2)$	2.14188×10^{-1}	8.91796×10^{-2}

Table 4 shows the MSVs of the FIR filtering errors $x_1(t+T)-\hat{x}_1(t,t+T)$ and $x_2(t+T)-\hat{x}_2(t,t+T)$ in the case of $T=1$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i\cdot h+T)-\hat{x}_1(i\cdot h,i\cdot h+T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i\cdot h+T,)-\hat{x}_2(i\cdot h,i\cdot h+T)\right)^2$, $h=0.001$.

Table 5 shows the MSVs of the filtering errors $x_1(t+T)-\hat{x}_1(0,t+T)$ and $x_2(t+T)-\hat{x}_2(0,t+T)$ in the case of $\mathrm{T}=1$ for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i\cdot h+T)-\hat{x}_1(0,i\cdot h+T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i\cdot h+T,)-\hat{x}_2(0,i\cdot h+T)\right)^2$, $h=0.001$.

Table 5. Mean-square values of the filtering errors $x_1(t+T)-\hat{x}_1(0,t+T)$ and $x_2(t+T)-\hat{x}_2(0,t+T)$, $\mathrm{T}=1$, for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.4^2)$, and $N(0,0.5^2)$

White Gaussian observation noise	MSV of $x_1(t+T)-\hat{x}_1(0,t+T)$	MSV of $x_2(t+T)-\hat{x}_2(0,t+T)$
$N(0,0.3^2)$	1.11533×10^{-1}	5.79816×10^{-2}
$N(0,0.4^2)$	1.60262×10^{-1}	6.46205×10^{-2}
$N(0,5^2)$	2.00979×10^{-1}	7.37283×10^{-2}

From Table 3 for $T=0.5$ and Table 4 for $T=1$, the MSVs of the robust RLS FIR filtering errors $x_1(t+T)-\hat{x}_1(t,t+T)$ and $x_2(t+T)-\hat{x}_2(t,t+T)$ for $T=1$ are smaller than those for $T=0.5$ in each observation noise. This result indicates that the estimation accuracy of the robust RLS FIR filter improves as the fixed interval T increases. The MSVs of the robust RLS FIR filter for $T=1$ in Table 4 are almost identical to those of the robust RLS filter in Table 5 for each observation noise. The results show that as the fixed interval T increases, the robust RLS FIR filter achieves accuracy similar to that of the robust RLS filter.

Example 2

Consider the second-order mass-spring system driven by zero-mean white Gaussian noise $w(t)$ [41].

$$\begin{aligned}
&y(t)=z(t)+v(t), z(t)=Hx(t),\\
&H=[1 \quad 0],\\
&\frac{dx(t)}{dt}=Ax(t)+\Gamma w(t),\\
&x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}, x(0)=\begin{bmatrix}1.0\\1.0\end{bmatrix},\\
&A=\begin{bmatrix}0 & 1\\-\omega_n^2 & -2\zeta\omega_n\end{bmatrix}, \omega_n=\sqrt{3},\\
&\zeta=\frac{2}{\omega_n}, \Gamma=\begin{bmatrix}0\\\omega_n^2\end{bmatrix},\\
&E[v(t)v(s)]=R\delta(t-s),\\
&E[w(t)w(s)]=Q\delta(t-s), Q=1,\\
&E[v(t)w(s)]=0, E[x(0)w(t)]=0,\\
&E[x(0)v(t)]=0.
\end{aligned} \tag{49}$$

Let the state-space model for the degraded observed value $\breve{y}(t)$ and the degraded state $\vec{x}(t)$ be given by (45). Figure 3 illustrates the FIR filtering estimate $\hat{x}_1(t,t+T)$, $T=1$, of the state variable $x_1(t+T)$ vs. t, $0\le \mathrm{t}\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$. Figure 1 shows that $\hat{x}_1(t,t+T)$ converges to $x_1(t+T)$ as t increases. Figure 2 illustrates the FIR filtering estimate $\hat{x}_2(t,t+T)$, $T=1$, of the state variable $x_2(t+T)$ vs. t, $0\le \mathrm{t}\le 1$, for the white Gaussian observation noise $N(0,0.3^2)$. Figure 2 shows that $\hat{x}_2(t,t+T)$ gradually converges to $x_2(t+T)$ as t increases.

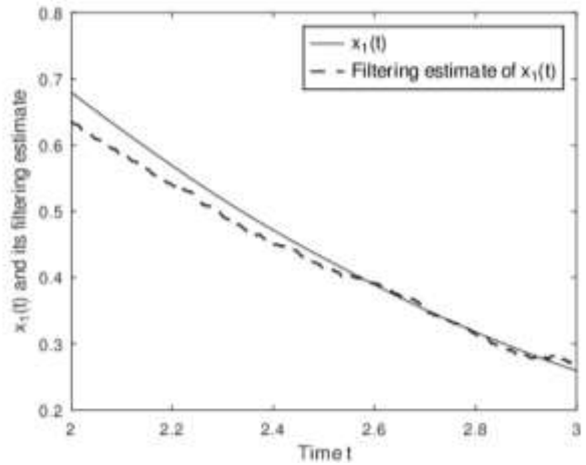

Fig. 3: FIR filtering estimate $\hat{x}_1(t, t+T)$, $T = 1$, of the state variable $x_1(t+T)$ vs. t, $0 \le t \le 1$, for the white Gaussian observation noise $N(0, 0.3^2)$

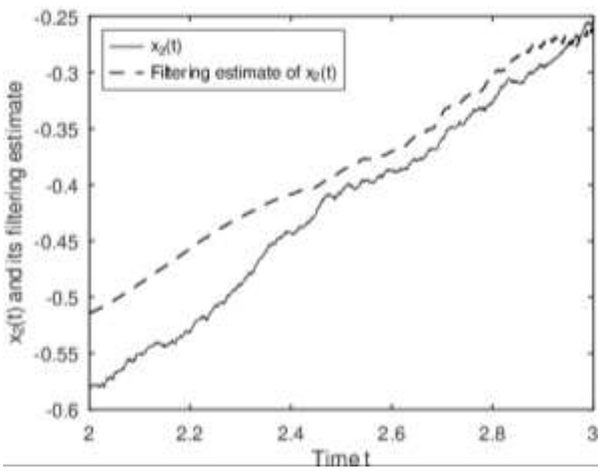

Fig. 4: FIR filtering estimate $\hat{x}_2(t, t+T)$, $T = 1$, of the state variable $x_2(t+T)$ vs. t, $0 \le t \le 1$, for the white Gaussian observation noise $N(0, 0.3^2)$

Figure 4 illustrates the FIR filtering estimate $\hat{x}_2(t, t+T)$, $T = 1$, of the state variable $x_2(t+T)$ vs. t, $0 \le t \le 1$, for the white Gaussian observation noise $N(0, 0.3^2)$. Figure 4 shows that $\hat{x}_2(t, t+T)$ gradually converges to $x_2(t+T)$ as t increases.Table 6 shows the MSVs of the filtering errors $x_1(t+T) - \hat{x}_1(t, t+T)$ and $x_2(t+T) - \hat{x}_2(t, t+T)$ in the case of $T = 0.5$ for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and $N(0, 0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i \cdot h + T) - \hat{x}_1(i \cdot h, i \cdot h + T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i \cdot h + T,) - \hat{x}_2(i \cdot h, i \cdot h + T)\right)^2$, $h = 0.001$.

Table 6. Mean-square values of the FIR filtering errors $x_1(t) - \hat{x}_1(t, t+T)$ and $x_2(t) - \hat{x}_2(t, t+T)$, $T = 0.5$, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and $N(0, 0.5^2)$

White Gaussian observation noise	MSV of $x_1(t) - \hat{x}_1(t, t+T)$	MSV of $x_2(t) - \hat{x}_2(t, t+T)$
$N(0, 0.3^2)$	1.42786×10^{-1}	1.00599×10^{-1}
$N(0, 0.4^2)$	2.29930×10^{-1}	9.83384×10^{-2}
$N(0, 0.5^2)$	3.20115×10^{-1}	1.10423×10^{-1}

Table 7. Mean-square values of the FIR filtering errors $x_1(t) - \hat{x}_1(t, t+T)$ and $x_2(t) - \hat{x}_2(t, t+T)$, $T = 1$, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and $N(0, 0.5^2)$

White Gaussian observation noise	MSV of $x_1(t) - \hat{x}_1(t, t+T)$	MSV of $x_2(t) - \hat{x}_2(t, t+T)$
$N(0, 0.3^2)$	1.40065×10^{-1}	9.80377×10^{-2}
$N(0, 0.4^2)$	2.23520×10^{-1}	9.51019×10^{-2}
$N(0, 0.5^2)$	3.10346×10^{-1}	1.07269×10^{-1}

Table 7 shows the MSVs of the FIR filtering errors $x_1(t+T) - \hat{x}_1(t, t+T)$ and $x_2(t+T) - \hat{x}_2(t, t+T)$ in the case of $T = 1$ for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and $N(0, 0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i \cdot h + T) - \hat{x}_1(i \cdot h, i \cdot h + T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i \cdot h + T,) - \hat{x}_2(i \cdot h, i \cdot h + T)\right)^2$, $h = 0.001$. From Table 6 for $T = 0.5$ and Table 7 for $T = 1$, the MSVs of the robust RLS FIR filtering errors $x_1(t+T) - \hat{x}_1(t, t+T)$ and $x_2(t+T) - \hat{x}_2(t, t+T)$ for $T = 1$ are smaller than those for $T = 0.5$ in each observation noise. This result indicates that the estimation accuracy of the robust RLS FIR filter improves as the fixed interval T increases.

Table 8 shows the MSVs of the filtering errors $x_1(t+T) - \hat{x}_1(0, t+T)$ and $x_2(t+T) - \hat{x}_2(0, t+T)$ in the case of $\mathrm{T} = 1$ for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and $N(0, 0.5^2)$. Here, the MSVs are calculated by $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_1(i \cdot h + T) - \hat{x}_1(0, i \cdot h + T)\right)^2$ and $\frac{1}{1000}\sum_{i=1}^{1000}\left(x_2(i \cdot h + T,) - \hat{x}_2(0, i \cdot h + T)\right)^2$, $h = 0.001$.

Table 8. Mean-square values of the filtering errors $x_1(t+T) - \hat{x}_1(0, t+T)$ and $x_2(t+T) - \hat{x}_2(0, t+T)$, $\mathrm{T} = 1$, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.4^2)$, and

$N(0, 0.5^2)$

White Gaussian observation noise	MSV of $x_1(t+T) - \hat{x}_1(0, t+T)$	MSV of $x_2(t+T) - \hat{x}_2(0, t+T)$
$N(0, 0.3^2)$	1.39703×10^{-1}	9.62447×10^{-2}
$N(0, 0.4^2)$	2.21413×10^{-1}	8.99001×10^{-2}
$N(0, 5^2)$	3.04095×10^{-1}	9.64902×10^{-2}

The MSVs of the robust RLS FIR filter for $T = 1$ in Table 7 are almost identical to those of the robust RLS filter in Table 8 for each observation noise. The results show that as the fixed interval T increases, the robust RLS FIR filter achieves accuracy similar to that of the robust RLS filter.

6 Conclusion

In my previous research, I designed the robust RLS Wiener filter for linear continuous-time stochastic systems with uncertainties. This paper presents the robust RLS FIR filter for linear continuous-time stochastic systems with uncertain parameters in both the system and observation matrices. One of the main features of the robust RLS FIR filter is the use of cross-covariance information between the system state and the degraded observed value, as well as the covariance function of the degraded state. For the robust filtering problems in linear continuous-time stochastic systems with uncertainties, this paper uses the state-space model of the observable companion form for the degraded signal. (6) and (7) give the estimates of the system and observation matrices for the uncertain state, respectively. Usages of the covariance information $\alpha(t)$, $\beta(t)$, $\breve{C}(t)$, and $\breve{D}(t)$ characterize the current robust RLS FIR filter as described in Theorem 1.

The numerical simulation examples have demonstrated the effectiveness of the proposed robust RLS FIR filtering algorithms. As the variance of the white Gaussian observation noise increases, the estimation accuracy of the robust RLS FIR filter decreases. The robust RLS FIR filter estimates the signal and the system state recursively based on observations over the finite time interval as time passes. As the length of the finite observation interval increases, the estimation accuracy of the robust RLS FIR filter improves. The MSVs of the robust RLS FIR filter for $T = 1$ in Table 4 and Table 7 are almost identical to those of the robust RLS filter in Table 5 and Table 8, respectively, for each observation noise. The results show that as the observation interval increases, the robust RLS FIR filter achieves accuracy similar to that of the robust RLS filter.

The second example is simulated on the second-order mass-spring system driven by zero-mean white Gaussian noise. The proposed robust FIR filter might be applicable to control problems with control input from the viewpoint of the separation principle between control and estimation.

References:

[1] E. G. Pale-Ramon, Y. S. Shmaliy, L. J. Morales-Mendoza, M. G. Lee, *Finite Impulse Response (FIR) Filters and Kalman Filter for Object Tracking Process*, In: S. Shmaliy, Y., Abdelnaby Zekry, A. (eds) *6th International Technical Conference on Advances in Computing, Control and Industrial Engineering (CCIE 2021). CCIE 2021, Hangzhou, China. Lecture Notes in Electrical Engineering,* Vol 920, 2022, Springer, Singapore. DOI: 10.1007/978-981-19-3927-3_66.

[2] P. S. Kim and M. E. Lee, A new FIR filter for state estimation and its application, *J Comput Sci Technol*, Vol. 22, 2007, pp. 779–784. DOI: 10.1007/s11390-007-9085-8.

[3] O. Ibarra-Manzano, J. A. Andrade-Lucio, Y. S. Shmaliy, Y. Xu, One-step predictive H2 FIR tracking under persistent disturbances and data errors, *WSEAS Transactions on Signal Processing*, Vol. 17, 2021, pp. 87-92, DOI:10.37394/232014.2021.17.12.

[4] O. Ibarra-Manzano, J. A. Andrade-Lucio, Y. S. Shmaliy, State estimation using UFIR and Kalman filtering in applications to local crystal and master clocks, *WSEAS Transactions on Circuits And Systems*, Vol. 18, 2019, pp. 96-101.

[5] S. Zhao, Y. S. Shmaliy, An approach to optimal filtering of time-variant systems via finite measurements, *Proceedings of the* 10^{th} *International Conference on Circuits, Systems, Signal and Telecommunications, CSST '16*, 2016, pp.80–85. Barcelona, Spain.

[6] W. Kwon, O. Kwon, FIR filters and recursive forms for continuous time-invariant state-space models, *IEEE Transactions on Automatic Control*, Vol. 32, No. 4, 1987, pp. 352–356, DOI: 10.1109/TAC.1987.1104606.

[7] Z. Quan, S. Han, J. H. Park, W. H. Kwon, Robust FIR filters for linear continuous-time state-space models with uncertainties, *IEEE Signal Processing Letters*, Vol. 15, 2008, pp. 621–624, DOI: 10.1109/LSP.2008.2004515.

[8] D. K. Lee, H. H. Kang, J. M. Pak, C. K. Ahn, Continuous-time optimal unbiased FIR filter

for input-delayed systems, *International Journal of Robust and Nonlinear Control*, Vol. 33, No. 2, 2023, pp. 1027–1042, DOI: 10.1002/rnc.6416.

[9] S. Nakamori, Design of RLS-FIR filter using covariance information in linear continuous-time stochastic systems, *Applied Mathematics and Computation*, Vol. 219, No. 17, 2013, pp. 9598–9608, DOI: 10.1016/j.amc.2013.03.022.

[10] W. H. Kwon, P. S. Kim, S. H. Han, A receding horizon unbiased FIR filter for discrete-time state space models, *Automatica*, Vol. 38, No. 3, 2002, pp. 545–551, DOI: 10.1016/S0005-1098(01)00242-4.

[11] S. Zhao, Y. S. Shmaliy, F. Liu, Fast computation of discrete optimal FIR estimates in white Gaussian noise, *IEEE Signal Process. Lett.*, Vol. 22, No. 6, 2015, pp. 718–722, DOI: 10.1109/LSP.2014.2368777.

[12] S. Zhao, Y. S. Shmaliy, F. Liu, Fast Kalman-like optimal unbiased FIR filtering with applications, *IEEE Transactions on Signal Processing*, Vol. 64, No. 9, 2016, pp. 2284–2297, DOI: 10.1109/TSP.2016.2516960.

[13] Y. S. Shmaliy, S. Khan, S. Zhao, Ultimate iterative UFIR filtering algorithm, *Measurement*, Vol. 92, 2016, pp. 236–242, DOI: 10.1016/j.measurement.2016.06.029.

[14] S. Zhao, Y. S. Shmaliy, G. Ji, S. H. Khan, Fast bias-constrained optimal FIR filtering for time-invariant state space models, *Adaptive Control & Signal*, Vol. 31, No. 7, 2017, pp. 1061–1076, DOI: 10.1002/acs.2747.

[15] S. Zhao, Y. Shmaliy, F. Liu, Optimal FIR filter for discrete-time LTV systems and fast iterative algorithm, *IEEE Transactions on Circuits and Systems II: Express Briefs*, Vol. 68, No. 4, 2021, pp. 1527–1531, DOI: 10.1109/TCSII.2020.3021674.

[16] Y. S. Shmaliy, Optimal gains of FIR estimators for a class of discrete-time state-space models, *IEEE Signal Processing Letters*, Vol. 15, 2008, pp. 517–520, DOI: 10.1109/LSP.2008.925746.

[17] Y. S. Shmaliy, Linear optimal FIR estimation of discrete time-invariant state-space models, *IEEE Transactions on Signal Processing*, Vol. 58, No. 6, 2010, pp. 3086–3096, DOI: 10.1109/TSP.2010.2045422.

[18] Y. S. Shmaliy, O. Ibarra-Manzano, Time-variant linear optimal finite impulse response estimator for discrete state-space models, *International Journal of Adaptive Control and Signal Processing*, Vol. 26, No. 2, 2012, pp. 95–104, DOI: 10.1002/acs.1274.

[19] Y. Levinson, L. Mirkin, L^2 Optimization in discrete FIR estimation: Exploiting state-space structure, *SIAM J. Control Optim.*, Vol. 51, No. 1, 2013, pp. 419–441, DOI: 10.1137/110845185.

[20] S. Zhao, F. Liu, Y. S. Shmaliy, Optimal FIR estimator for discrete time-variant state-space model, *2014 11th International Conference on Electrical Engineering, Computing Science and Automatic Control*, *CCE 2014*, 2014, Ciudad del Carmen, Campeche. México. DOI: 10.1109/ICEEE.2014.6978270.

[21] S. Zhao, Y. S. Shmaliy, B. Huang, F. Liu, Minimum variance unbiased FIR filter for discrete time-variant systems, *Automatica*, Vol. 53, 2015, pp. 355–361, DOI: 10.1016/j.automatica.2015.01.022.

[22] C. K. Ahn, Strictly passive FIR filtering for state-space models with external disturbance, *AEU - International Journal of Electronics and Communications*, Vol. 66, No. 11, 2012, pp. 944–948, DOI: 10.1016/j.aeue.2012.04.002.

[23] C. K. Ahn, P. S. Kim, Fixed-lag maximum likelihood FIR smoother for state-space models, *IEICE Electron. Express*, Vol. 5, No. 1, 2008, pp. 11–16, DOI: 10.1587/elex.5.11.

[24] S. Nakamori, Robust RLS Wiener FIR filter for signal estimation in linear discrete-time stochastic systems with uncertain parameters, *Frontiers in Signal Processing*, 2019, Vol. 3, No. 2, pp. 19–36, [Online]. https://www.academia.edu/94119566/Robust_RLS_Wiener_FIR_Filter_for_Signal_Estimation_in_Linear_Discrete_Time_Stochastic_Systems_with_Uncertain_Parameters (Accessed Date: December 28, 2024) .

[25] Z. Pan, B. Huang, F. Liu, A Koopman, Operator-based finite impulse response filter for nonlinear systems, *2023 62nd IEEE Conference on Decision and Control* (*CDC*), 2023, pp. 2159–2165. Marina Bay Sands, Singapore. https://doi.org/10.1109/CDC49753.2023.10383222.

[26] Z. Pan, S. Zhao, B. Huang, F. Liu, Confidence set-membership FIR filter for discrete time-variant systems, *Automatica*, Vol. 157, 2023, Article 111231, DOI: 10.1016/j.automatica.2023.111231.

[27] D. Simon, Y. S. Shmaliy, Unified forms for Kalman and finite impulse response filtering and smoothing, *Automatica*, Vol. 49, 2013, pp. 1892–1899, DOI: 10.1016/j.automatica.2013.02.026.

[28] K. J. Uribe-Murcia, Y. S. Shmaliy, C. K. Ahn,

S. Zhao, Unbiased FIR filtering for time-stamped discretely delayed and missing data, *IEEE Trans. Automat. Contr.*, Vol. 65, No. 5, 2020, pp. 2155–2162, DOI: 10.1109/TAC.2019.2937850.

[29] K. Uribe-Murcia, Y. S. Shmaliy, UFIR state estimator for network systems with two-step delayed and lost data, *WSEAS Transactions on Signal Processing*, Vol. 17, 2021, pp. 81-86, DOI:10.37394/232014.2021.17.11.

[30] Y. Zhu, S. Zhao, X. Luan, F. Liu, Iterative maximum likelihood FIR filter for state-space models with time-stamped delayed and missing data, *Circuits Syst Signal Process*, Vol. 41, No. 9, 2022, pp. 5065–5083, DOI: 10.1007/s00034-022-02021-7.

[31] Y. Zhu, S. Zhao, X. Luan, F. Liu, Maximum likelihood FIR filter for systems with Bernoulli distributed random delays and measurement failures, *Digital Signal Processing*, Vol. 134, 2023, Article 103904, DOI: 10.1016/j.dsp.2022.103904.

[32] S. Zhao, Y. S. Shmaliy, F. Liu, S. H. Khan, Unbiased, optimal, and in-betweens: the trade-off in discrete finite impulse response filtering, *IET Signal Processing*, Vol. 10, No. 4, 2016, pp. 325–334, DOI: 10.1049/iet-spr.2015.0360.

[33] S. Zhao, Y. S. Shmaliy, J. A. Andrade-Lucio, Backward optimal FIR filtering and recursive forms for discrete LTV processes, *Signal Processing*, Vol. 180, 2021, Article 107857, DOI: 10.1016/j.sigpro.2020.107857.

[34] S. Nakamori, Least-squares finite impulse response smoother estimating signal at start time of fixed interval in linear discrete-time stochastic systems, *Scholars Journal of Research in Mathematics and Computer Science*, Vol. 2, No. 2, 2018, pp. 159-172, [Online]. https://www.academia.edu/103125528/SCHOLARS_SCITECH_RESEARCH_ORGANIZATION_Least_Squares_Finite_Impulse_Response_Smoother_Estimating_Signal_at_Start_Time_of_Fixed_Interval_in_Linear_Discrete_Time_Stochastic_Systems (Accessed Date: December 28, 2024).

[35] Y. S. Shmaliy, S. Zhao, *Optimal and Robust State Estimation: Finite Impulse Response (FIR) and Kalman Approaches*. John Wiley & Sons, Inc., 2022, DOI: 10.1002/9781119863106.

[36] S. Nakamori, Robust recursive least-squares Wiener filter for linear continuous-time uncertain stochastic systems, *WSEAS Transactions on Signal Processing*, Vol. 19, 2023, pp. 108-117, https://doi.org/10.37394/232014.2023.19.12.

[37] C.K. Chui, G. Chen, Orthogonal Projection and Kalman Filter. In: *Kalman Filtering*. Springer Series in Information Sciences, Vol. 17, Springer, Berlin, Heidelberg, 1999, https://doi.org/10.1007/978-3-662-03859-8_3.

[38] F. Lewis, L. Xie, D. Popa, *Optimal and Robust Estimation*, 2nd Edition, CRC Press, Boca Raton, FL, 2008, DOI: 10.1201/9781315221656.

[39] D. Simon, *Optimal State Estimation: Kalman, H∞, and Nonlinear Approaches*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2006, DOI: 10.1002/0470045345.

[40] A. P. Sage, J. L. Melsa, *Estimation Theory with Applications to Communication and Control*, *McGraw-Hill series in systems science*, McGraw-Hill, Inc., New York, 1971.

[41] M. S. Grewal, A. P. Andrews, *Kalman Filtering: Theory and Practice Using Matlab*, Fourth Edition, John Wiley & Sons, Inc., New York, 2015.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDIX

Proof of Theorem 1

Introducing an integral equation

$$J(t+T,s)R=\beta^T(s)-\int_t^{t+T}J(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau, \quad \text{(A-1)}$$

from (2), we obtain (A-2) for the optimal impulse response function.

$$h(t+T,s))=\alpha(t+T)J(t+T,s) \quad \text{(A-2)}$$

Differentiating (A-1) with respect to t, we have

$$\frac{\partial J(t+T,s)}{\partial t}=-J(t+T,t+T)\breve{H}\breve{C}(t+T)L(t+T,s)+J(t+T,t)\breve{H}\breve{D}(t)\breve{J}(t+T,s). \quad \text{(A-3)}$$

Here, $L(t+T,s)$ satisfies

$$L(t+T,s)R=\breve{D}^T(s)\,\breve{H}^T-\int_t^{t+T}L(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-4)}$$

$\breve{J}(t+T,s)$ satisfies

$$\breve{J}(t+T,s)R=\breve{C}^T(s)\,\breve{H}^T-\int_t^{t+T}\breve{J}(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-5)}$$

From (A-1), $J(t+T,t+T)$ satisfies

$$J(t+T,t+T)R=\beta^T(t+T)-\int_t^{t+T}J(t+T,\tau)\breve{H}\,K_{\breve{x}}(\tau,t+T)\breve{H}^T d\tau. \quad \text{(A-6)}$$

From (14), (A-6) becomes

$$J(t+T,t+T)R=\beta^T(t+T)-\int_t^{t+T}J(t+T,\tau)\breve{H}\,\breve{D}(\tau)\breve{C}^T(t+T)\breve{H}^T d\tau. \quad \text{(A-7)}$$

Introducing

$$r(t,t+T)=\int_t^{t+T}J(t+T,\tau)\breve{H}\breve{D}(\tau)d\tau, \quad \text{(A-8)}$$

(A-7) becomes

$$J(t+T,t+T)R=\beta^T(t+T)-r(t,t+T)\breve{C}^T(t+T)\breve{H}^T. \quad \text{(A-9)}$$

From (A-1), $J(t+T,t)$ satisfies

$$J(t+T,t)R=\beta^T(t)-\int_t^{t+T}J(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,t)\breve{H}^T d\tau. \quad \text{(A-10)}$$

From (14), (A-10) becomes

$$J(t+T,t)R=\beta^T(t)-\int_t^{t+T}J(t+T,\tau)\breve{H}\breve{C}(\tau)d\tau\breve{D}^T(t)\breve{H}^T. \quad \text{(A-11)}$$

Introducing

$$\breve{r}(t,t+T)=\int_t^{t+T}J(t+T,\tau)\breve{H}\breve{C}(\tau)d\tau, \quad \text{(A-12)}$$

(A-11) becomes

$$J(t+T,t)R=\beta^T(t)-\breve{r}(t,t+T)\breve{D}^T(t)\breve{H}^T. \quad \text{(A-13)}$$

Differentiating (A-8) with respect to t, we have

$$\frac{dr(t,t+T)}{dt}=J(t+T,t+T)\breve{H}\breve{D}(t+T)-J(t+T,t)\breve{H}\breve{D}(t)+\int_t^{t+T}\frac{\partial J(t+T,\tau)}{\partial t}\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-14)}$$

Substituting (A-3) into (A-14) and introducing functions

$$S(t,t+T)=\int_t^{t+T}L(t+T,\tau)\breve{H}\breve{D}(\tau)d\tau \quad \text{(A-15)}$$

and

$$\overleftrightarrow{r}(t,t+T)=\int_t^{t+T}\breve{J}(t+T,\tau)\breve{H}\breve{D}(\tau)d\tau, \quad \text{(A-16)}$$

we obtain

$$\frac{dr(t,t+T)}{dt}=J(t+T,t+T)(\breve{H}\breve{D}(t+T)-\breve{H}\breve{C}(t+T)S(t,t+T))-J(t+T,t)(\breve{H}\breve{D}(t)-\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T)). \quad \text{(A-17)}$$

Differentiating (A-12) with respect to t, we have

$$\frac{d\breve{r}(t,t+T)}{dt}=J(t+T,t+T)\breve{H}\breve{C}(t+T)-J(t+T,t)\breve{H}\breve{C}(t)+\int_t^{t+T}\frac{\partial J(t+T,\tau)}{\partial t}\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-18)}$$

Substituting (A-3) into (A-18) and introducing

$$\breve{p}(t,t+T)=\int_t^{t+T}L(t+T,\tau)\breve{H}\breve{C}(\tau)d\tau \quad \text{(A-19)}$$

and

$$\overleftrightarrow{\Gamma}(t,t+T)=\int_t^{t+T}\check{J}(t+T,\tau)\breve{H}\check{C}(\tau)d\tau, \quad \text{(A-20)}$$

we obtain

$$\begin{aligned}&\frac{d\check{r}(t,t+T)}{dt}\\&= J(t+T,t+T)(\breve{H}\check{C}(t+T)\\&-\breve{H}\check{C}(t+T)\breve{p}(t,t+T))\\&-J(t+T,t)(\breve{H}\check{C}(t)\\&\qquad -\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T)).\end{aligned} \quad \text{(A-21)}$$

Differentiating (A-4) with respect to t, we have

$$\begin{aligned}&\frac{\partial L(t+T,s)}{\partial t}R\\&=-L(t+T,t+T)\breve{H}K_{\breve{x}}(t+T,s)\breve{H}^T\\&+L(t+T,t)\breve{H}K_{\breve{x}}(t,s)\breve{H}^T\\&-\int_t^{t+T}\frac{\partial L(t+T,\tau)}{\partial t}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau.\end{aligned} \quad \text{(A-22)}$$

From (14) and (A-5), we obtain

$$\begin{aligned}&\frac{\partial L(t+T,s)}{\partial t}\\&=-L(t+T,t+T)\breve{H}\check{C}(t+T)L(t\\&\qquad +T,s)\\&+L(t+T,t)\breve{H}\breve{D}(t)\check{J}(t+T,s).\end{aligned} \quad \text{(A-23)}$$

Differentiating (A-5) with respect to t, we have

$$\begin{aligned}&\frac{\partial \check{J}(t+T,s)}{\partial t}R\\&=-\check{J}(t+T,t+T)\breve{H}K_{\breve{x}}(t+T,s)\breve{H}^T\\&+\check{J}(t+T,t)\breve{H}K_{\breve{x}}(t,s)\breve{H}^T\\&-\int_t^{t+T}\frac{\partial \check{J}(t+T,\tau)}{\partial t}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau.\end{aligned} \quad \text{(A-24)}$$

From (14), (A-4) and (A-5), we obtain

$$\begin{aligned}&\frac{\partial \check{J}(t+T,s)}{\partial t}\\&=-\check{J}(t+T,t+T)\breve{H}\check{C}(t+T)L(t\\&\qquad +T,s)\\&+\check{J}(t+T,t)\breve{H}\breve{D}(t)\check{J}(t+T,s).\end{aligned} \quad \text{(A-25)}$$

From (A-5), $\check{J}(t+T,t+T)$ satisfies

$$\begin{aligned}&\check{J}(t+T,t+T)R=\check{C}^T(t+T)\breve{H}^T\\&-\int_t^{t+T}\check{J}(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,t\\&\qquad +T)\breve{H}^T d\tau.\end{aligned} \quad \text{(A-26)}$$

Substituting (14) into (A-26) and using (A-16), (A-26) becomes

$$\begin{aligned}&\check{J}(t+T,t+T)R=\check{C}^T(t+T)\breve{H}^T\\&-\overleftrightarrow{r}(t,t+T)\check{C}^T(t+T)\breve{H}^T.\end{aligned} \quad \text{(A-27)}$$

From (A-5), $\check{J}(t+T,t)$ satisfies

$$\begin{aligned}&\check{J}(t+T,t)R=\check{C}^T(t)\ \breve{H}^T\\&-\int_t^{t+T}\check{J}(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,t)\breve{H}^T d\tau.\end{aligned} \quad \text{(A-28)}$$

Substituting (14) into (A-28) and introducing

$$\begin{aligned}\overleftrightarrow{\Gamma}(t,t+T)&=\int_t^{t+T}\check{J}(t\\&\qquad +T,\tau)\breve{H}\breve{A}(\tau)d\tau,\end{aligned} \quad \text{(A-29)}$$

(A-28) becomes

$$\begin{aligned}&\check{J}(t+T,t)R=\check{C}^T(t)\ \breve{H}^T\\&-\overleftrightarrow{\Gamma}(t,t+T)\breve{D}^T(t)\breve{H}^T.\end{aligned} \quad \text{(A-30)}$$

Differentiating (A-16) with respect to t, we have

$$\begin{aligned}&\frac{d\overleftrightarrow{r}(t,t+T)}{dt}\\&= \check{J}(t+T,t+T)\breve{H}\breve{D}(t+T)\\&-\check{J}(t+T,t)\breve{H}\breve{D}(t)\\&+\int_t^{t+T}\frac{\partial \check{J}(t+T,\tau)}{\partial t}\breve{H}\breve{D}(\tau)d\tau.\end{aligned} \quad \text{(A-31)}$$

Substituting (A-25) into (A-31) and introducing

$$\begin{aligned}S(t,t+T)&=\int_t^{t+T}L(t\\&\qquad +T,\tau)\breve{H}\breve{D}(\tau)d\tau,\end{aligned} \quad \text{(A-32)}$$

we obtain

$$\begin{aligned}&\frac{d\overleftrightarrow{r}(t,t+T)}{dt}\\&= \check{J}(t+T,t+T)(\breve{H}\breve{D}(t+T)\\&-\breve{H}\check{C}(t+T)S(t,t+T))\\&-\check{J}(t+T,t)(\breve{H}\breve{D}(t)\\&\qquad -\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T)).\end{aligned} \quad \text{(A-33)}$$

Differentiating (A-19) with respect to t, we have

$$\begin{aligned}&\frac{d\breve{p}(t,t+T)}{dt}\\&= L(t+T,t+T)\breve{H}\check{C}(t+T)\\&-L(t+T,t)\breve{H}\check{C}(t)\\&+\int_t^{t+T}\frac{\partial L(t+T,\tau)}{\partial t}\breve{H}\check{C}(\tau)d\tau.\end{aligned} \quad \text{(A-34)}$$

Substituting (A-22) into (A-34) and using (A-20), we obtain

$$\begin{aligned}&\frac{d\breve{p}(t,t+T)}{dt}\\&= L(t+T,t+T)(\breve{H}\check{C}(t+T)\\&-\breve{H}\check{C}(t+T)\breve{p}(t,t+T))\\&-L(t+T,t)(\breve{H}\check{C}(t)\\&-\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T)).\end{aligned} \quad \text{(A-35)}$$

Differentiating (A-20) with respect to t, we have

$$\begin{aligned}\frac{d\overleftrightarrow{\Gamma}(t,t+T)}{dt} &= \breve{J}(t+T,t+T)\breve{H}\breve{C}(t+T)\\ &-\breve{J}(t+T,t)\breve{H}\breve{C}(t)\\ &+\int_{t}^{t+T}\frac{\partial\breve{J}(t+T,\tau)}{\partial t}\breve{H}\breve{C}(\tau)d\tau.\end{aligned} \tag{A-36}$$

From (A-25), (A-36) becomes

$$\begin{aligned}\frac{d\overleftrightarrow{\Gamma}(t,t+T)}{dt} &= \breve{J}(t+T,t+T)(\breve{H}\breve{C}(t+T)\\ &-\breve{H}\breve{C}(t+T)\breve{p}(t,t+T))\\ &-\breve{J}(t+T,t)(\breve{H}\breve{C}(t)\\ &-\breve{H}\breve{D}(t)\overleftrightarrow{\Gamma}(t,t+T)).\end{aligned} \tag{A-37}$$

Differentiating (A-32) with respect to t, we have

$$\begin{aligned}\frac{dS(t,t+T)}{dt} &= L(t+T,t+T)\breve{H}\breve{D}(t+T)\\ &-L(t+T,t)\breve{H}\breve{D}(t)\\ &+\int_{t}^{t+T}\frac{\partial L(t+T,\tau)}{\partial t}\breve{H}\breve{D}(\tau)d\tau.\end{aligned} \tag{A-38}$$

Substituting (A-23) into (A-38) and using (A-16), we obtain

$$\begin{aligned}\frac{dS(t,t+T)}{dt} &= L(t+T,t+T)(\breve{H}\breve{D}(t+T)\\ &-\breve{H}\breve{C}(t+T)S(t,t+T))\\ &-L(t+T,t)(\breve{H}\breve{D}(t)\\ &-\breve{H}\breve{D}(t)\overleftrightarrow{r}(t,t+T)).\end{aligned} \tag{A-39}$$

From (A-4), $L(t+T,t+T)$ satisfies

$$\begin{aligned}L(t+T,t+T)R &= \breve{D}^T(t+T)\,\breve{H}^T\\ &-\int_{t}^{t+T}L(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,t+T)\breve{H}^T d\tau.\end{aligned} \tag{A-40}$$

From (14) and (A-32), (A-40) becomes

$$\begin{aligned}L(t+T,t+T)R &= \breve{D}^T(t+T)\,\breve{H}^T\\ &-S(t,t+T)\breve{C}^T(t+T)\breve{H}^T.\end{aligned} \tag{A-41}$$

From (A-4), $L(t+T,t)$ satisfies

$$\begin{aligned}L(t+T,t)R &= \breve{D}^T(t)\,\breve{H}^T\\ &-\int_{t}^{t+T}L(t+T,\tau)\breve{H}K_{\breve{x}}(\tau,t)\breve{H}^T d\tau.\end{aligned} \tag{A-42}$$

From (14) and (A-19), (A-42) becomes

$$\begin{aligned}L(t+T,t)R &= \breve{D}^T(t)\,\breve{H}^T\\ &-\breve{p}(t,t+T)\breve{D}^T(t)\breve{H}^T.\end{aligned} \tag{A-43}$$

The FIR filtering estimate $\hat{x}(t,t+T)$ of the state $x(t+T)$ is given by (8). Introducing a function

$$e(t,t+T) = \int_{t}^{t+T}J(t+T,s)\breve{y}(s)ds, \tag{A-44}$$

from (A-2), (8) becomes

$$\hat{x}(t,t+T) = \alpha(t+T)e(t,t+T). \tag{A-45}$$

Differentiating (A-44) with respect to t, we have

$$\begin{aligned}\frac{de(t,t+T)}{dt} &= J(t+T,t+T)\breve{y}(t+T)\\ &-J(t+T,t)\breve{y}(t)\\ &+\int_{t}^{t+T}\frac{\partial J(t+T,s)}{\partial t}\breve{y}(s)ds.\end{aligned} \tag{A-46}$$

Substituting (A-3) into (A-46) and introducing functions

$$f(t,t+T) = \int_{t}^{t+T}L(t+T,s)\breve{y}(s)ds \tag{A-47}$$

and

$$\breve{e}(t,t+T) = \int_{t}^{t+T}\breve{J}(t+T,s)\breve{y}(s)ds, \tag{A-48}$$

we obtain

$$\begin{aligned}\frac{de(t,t+T)}{dt} &= J(t+T,t+T)(\breve{y}(t+T)\\ &-\breve{H}\breve{C}(t+T)f(t,t+T))\\ &-J(t+T,t)(\breve{y}(t)\\ &-\breve{H}\breve{D}(t)\breve{e}(t,t+T)).\end{aligned} \tag{A-49}$$

Differentiating (A-47) with respect to t, we have

$$\begin{aligned}\frac{df(t,t+T)}{dt} &= L(t+T,t+T)\breve{y}(t+T)\\ &-L(t+T,t)\breve{y}(t)\\ &+\int_{t}^{t+T}\frac{\partial L(t+T,s)}{\partial t}\breve{y}(s)ds.\end{aligned} \tag{A-50}$$

Substituting (A-23) into (A-50) and using (A-48), we obtain

$$\begin{aligned}\frac{df(t,t+T)}{dt} &= L(t+T,t+T)(\breve{y}(t+T)\\ &-\breve{H}\breve{C}(t+T)f(t,t+T))\\ &-L(t+T,t)(\breve{y}(t)-\breve{H}\breve{D}(t)\breve{e}(t,t+T)).\end{aligned} \tag{A-51}$$

Differentiating (A-48) with respect to t, we have

$$\frac{d\breve{e}(t,t+T)}{dt} \tag{A-52}$$

$$= \breve{J}(t+T,t+T)\breve{y}(t+T) - \breve{J}(t+T,t)\breve{y}(t) + \int_t^{t+T} \frac{\partial \breve{J}(t+T,s)}{\partial t}\breve{y}(s)ds.$$

Substituting (A-25) into (A-52) and using (A-47), we obtain

$$\frac{d\breve{e}(t,t+T)}{dt} = \breve{J}(t+T,t+T)(\breve{y}(t+T) - \breve{H}\breve{C}(t+T)f(t,t+T)) - \breve{J}(t+T,t)(\breve{y}(t) - \breve{H}\breve{D}(t)\breve{e}(t,t+T)). \quad \text{(A-53)}$$

From (A-1), $J(T,s)$ satisfies

$$J(T,s)R = \beta^T(s) - \int_0^T J(T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-54)}$$

Differentiating (A-54) with respect to T, we have

$$\frac{dJ(T,s)}{dT}R = -J(T,T)\breve{H}K_{\breve{x}}(T,s)\breve{H}^T - \int_0^T \frac{\partial J(T,\tau)}{\partial T}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-55)}$$

By putting $t=0$ in (A-4), we have

$$L(T,s)R = \breve{D}^T(s)\,\breve{H}^T - \int_0^T L(T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-56)}$$

From (14), (A-55) and (A-56), we obtain

$$\frac{dJ(T,s)}{dT} = -J(T,T)\breve{H}\breve{C}(T)L(T,s). \quad \text{(A-57)}$$

From (A-8), $r(0,T)$ is given by

$$r(0,T) = \int_0^T J(T,\tau)\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-58)}$$

Differentiating (A-58) with respect to T we have

$$\frac{dr(0,T)}{dT} = J(T,T)\breve{H}\breve{D}(T) + \int_0^T \frac{\partial J(T,s)}{\partial T}\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-59)}$$

Substituting (A-57) into (A-59), we obtain

$$\frac{dr(0,T)}{dT} = J(T,T)(\breve{H}\breve{D}(T) - \breve{H}\breve{C}(T)S(0,T), \quad r(0,0)=0. \quad \text{(A-60)}$$

Here, from (A-32), $S(0,T)$ is given by

$$S(0,T) = \int_0^T L(T,\tau)\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-61)}$$

Differentiating (A-56) with respect to T, we have

$$\frac{L(T,s)}{\partial T}R = -L(T,T)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T - \int_0^T \frac{\partial L(T,\tau)}{\partial T}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-62)}$$

From (A-56) and (A-62), we obtain

$$\frac{L(T,s)}{\partial T} = -L(T,T)\breve{H}\breve{C}(T)L(T,s). \quad \text{(A-63)}$$

Differentiating (A-61) with respect to T, we have

$$\frac{dS(0,T)}{dT} = L(T,T)\breve{H}\breve{D}(T) + \int_0^T \frac{\partial L(T,\tau)}{\partial T}\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-64)}$$

Substituting (A-63) into (A-64) and using (A-61), we obtain

$$\frac{dS(0,T)}{dT} = L(T,T)(\breve{H}\breve{D}(T) - \breve{H}\breve{C}(T)S(0,T)),\ S(0,0)=0 \quad \text{(A-65)}$$

By putting $s=T$ in (A-54), we have

$$J(T,T)R = \beta^T(T) - \int_0^T J(T,\tau)\breve{H}K_{\breve{x}}(\tau,T)\breve{H}^T d\tau. \quad \text{(A-66)}$$

From (14) and (A-58), we obtain

$$J(T,T)R = \beta^T(T) - r(0,T)\breve{C}^T(T)\breve{H}^T. \quad \text{(A-67)}$$

By putting $s=T$ in (A-56), we have

$$L(T,T)R = \breve{D}^T(T)\,\breve{H}^T - \int_0^T L(T,\tau)\breve{H}K_{\breve{x}}(\tau,T)\breve{H}^T d\tau. \quad \text{(A-68)}$$

From (14) and (A-61), we obtain

$$L(T,T)R = \breve{D}^T(T)\,\breve{H}^T - S(0,T)\breve{C}^T(T)\breve{H}^T. \quad \text{(A-69)}$$

By putting $t=0$ in (A-5), we have

$$\breve{J}(T,s)R = \breve{C}^T(s)\,\breve{H}^T - \int_0^T \breve{J}(T,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-70)}$$

Differentiating (A-70) with respect to T, we have

$$\frac{d\breve{J}(T,s)}{dT} \quad \text{(A-71)}$$

$$= -\breve{J}(T,T)\breve{H}K_{\breve{x}}(T,s)\breve{H}^T - \int_0^T \frac{\partial \breve{J}(T,\tau)}{\partial T}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau.$$

From (A-56) and (A-71), we obtain

$$\frac{d\breve{J}(T,s)}{dT} = -\breve{J}(T,T)\breve{H}\breve{C}(T)L(T,s). \quad \text{(A-72)}$$

By putting t =0 in (A-16), $\overleftrightarrow{r}(0,T)$ is given by

$$\overleftrightarrow{r}(0,T) = \int_0^T \breve{J}(T,\tau)\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-73)}$$

Differentiating (A-73) with respect to T, we have

$$\frac{d\overleftrightarrow{r}(0,T)}{dT} = \breve{J}(T,T)\breve{H}\breve{D}(T) + \int_0^T \frac{\partial \breve{J}(T,\tau)}{\partial T}\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-74)}$$

Substituting (A-72) into (A-74) and using (A-61), we obtain

$$\frac{d\overleftrightarrow{r}(0,T)}{dT} = \breve{J}(T,T)(\breve{H}\breve{D}(T) - \breve{H}\breve{C}(T)S(0,T)),\ \overleftrightarrow{r}(0,0) = 0. \quad \text{(A-75)}$$

By putting t =0 in (A-19), $\breve{p}(0,T)$ is given by

$$\breve{p}(0,T) = \int_0^T L(T,\tau)\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-76)}$$

Differentiating (A-76) with respect to T, we have

$$\frac{d\breve{p}(0,T)}{dT} = L(T,T)\breve{H}\breve{C}(T) + \int_0^T \frac{\partial L(T,\tau)}{\partial T}\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-77)}$$

Substituting (A-63) into (A-77) and using (A-76), we obtain

$$\frac{d\breve{p}(0,T)}{dT} = L(T,T)(\breve{H}\breve{C}(T) - \breve{H}\breve{C}(T)\breve{p}(0,T)),\ \breve{p}(0,0) = 0. \quad \text{(A-78)}$$

By putting t =0 in (A-12), $\check{r}(0,T)$ is given by

$$\check{r}(0,T) = \int_0^T J(T,\tau)\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-79)}$$

Differentiating (A-79) with respect to T, we have

$$\frac{d\check{r}(0,T)}{dT} = J(T,T)\breve{H}\breve{C}(T) + \int_0^T \frac{\partial J(T,\tau)}{\partial T}\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-80)}$$

Substituting (A-57) into (A-80) and using (A-76), we obtain

$$\frac{d\check{r}(0,T)}{dT} = J(T,T)(\breve{H}\breve{C}(T) - \breve{H}\breve{C}(T)\breve{p}(0,T)),\ \check{r}(0,0) = 0. \quad \text{(A-81)}$$

By putting t =0 in (A-20), $\overleftrightarrow{\Gamma}(0,T)$ is given by

$$\overleftrightarrow{\Gamma}(0,T) = \int_0^T \breve{J}(T,\tau)\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-82)}$$

Differentiating (A-82) with respect to T, we have

$$\frac{d\overleftrightarrow{\Gamma}(0,T)}{dT} = \breve{J}(T,T)\breve{H}\breve{C}(T) + \int_0^T \frac{\partial \breve{J}(T,\tau)}{\partial T}\breve{H}\breve{C}(\tau)d\tau. \quad \text{(A-83)}$$

Substituting (A-72) into (A-83) and using (A-76), we obtain

$$\frac{d\overleftrightarrow{\Gamma}(0,T)}{dT} = \breve{J}(T,T)(\breve{H}\breve{C}(T) - \breve{H}\breve{C}(T)\breve{p}(0,T)),\ \overleftrightarrow{\Gamma}(0,0) = 0. \quad \text{(A-84)}$$

By putting $t = 0$ in (A-26), we have

$$\breve{J}(T,T)R = \breve{C}^T(T)\breve{H}^T - \int_0^T \breve{J}(T,\tau)\breve{H}K_{\breve{x}}(\tau,T)\breve{H}^T d\tau. \quad \text{(A-85)}$$

From (14) and (A-73), we obtain

$$\breve{J}(T,T)R = \breve{C}^T(T)\breve{H}^T - \overleftrightarrow{r}(0,T)\breve{C}^T(T)\breve{H}^T. \quad \text{(A-86)}$$

From (A-45), $\hat{x}(0,T)$ is given by

$$\hat{x}(0,T) = \alpha(T)e(0,T). \quad \text{(A-87)}$$

From (A-44), $e(0,T)$ is given by

$$e(0,T) = \int_0^T J(T,s)\breve{y}(s)ds. \quad \text{(A-88)}$$

Differentiating (A-88) with respect to T, we have

$$\frac{de(0,T)}{dT} = J(T,T)\breve{y}(T) + \int_0^T \frac{\partial J(T,s)}{\partial T}\breve{y}(s)ds. \quad \text{(A-89)}$$

Substituting (A-57) into (A-89) and introducing

$$f(0,T) = \int_0^T L(T,s)\breve{y}(s)ds, \quad \text{(A-90)}$$

we obtain

$$\frac{de(0,T)}{dT} = J(T,T)(\breve{y}(T) \quad \text{(A-91)}$$

DOI: 10.37394/232014.2024.20.11

$-\breve{H}\breve{C}(T)f(0,T)), e(0,0)=0.$

Differentiating (A-90) with respect to T, we have

$$\frac{df(0,T)}{dT} = L(T,T)\breve{y}(T) + \int_0^T \frac{\partial L(T,s)}{\partial T}\breve{y}(s)ds, \quad \text{(A-92)}$$

Substituting (A-63) into (A-92) and using (A-90), we obtain

$$\frac{df(0,T)}{dT} = L(T,T)\left(\breve{y}(T) - \breve{H}\breve{C}(T)f(0,T)\right), \quad \text{(A-93)}$$
$$f(0,0)=0.$$

By putting $t=0$ in (A-48), we have

$$\breve{e}(0,T) = \int_0^T \breve{J}(T,s)\breve{y}(s)ds. \quad \text{(A-94)}$$

Differentiating (A-94) with respect to T, we have

$$\frac{d\breve{e}(0,T)}{dT} = \breve{J}(T,T)\breve{y}(T) + \int_0^T \frac{\partial \breve{J}(T,s)}{\partial T}\breve{y}(s)ds. \quad \text{(A-95)}$$

Substituting (A-72) into (A-95) and using (A-90), we obtain

$$\frac{d\breve{e}(0,T)}{dT} = \breve{J}(T,T)\left(\breve{y}(T) - \breve{H}\breve{C}(T)f(0,T)\right), \quad \text{(A-96)}$$
$$\breve{e}(0,0)=0.$$

(Q.E.D.)