Stability Analysis of Linear Time-Delay Systems

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Abstract: - The time-delay issue has been a hot topic over the past two decades. In this paper, the problem of stability of linear systems having time delay is addressed. Presently, the system can have parameter uncertainties. The problem of parameter uncertainties is not widely addressed. The suggested method is easy and differs from the available solutions. In this paper, the stability of the system is dealt with using Lyapunov functions or functionals. Finally, several simulation examples have been established to highlight the performances of the suggested method.

Key-Words: - Linear system, time-delay system, variable time delay, functional approach, Lyapunov functions, stability analysis, delay compensation, Linear Matrix Inequalities (LMI).

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1 Introduction

The problem of time-delay systems has been a hot topic over the past two decades, [1], [2]. Time delay can have a significant impact on the system. It can have a stabilizing effect, [3]. However, it is, in many cases, a source of instability. Then, the study of time-delay systems with parameter uncertainties is more complicated. Subsequently, they are very little studied. Then, several works have been focused on the time-delay system issue, [4].

Time delays appear in many industrial systems, e.g., in medicine and chemical control systems, [5], [6]. The presence of non-linearity in the system complicates the study and the parameter estimation of systems, [7], [8], [9].

Delays are often observed in sensitive areas of security and communication technologies, [10]. The problem of parameter estimation of a system is very interesting and has been extensively studied, [11], [12].

Among the effective methods for studying system stability, is the one using the Lyapunov approach. For stability analysis of time-delay systems, an efficient method is based on the Lyapunov method, [13]. The stability of time-delay systems using the Lyapunov methods can be divided into two main classes. The first one is based on the Krasovskii method of Lyapunov functionals, and the second class is based on the Razumikhin method of Lyapunov functions, [14]. These two Lyapunov method classes for stability analysis of time-delay linear systems result generally in linear matrix inequality conditions. The linear matrix inequality to deal with the stability issue of time-delay linear systems provides constructive fine-dimensional conditions, despite significant model uncertainties.

Among the methods used for stability of time-delay systems are those based on observers, [15], [16]. In [17], the problem of multiple time-varying delays for linear systems is discussed. Nonlinear discrete-time system with time-varying delay has been dealt in [18]. In [19], the problem of impulsive stabilization for a class of time-delay systems in the presence of input disturbances has been addressed.

The delay time problem has been in several areas using a variety of techniques, [20], [21].

The presence of time delay in systems having parameter uncertainties can lead to system instability and make their study very difficult. In this paper, the problem of time-delay systems with parameter uncertainties is proposed.

The remainder of this paper is organized as follows. The problem statement is described in Section 2. The problem of stability of the time-delay system using the Lyapunov method is proposed in Section 3. The stability analysis using the Razumikhin method is discussed in Section 4. Finally, to show the effectiveness of the obtained results, examples of simulations and some concluding remarks are given in Section 5.

2 Problem Statement

Presently, the stability problem of time-delay systems is addressed. This study is considered when the system parameters are not constant. Then, the system under study is described as follows:

$$\dot{x}(t) = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + A_d\boldsymbol{x}\big(\boldsymbol{t} - \boldsymbol{d}(\boldsymbol{t})\big) \quad (1)$$

where $\boldsymbol{x}(\boldsymbol{t}) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{A}(\boldsymbol{t}) \in \mathbb{R}^{n\times n}$ is a matrix of uncertain arguments, $A_d \in \mathbb{R}^{n\times n}$ is a matrix of constant elements, and $\boldsymbol{d}(\boldsymbol{t}) \in \mathbb{R}$ is the time-delay, that is variable with respect to time. The time-delay is supposed to be bounded and time-varying function $\boldsymbol{d}(\boldsymbol{t}) \in [0 \quad D]$. It is shown that the time-delay affects the stability of the system. Specifically, let consider the following time-delay system:

$$\dot{x}(t) = A_d x\big(t - d(t)\big) \quad (2)$$

It is readily seen that the letter is asymptotically stable for $\boldsymbol{d}(\boldsymbol{t}) \in [0 \quad \pi/2]$. This system (described by (2)) becomes unstable for $\boldsymbol{d}(\boldsymbol{t}) > \pi/2$. Furthermore, the presence of a time-delay in the system can stabilize this system. For instance, let us consider the following time-delay system:

$$\ddot{y}(t) = -y(t) + y\big(t - \boldsymbol{d}(\boldsymbol{t})\big) \quad (3)$$

It is readily seen that, for $\boldsymbol{d}(\boldsymbol{t}) = \boldsymbol{0}$ (i.e., system without time delay), the system described by (3) is not stable. Unlike the case of $\boldsymbol{d}(\boldsymbol{t}) = \boldsymbol{0}$, the system becomes stable for $\boldsymbol{d}(\boldsymbol{t}) = \boldsymbol{1}$.

3 Stability Analysis using Lyapunov Method

In this section, the stability problem of time-delay system described by (1) is dealt. In this respect, method based on Lyapunov functions is considered to deal the system stability problem. Then, at this stage a crucial step in the stability study of system (1) consists of the choice of Lyapunov functional. This study is considered when the system parameters are not constant.

In this work, the following Lyapunov functional is proposed:

$$V\big(t, x(t)\big) = x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Qx(s)\,ds \quad (4)$$

where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are any positive defined matrices, noted $P > 0$ and $Q > 0$ respectively.

Lemma 1. Let consider the time-delay systems described by (1). Let suppose that the time-delay $\boldsymbol{d}(\boldsymbol{t})$ is differentiable function satisfying the condition $\dot{d}(t) \leq \bar{d} < 1$, i.e., the derivative $\dot{d}(t)$ is bounded of upper bound $\bar{d}$, which less than 1.

If there exist two positive matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that the following linear matrix inequality is satisfied:

$$\begin{bmatrix} PA + A^TP + Q & PA_d \\ {A_d}^TP & -(1-D)Q \end{bmatrix} < 0 \quad (5)$$

the time-delay system, given in (1), is thus asymptotically stable. □

Proof. Let consider the Lyapunov functional $V\big(t, x(t)\big)$ defined in (4). To show that the system (1) is asymptotically stable, it is sufficient to show that the derivative $\dot{V}\big(t, x(t)\big)$ of the Lyapunov functional $V\big(t, x(t)\big)$ under the assumption (5) is negative. In this respect, by differentiating this Lyapunov functional $V\big(t, x(t)\big)$ with respect to time, one gets:

$$\dot{V}\big(t, x(t)\big) = 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - \Big(1 - \dot{d}(t)\Big)x^T\big(t - d(t)\big)Qx\big(t - d(t)\big) \quad (6)$$

It is readily seen that the derivative $\dot{V}\big(t, x(t)\big)$ satisfies:

$$\dot{V}\big(t, x(t)\big) \leq 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - (1 - \bar{d})x^T\big(t - d(t)\big)Qx\big(t - d(t)\big) \quad (7)$$

The latter can be rewritten as follows:

$$\dot{V}\big(t, x(t)\big) \leq z^T\big(t, d(t)\big)\, R\, z\big(t, d(t)\big) \quad (8)$$

where:

$$z\big(t, d(t)\big) = [x^T(t) \quad x^T(t-\tau)]^T \quad (9a)$$

and:

$$R = \begin{bmatrix} PA + A^TP + Q & PA_d \\ {A_d}^TP & -(1-D)Q \end{bmatrix} \quad (9b)$$

The first submatrix $PA + A^TP + Q$ concerns the condition of the stability of the system dynamics without delay. If it is defined negative, the system is stable regardless of the delayed term. The two sub-matrices PA_d and ${A_d}^TP$ are related to the effect of time delay on our proposed system. More precisely, they represent the influence of the delayed state $x(t - d(t))$ on the system dynamics. The last term $-(1-D)Q$ controls the influence of time delay on stability and ensures that the amortization (introduced by Q) is strong enough to offset the effect of the delay.

To ensure system stability, matrix R must be negative defined. This means that the energies introduced by the delay and dynamics of the system are always offset by the global amortization.

It follows (5) (i.e., the matrix $R < 0$), one concludes that $\dot{V}\big(t, x(t)\big)$ is negative, which completes the proof of Lemma1. ■

4 Stability Analysis using Razumikhin Method

Presently, the stability problem of time-delay system (1) is addressed. In this section, a solution using Lyapunov based on Razumikhin method is presented.

The system under study is subject to the following assumptions:

Assumptions.

A1. Firstly, let consider the candidate Lyapunov function:

$$V\big(t, x(t)\big) = x^T(t) P x(t) \tag{10}$$

where this Lyapunov function is differentiable for any matrix $P > 0$.

A2. Assume that there exist two continuous positive functions $u\big(t, x(t)\big)$ and $v\big(t, x(t)\big)$, ($\mathbb{R}^+ \to \mathbb{R}^+$), where $v\big(t, x(t)\big)$ is strictly increasing, satisfying the following condition:

$$u\big(t, x(t)\big) \leq V\big(t, x(t)\big) \leq v\big(t, x(t)\big) \tag{11}$$

Lemma 2. Let's consider the time-delay systems described by (1) subjected to assumptions A1-2. If the Lyapunov function, defined by (10), satisfies the following condition:

$$V\big(t-\tau, x(t-\tau)\big) \leq V\big(t, x(t)\big)$$
$$for\ any\ \tau \in [0 \quad D] \tag{12}$$

where D is an upper bound of time-delay $\boldsymbol{d(t)}$ ($\boldsymbol{d(t)} \in [0 \quad D]$).

If there exists a continuous nondecreasing function $w\big(t, x(t)\big)$, which is positive such that:

$$w\big(t, x(t)\big) \leq -\dot{V}(t, x) \tag{13}$$

The time-delay system (1) is uniformly stable. ■

5 Simulation

To show the effectiveness of this study, numerical examples are established in this section.

Then, let's consider the following time-delay system:

$$\dot{x}(t) = \boldsymbol{A(t)x(t)} + A_d \boldsymbol{x\big(t - d(t)\big)} \tag{14}$$

where:

$$\boldsymbol{A(t)} = \boldsymbol{A} + A_\varepsilon(\boldsymbol{t}) \tag{15a}$$

That is, it is equal to a constant matrix $\boldsymbol{A}$ plus a variable matrix $A_\varepsilon(\boldsymbol{t})$, modeling the parameter uncertainties. In the simulation part, the state vector $\boldsymbol{x(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2$ is:

In this simulation, the matrices $\boldsymbol{A}$, $A_\varepsilon(\boldsymbol{t})$, and $A_d(\boldsymbol{t})$ are set, respectively, to:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{-2} \end{bmatrix};$$
$$A_\varepsilon(\boldsymbol{t}) = \begin{bmatrix} \boldsymbol{0.5sin(t)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0.5cos(t)} \end{bmatrix};$$
$$\boldsymbol{A_d} = \begin{bmatrix} \boldsymbol{0.3} & \boldsymbol{0} \\ \boldsymbol{0.2} & \boldsymbol{0.3} \end{bmatrix} \tag{15b}$$

The varying time delay is given by:

$$\boldsymbol{d(t) = 0.9 + 0.05\sin(0.2\pi t)} \tag{15c}$$

Firstly, suppose that the system is free time-delay, i.e., $\boldsymbol{d(t) = 0}$. Then, for an initial state of $\boldsymbol{x(0)} = \begin{pmatrix} x_{0_1} \\ x_{0_2} \end{pmatrix} = \begin{pmatrix} 0.2 \\ 1.02 \end{pmatrix}$, Figure 1 and Figure 2 show the evolution of the state $x_1(\boldsymbol{t})$ and $x_2(\boldsymbol{t})$ over time, respectively.

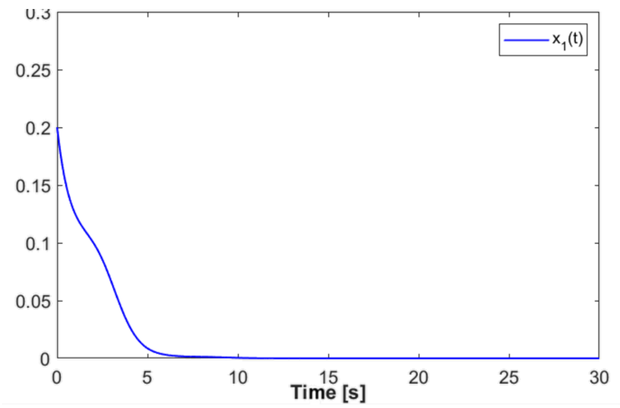

Fig. 1: Evolution of the state $x_1(t)$ versus time

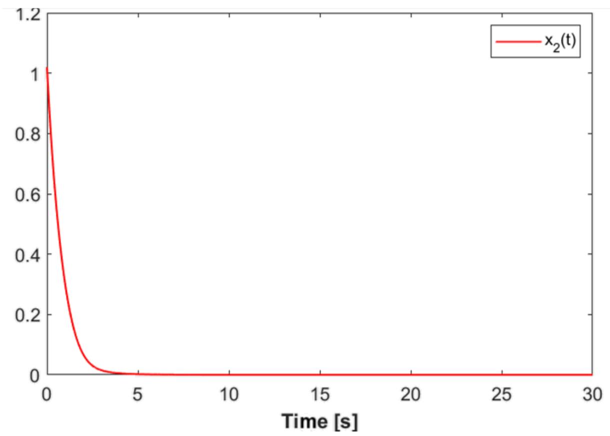

Fig. 2: Evolution of the state $x_2(t)$ versus time

In the following part of the simulation, a variable delay has been considered as described in (15c). The evolution of $\boldsymbol{x_1(t)}$ and $\boldsymbol{x_2(t)}$ are illustrated in Figure 3 and Figure 4.

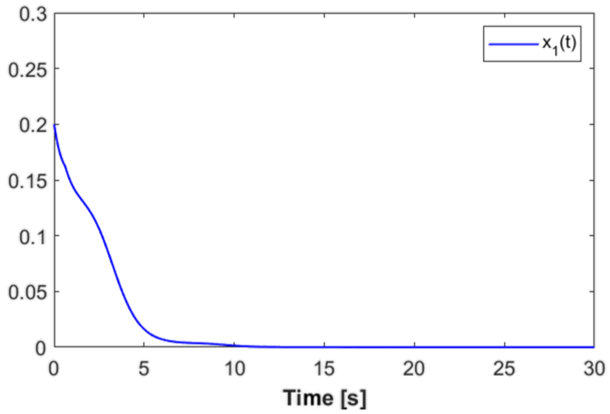

Fig. 3: Evolution of the state $x_1\ (t)$ versus time

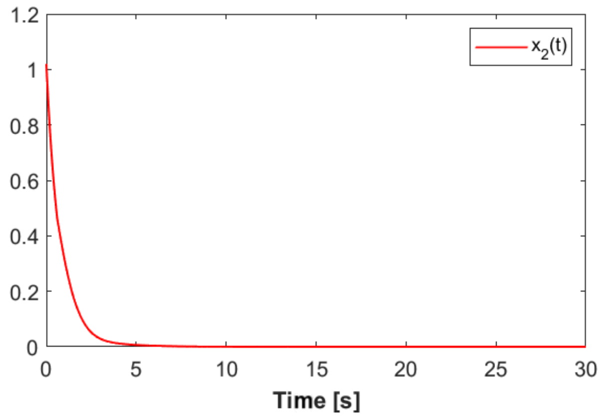

Fig. 4: Evolution of the state $x_2\ (t)$ versus time

Under the obtained condition, when the proposed system $(\mathbf{1})$ is free time-delay, Figure 1 and Figure 2 show that $\boldsymbol{x_1(t)}$ stabilizes after 9.3s and $\boldsymbol{x_2(t)}$ after 4.2s. In the case of system having a variable state delay, $\boldsymbol{x_1(t)}$ stabilizes after 10.5s and $\boldsymbol{x_2(t)}$after 5.6s as shown in Figure 3 and Figure 4. Figure 3 and Figure 4 demonstrate, respecting the condition on delay, that the time delay does not affect the system's stability when the time t approaches infinity. Which confirms the effectiveness of the proposed study.

6 Conclusion

In the present work, the problem of stability of linear systems having time delay is discussed.

Presently, the system can have parameter uncertainties. The problem of parameter uncertainties is not widely addressed. The suggested method is easy and differs from the available solutions. In this paper, the stability of system is dealt with using Lyapunov functions or functionals.

References:

[1] E. Fridman, Tutorial on Lyapunov-based methods for time-delay systems, *European Journal of Control*, vol. 20, no. 6, pp. 271-283, 2014. DOI: 10.1016/j.ejcon.2014.10.001.

[2] H. Zhang, Y. Li, X. Xie, J. Xia, A novel stability criterion and precise control for linear time delay systems based on regional configuration of the poles, *ISA Transactions*, vol. 148, pp. 349-357, 2024. DOI: 10.1016/j.isatra.2024.03.008.

[3] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*, Springer, 2014.

[4] T. Ahmed-Ali, I. Karafyllis, and F. Giri, Sampled-data observers for delay systems and hyperbolic PDE-ODE loops, *Automatica*, vol. 123, no. 109349, 2021. DOI: 10.1016/j.automatica.2020.109349.

[5] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[6] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer, Dordrecht, 1999.

[7] A. Brouri, Wiener-Hammerstein nonlinear system identification using spectral analysis, *International Journal of Robust and Nonlinear Control*, vol. 32, no. 10, pp, 2022. 6184-6204. DOI: 10.1002/rnc.6135.

[8] A. Brouri and F. Giri, Identification of series-parallel systems composed of linear and nonlinear blocks, *International Journal of Adaptive Control and Signal Processing*, vol. 37, no. 8, pp. 2021-2040, 2023. DOI: 10.1002/acs.3624.

[9] F. Li, L. Jia, Parameter estimation of Hammerstein–Wiener nonlinear system with noise using special test signals, *Neurocomputing*, vol. 344, pp. 37-48, 2019. DOI: 10.1016/j.neucom.2018.02.108.

[10] J.P. Richard, Time-delay systems: an overview of some recent advances and open problems, *Automatica*, vol. 39, pp. 1667-1694, 2003. DOI: 10.1016/S0005-1098(03)00167-5.

[11] Krasovskii, *Stability of Motion,* Stanford University Press, Redwood City, 1963.

[12] B. Razumikhin, On the stability of systems with a delay, *Prikladnaya Matematika i Mekhanika*, vol. 20, pp. 500-512, 1956.

[13] H.-B. Zeng, J.H. Park, J.-W. Xia, S.-P. Xiao, Improved delay-dependent stability criteria for T-S fuzzy systems with time-varying delay, *Applied Mathematics and Computation*,

vol. 235, pp. 492-501, 2014. DOI: 10.1016/j.amc.2014.03.005.

[14] Tarek Ahmed-Ali, Qinghua Zhang, Fouad Giri, Xingwen Liu. Transforming Time-Delay System Observers to Adaptive Observers. *IFAC 2020- IFAC World Congress,* Berlin, Germany. pp.1-5, 2020. hal-02971638.

[15] S. Syafiie, Controllers and observer synthesis for linear systems with multiple time-varying delays in range, *IFAC Journal of Systems and Control*, vol. 27, no. 2, 100257, 2024. DOI: 10.1016/j.ifacsc.2024.100257.

[16] D. Yali, C. Laijun, & M. Shengwei, Stability analysis and observer design for discrete-time systems with interval time-varying delay, *Optimal Control Applications & Methods*, 37, 340-358, 2016. DOI: 10.1002/oca.2171.

[17] W.H. Chen, W. Xu, W. X. Zheng, Sliding-mode-based impulsive control for a class of time-delay systems with input disturbance, *Automatica*, vol. 164, 111633, 2024. DOI: 10.1016/j.automatica.2024.111633.

[18] Z. Xindan, Z. Jianping, H. Yanren, Finite element error estimation for parabolic optimal control problems with time delay, *Applied Numerical Mathematics,* vol. 212, pp. 176-196, 2025. DOI: 10.1016/j.apnum.2025.02.002.

[19] Y.C. Kutlu, V. Tomáš, B. Jaroslav, H. Martin, N. Silviu-Iulian, A spectrum-based filter design for periodic control of systems with time delay, *Journal of Sound and Vibration*, vol. 604, 118959, 2025. DOI: 10.1016/j.jsv.2025.118959.

[20] F. Liu, G. Liu, H. Liu, S. Huo, Extended dissipative analysis of integral sliding mode control for singular interval type-2 fuzzy systems with time delay, *Applied Mathematics and Computation*, vol. 493, 12925, 2025. DOI: 10.1016/j.amc.2024.129251.

[21] D. Wang, H. Yang, L. Yang, Chaos and bifurcation analysis of tumor-immune controlled system with time delay, *Alexandria Engineering Journal*, vol. 119, pp. 267-284, 2025. DOI: 10.1016/j.aej.2025.01.081.

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Conflict of Interest

The authors have no conflicts of interest to declare.