Complex Information Filter and Complex Kalman Filter Comparison: Selection of the Faster Filter

ATHANASIOS POLYZOS[1], CHRISTOS TSINOS[1], MARIA ADAM[2], NICHOLAS ASSIMAKIS[1]
[1]Department of Digital Industry Technologies,
National and Kapodistrian University of Athens,
34400 Psachna Evias,
GREECE

[2]Department of Computer Science and Biomedical Informatics,
University of Thessaly,
2-4 Papasiopoulou Str., 35131, Lamia,
GREECE

Abstract: - Complex Kalman filters are used in complex signal processing. A comparison between the complex Kalman filter and the complex Information filter is presented in the general case of discrete-time widely linear models. The complex Kalman filter and the complex Information filter compute iteratively the same estimations. The computational burdens of these complex filters are determined and a method is derived to decide which filter is the faster one, taking into account only the model dimensions.

Key-Words: - Linear estimation, Widely linear model, Complex Kalman filter, Complex Information filter, Complex signals, Computational burden.

Received: April 19, 2024. Revised: August 24, 2024. Accepted: September 15, 2024. Published: November 13, 2024.

1 Introduction

Kalman filter [1], [2] and Information filter [2], [3] are known estimation algorithms that have been successfully used in many linear estimation problems: applications of Kalman filter are referred in [2], [4], [5], [6] and applications of Information filter are referred in [3], [7], [8], [9], [10], [11]. Applications of complex Kalman filter include tracking, oceanography, array processing, communications, biomedicine [6], [12], tracking for Global Navigation Satellite System (GNSS) meta-signals [13], power system frequency [14], [15], unbalanced grids [16], proper and improper signals applications [17], two-dimensional local navigation system using complex Kalman and particle filters [18].

The basic statistical properties of a complex signal are (a) the covariance matrix that concerns the total power of the complex signal and (b) the pseudo-covariance matrix (complementary covariance) that concerns the correlations between the real and imaginary parts of the complex signal, [12]. In linear estimation, the complex Kalman filter is derived assuming (a) the traditional state space model which takes into account the covariance matrix only; the derived conventional complex Kalman filter (CCKF) takes into account the covariance matrix only and hence it is applicable to proper (circular) signals (b) the augmented model or widely linear model, which takes into account the covariance as well as the pseudo-covariance matrices; the derived augmented complex Kalman filter (ACKF) is applicable to improper (non-circular) complex signals that are correlated with their complex conjugates. It is worth noting that the use of the pseudo-covariance matrix in ACKF can improve the performance of CCKF, [19].

Motivated by minimizing the computational time, the paper derives the augmented complex Information filter (ACIF) from the equations of the augmented complex Kalman filter. A comparison between the complex Kalman filter and the complex Information filter is presented in this paper. The origin of this idea is a comparison of the corresponding filters in linear estimation, where real signals are involved, [3]. The key contributions of this paper are as follows: a) the calculation burdens of the augmented complex Kalman and complex Information filters are derived, b) a method is derived to determine the faster complex filter.

2 Augmented Model

Let a complex variable x. The complex conjugate is denoted as $\bar{x}$. Let a complex matrix M. The transpose matrix is denoted as M^T and the conjugate transpose matrix as M^*. The augmented matrix is $M^a = \begin{bmatrix} M & N \\ \bar{N} & \overline{M^*} \end{bmatrix}$ and $M^{a*} = \begin{bmatrix} M & N \\ \bar{N} & \bar{M} \end{bmatrix}^* = \begin{bmatrix} M^* & \overline{N^*} \\ N^* & \overline{M^*} \end{bmatrix}$.

The augmented matrix inversion is $\begin{bmatrix} X & Y \\ \bar{Y} & \bar{X} \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}$ with $A = (X - Y\bar{X}^{-1}\bar{Y})^{-1}, B = -AY\bar{X}^{-1}$

The following widely linear model [19] is used in complex Kalman filters:

$x(k) = F(k)x(k-1) + A(k)\bar{x}(k-1) + w(k)$

$z(k) = H(k)x(k) + B(k)\bar{x}(k) + v(k)$

Here, $x(k)$ is the $n \times 1$ state complex vector, $z(k)$ is the $m \times 1$ measurement complex vector, $F(k)$ and $A(k)$ are the $n \times n$ transition complex matrices, $H(k)$ and $B(k)$ are is the $m \times n$ output complex matrices, $w(k)$ is the $n \times 1$ state noise complex vector and $v(k)$ is the $m \times 1$ measurement noise complex vector at (discrete) time k.

Furthermore, the state noise $w(k)$ is a Gaussian process with zero mean and known $n \times n$ dimensional covariance $Q(k)$ and pseudo-covariance $U(k)$. The measurement noise $v(k)$ is a Gaussian process with zero mean and known $m \times m$ dimensional known covariance $R(k)$ and pseudo-covariance $V(k)$.

The initial state $x(0)$ is Gaussian with known mean x_0, covariance P_0 and pseudo-covariance Π_0.

Consider the $2n \times 1$ augmented state vector $x^a(k) = \begin{bmatrix} x(k) \\ \bar{x}(k) \end{bmatrix}$ and the $2m \times 1$ augmented measurement vector $z^a(k) = \begin{bmatrix} z(k) \\ \bar{z}(k) \end{bmatrix}$.

Using the complex augmented state and measurement vectors, the augmented model (or widely linear model) becomes:

$$x^a(k) = F^a(k)x^a(k-1) + w^a(k) \quad (1)$$

$$z^a(k) = H^a(k)x^a(k) + v^a(k) \quad (2)$$

Here, the augmented noise vectors are $w^a(k) = \begin{bmatrix} w(k) \\ \bar{w}(k) \end{bmatrix}$ and $v^a(k) = \begin{bmatrix} v(k) \\ \bar{v}(k) \end{bmatrix}$ and the augmented matrices are $F^a(k) = \begin{bmatrix} F(k) & A(k) \\ \bar{A}(k) & \bar{F}(k) \end{bmatrix}$ and $H^a(k) = \begin{bmatrix} H(k) & B(k) \\ \bar{B}(k) & \bar{H}(k) \end{bmatrix}$.

Furthermore assume that:

- the augmented state noise process $w^a(k)$ is non-circular Gaussian with zero mean and covariance matrix $Q^a(k) = \begin{bmatrix} Q(k) & U(k) \\ \bar{U}(k) & \bar{Q}(k) \end{bmatrix}$,
- the augmented measurement noise process $v^a(k)$ is non-circular Gaussian with zero mean and covariance matrix $R^a(k) = \begin{bmatrix} R(k) & V(k) \\ \bar{V}(k) & \bar{R}(k) \end{bmatrix}$,
- the augmented initial state $x^a(0)$ is non-circular Gaussian with mean ${x_0}^a = \begin{bmatrix} x_0 \\ \bar{x}_0 \end{bmatrix}$ and covariance matrix ${P_0}^a = \begin{bmatrix} P_0 & \Pi_0 \\ \bar{\Pi}_0 & \bar{P}_0 \end{bmatrix}$.

It is worth to note that a) $Q^a(k)$ and $R^a(k)$ are Hermitian matrices (M is a Hermitian matrix when $M^* = M$), as covariance matrices and b) $U(k)$ and $V(k)$ are symmetric matrices (N is a symmetric matrix when $N^T = N$), as pseudo-covariance matrices, [9].

3 Augmented Complex Kalman Filter

Let denote a) the state prediction as $x(k|k-1)$ with covariance $P(k|k-1)$ and pseudo-covariance $\Pi(k|k-1)$; then, the augmented state prediction is $x^a(k|k-1)$ with covariance matrix $P^a(k|k-1) = \begin{bmatrix} P(k|k-1) & \Pi(k|k-1) \\ \bar{\Pi}(k|k-1) & \bar{P}(k|k-1) \end{bmatrix}$, b) the state estimation as $x(k|k)$ with covariance $P(k|k)$ and pseudo-covariance $\bar{P}(k|k)$; then the augmented state estimation is $x^a(k|k)$ with covariance matrix $P^a(k|k) = \begin{bmatrix} P(k|k) & \Pi(k|k) \\ \bar{\Pi}(k|k) & \bar{P}(k|k) \end{bmatrix}$. Let denote the augmented Kalman filter gain as $K^a(k|k) = \begin{bmatrix} K(k|k) & G(k|k) \\ \bar{G}(k|k) & \bar{K}(k|k) \end{bmatrix}$.

The augmented (widely linear) complex Kalman filter (ACKF) computes the augmented state prediction and estimation and the corresponding covariances, using the augmented Kalman filter gain, which is derived by minimizing the cost function based on the MSE criterion, and is summarized as follows, [6]:

Augmented Complex Kalman Filter (ACKF)

initial conditions

$x^a(0|-1) = {x_0}^a, P^a(0|-1) = {P_0}^a$

iterations $k = 0,1, \ldots$

$K^a(k) = P^a(k|k-1)H^{a*}(k)$
$\quad [H^a(k)P^a(k|k-1)H^{a*}(k) + R^a(k)]^{-1}$

$x^a(k|k) = [I^a - K^a(k)H^a(k)]x^a(k|k-1) + K^a(k)z^a(k)$

$P^a(k|k) = [I^a - K^a(k)H^a(k)]P^a(k|k-1)$

$x^a(k+1|k) = F^a(k)x^a(k|k)$
$P^a(k+1|k) = Q^a(k) + F^a(k)P^a(k|k)F^{a*}(k)$

Here, $I^a = I_{2n}$ is the $2n \times 2n$ identity matrix.

In the special case of time-invariant models, where the augmented model parameters $F^a(k), H^a(k), Q^a(k), R^a(k)$ are constant matrices, the Time Invariant Augmented Complex Kalman Filter is derived.

4 Augmented Complex Information Filter

The idea [2], [3] is the use of the Information matrix, which is the inverse of the covariance matrix and the information state vector which is connected to the estimation vector through the information matrix. Let define:

$$S^a(k|k) = P^{a^{-1}}(k|k)$$
$$y^a(k|k) = P^{a^{-1}}(k|k)x^a(k|k)$$
$$S^a(k+1|k) = P^{a^{-1}}(k+1|k)$$
$$y^a(k+1|k) = P^{a^{-1}}(k+1|k)x^a(k+1|k)$$

Then we can derive the augmented complex Information filter equations strictly from the augmented complex Kalman filter equations and using the matrix inversion lemma. In fact:

- concerning the Kalman filter gain, we get

$$K^a(k) = P^a(k|k-1)H^{a*}(k) [H^a(k)P^a(k|k-1)H^{a*}(k) + R^a(k)]^{-1}$$
$$K^a(k)[H^a(k)P^a(k|k-1)H^{a*}(k) + R^a(k)] = P^a(k|k-1)H^{a*}(k)$$
$$K^a(k)H^a(k)P^a(k|k-1)H^{a*}(k) + K^a(k)R^a(k) = P^a(k|k-1)H^{a*}(k)$$
$$K^a(k)R^a(k) = P^a(k|k-1)H^{a*}(k) - K^a(k)H^a(k)P^a(k|k-1)H^{a*}(k)$$
$$K^a(k)R^a(k) = [I^a - K^a(k)H^a(k)]P^a(k|k-1)H^{a*}(k)$$
$$K^a(k)R^a(k) = P^a(k|k)H^{a*}(k)$$
$$K^a(k) = P^a(k|k)H^{a*}(k)R^{a^{-1}}(k) = S^{a^{-1}}(k|k)H^{a*}(k)R^{a^{-1}}(k)$$

- concerning the estimation, we get

$$P^a(k|k) = [I^a - K^a(k)H^a(k)]P^a(k|k-1)$$
$$P^a(k|k) = P^a(k|k-1) - K^a(k)H^a(k)P^a(k|k-1)$$
$$P^a(k|k) = P^a(k|k-1) - P^a(k|k-1)H^{a*}(k)[H^a(k)P^a(k|k-1)H^{a*}(k) + R^a(k)]^{-1}H^a(k)P^a(k|k-1)$$
$$P^a(k|k) = \left[P^{a^{-1}}(k|k-1) + H^{a*}(k)R^{a^{-1}}(k)H^a(k)\right]^{-1}$$
$$P^{a^{-1}}(k|k) = P^{a^{-1}}(k|k-1) + H^{a*}(k)R^{a^{-1}}(k)H^a(k)$$
$$S^a(k|k) = S^a(k|k-1) + H^{a*}(k)R^{a^{-1}}(k)H^a(k)$$

and

$$x^a(k|k) = [I^a - K^a(k)H^a(k)]x^a(k|k-1) + K^a(k)z^a(k)$$
$$x^a(k|k) = P^a(k|k)P^{a^{-1}}(k|k-1)x^a(k|k-1) + K^a(k)z^a(k)$$
$$x^a(k|k) = S^{a^{-1}}(k|k)S^a(k|k-1)x^a(k|k-1) + K^a(k)z^a(k)$$
$$S^a(k|k)x^a(k|k) = S^a(k|k-1)x^a(k|k-1) + S^a(k|k)K^a(k)z^a(k)$$
$$y^a(k|k) = y^a(k|k-1) + S^a(k|k)S^{a^{-1}}(k|k)H^{a^H}(k)R^{a^{-1}}(k)z^a(k)$$
$$y^a(k|k) = y^a(k|k-1) + H^{a^H}(k)R^{a^{-1}}(k)z^a(k)$$

- concerning the prediction, we get

$$x^a(k+1|k) = F^a(k)x^a(k|k)$$
$$P^a(k+1|k)y^a(k+1|k) = F^a(k)P^a(k|k)y^a(k|k)$$
$$S^{a^{-1}}(k+1|k)y^a(k+1|k) = F^a(k)S^{a^{-1}}(k|k)y^a(k|k)$$
$$y^a(k+1|k) = S^a(k+1|k)F^a(k)S^{a^{-1}}(k|k)y^a(k|k)$$

Thus, the augmented complex Information filter and is summarized as follows:

Augmented Complex Information Filter (ACIF)
initial conditions
$y^a(0|-1) = P^{a^{-1}}(0|-1)x^a(0|-1) = P_0{}^{a^{-1}}x_0{}^a$
$S^a(0|-1) = P^{a^{-1}}(0|-1) = P_0{}^{a^{-1}}$
iterations $k = 0,1,\ldots$
$y^a(k|k) = y^a(k|k-1) + H^{a*}(k)R^{a^{-1}}(k)z^a(k)$
$S^a(k|k) = S^a(k|k-1) + H^{a*}(k)R^{a^{-1}}(k)H^a(k)$
$P^a(k|k) = S^{a^{-1}}(k|k)$
$x^a(k|k) = P^a(k|k)y^a(k|k)$
$K^a(k) = P^a(k|k)H^{a*}(k)R^{a^{-1}}(k)$
$P^a(k+1|k) = Q^a(k) + F^a(k)P^a(k|k)F^{a*}(k)$
$S^a(k+1|k) = P^{a^{-1}}(k+1|k)$
$y^a(k+1|k) = S^a(k+1|k)F^a(k)P^a(k|k)y^a(k|k)$
$x^a(k+1|k) = P^a(k|k)y^a(k+1|k)$

In the special case of time-invariant models, where the augmented model parameters $F^a(k), H^a(k), Q^a(k), R^a(k)$ are constant matrices, the Time Invariant Augmented Complex Information Filter is derived. Note that in this case the matrices $R^{a^{-1}}, H^{a*}R^{a^{-1}}, H^{a*}R^{a^{-1}}H^a$ are calculated off-line.

5 Computational Requirements

It is shown that ACKF and ACIF are algebraically equivalent filters. Then it becomes clear that the two filters calculate the same state estimations and state predictions. Moreover, it is obvious that the two filters are iterative algorithms. As a result, the comparison of the filters' computational time, is equivalent to the comparison of their per iteration calculation burden (CB); the calculation burden of the off-line calculations is not taken into account.

Table 1. ACKF per iteration calculation burden

Augmented Complex Kalman Filter (ACKF)	
Matrix Operation	**Calculation Burden**
$H^a(k)P^a(k\|k-1)$	$32n^2m-12nm$
$H^a(k)P^a(k\|k-1)H^{a*}(k)$	$32nm^2-6m^2+m$
$H^a(k)P^a(k\|k-1)H^{a*}(k)+R^a(k)$	$2m^2+m$
$[H^a(k)P^a(k\|k-1)H^{a*}(k)+R^a(k)]^{-1}$	$\frac{1}{6}(208m^3-96m^2+8m)$
$K^a(k)=P^a(k\|k-1)H^{a*}(k)$ $[H^a(k)P^a(k\|k-1)H^{a*}(k)+R^a(k)]^{-1}$	$32nm^2-12nm$
$H^a(k)x^a(k\|k-1)$	$16nm-2m$
$H^a(k)x^a(k\|k-1)-z^a(k)$	$2m$
$K^a(k)[H^a(k)x^a(k\|k-1)-z^a(k)]$	$16nm-2n$
$x^a(k\|k)=x^a(k\|k-1)$ $+K^a(k)[H^a(k)x^a(k\|k-1)-z^a(k)]$	$2n$
$K^a(k)H^a(k)P^a(k\|k-1)$	$32n^2m-6n^2+n$
$P^a(k\|k)=P^a(k\|k-1)$ $-K^a(k)H^a(k)P^a(k\|k-1)$	$2n^2+n$
$x^a(k+1\|k)=F^a(k)x^a(k\|k)$	$16n^2-2n$
$F^a(k)P^a(k\|k)$	$32n^3-12n^2$
$F^a(k)P^a(k\|k)F^{a*}(k)$	$32n^3-6n^2+n$
$P^a(k+1\|k)=Q^a(k)+F^a(k)P^a(k\|k)F^{a*}(k)$	$2n^2+n$

Note that in the time-invariant case, the calculation burden remains the same as in the time-varying case.

Table 2. ACIF per iteration calculation burden

Augmented Complex Information Kalman Filter (ACIF)	
Matrix Operation	**Calculation Burden**
$R^{a^{-1}}(k)$	$\frac{1}{6}(208m^3-96m^2+8m)$
$H^{a*}(k)R^{a^{-1}}(k)$	$32nm^2-12nm$
$H^{a*}(k)R^{a^{-1}}(k)H^a(k)$	$32n^2m-6n^2+n$
$H^{a*}(k)R^{a^{-1}}(k)z^a(k)$	$16nm-2n$
$y^a(k\|k)=y^a(k\|k-1)$ $+H^{a*}(k)R^{a^{-1}}(k)z^a(k)$	$2n$
$S^a(k\|k)=S^a(k\|k-1)$ $+H^{a*}(k)R^{a^{-1}}(k)H^a(k)$	$2n^2+n$
$P^a(k\|k)=S^{a^{-1}}(k\|k)$	$\frac{1}{6}(208n^3-96n^2+8n)$
$x^a(k\|k)=P^a(k\|k)y^a(k\|k)$	$16n^2-6n$
$K^a(k)=P^a(k\|k)H^{a*}(k)R^{a^{-1}}(k)$	$32n^2m-12nm$
$F^a(k)P^a(k\|k)$	$32n^3-12n^2$
$F^a(k)P^a(k\|k)F^{a*}(k)$	$32n^3-6n^2+n$
$P^a(k+1\|k)=Q^a(k)$ $+F^a(k)P^a(k\|k)F^{a*}(k)$	$2n^2+n$
$S^a(k+1\|k)=P^{a^{-1}}(k+1\|k)$	$\frac{1}{6}(208n^3-96n^2+8n)$
$F^a(k)P^a(k\|k)y^a(k\|k)$	$16n^2-2n$
$y^a(k+1\|k)=S^a(k+1\|k)$ $F^a(k)P^a(k\|k)y^a(k\|k)$	$16n^2-6n$
$x^a(k+1\|k)=P^a(k\|k)y^a(k+1\|k)$	$16n^2-6n$

It is evident that the calculation burdens of the two complex filters involve complex matrix operations, the calculation burdens of which are given in the Appendix.

The per iteration calculation burdens of the augmented complex Kalman filter and the complex Information filter are analytically calculated in Table 1 and Table 2, respectively.

Note that in the time-invariant case, the matrices $R^{a^{-1}}, H^{a^*}R^{a^{-1}}, H^{a^*}R^{a^{-1}}H^a$ are calculated off-line.

The per iteration calculation burdens of the augmented complex Kalman filter and the complex Information filter are and summarized in Table 3.

Table 3. ACKF and ACIF calculation burdens

Model	Filter	Per Iteration Calculation Burden
time varying	Kalman	$CB_{ACKFtv}=64n^3-4n^2+2n$ $+64n^2m+8nm+64nm^2$ $+\frac{1}{6}(208m^3-120m^2+20m)$
	Information	$CB_{ACIFtv}=\frac{1}{6}(800n^3+72n^2-80n)$ $+64n^2m-8nm+32nm^2$ $+\frac{1}{6}(208m^3-96m^2+8m)$
time invariant	Kalman	$CB_{ACKFti}=64n^3-4n^2+2n$ $+64n^2m+8nm+64nm^2$ $+\frac{1}{6}(208m^3-120m^2+20m)$
	Information	$CB_{ACIFti}=\frac{1}{6}(800n^3+108n^2-86n)$ $+32n^2m+4nm$

6 Selection of the Faster Filter

In this section, a method is derived to select the faster complex filter. From Table 3, it is obvious that the calculation burdens of the two complex filters depend on the state vector dimension n and the measurement vector dimension m. Hence, the selection of the faster complex filter depends on the relation between the known dimensions n and m.

In the general *time-varying models* case, we get:

$$CB_{ACKFtv}-CB_{ACIFtv} = (32n-4)m^2+(16n+2)m + \frac{1}{6}(-416n^3-96n^2+92n)$$

Figure 1 presents the areas where the complex Information filter or the complex Kalman filter is faster; subject to the model dimensions. The obtained Rule of Thumb for time-varying models has as follows:

The ACIF is faster than ACKF, when $m/n>1.5$

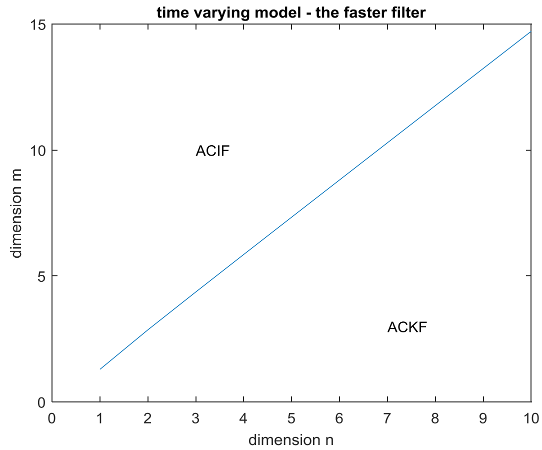

Fig. 1: The faster filter – time-varying model

In the special *time-invariant models* case, we get:

$$CB_{ACKFti} - CB_{ACIFti} = \frac{1}{6}208m^3 + (64n - 20)m^2 + \frac{1}{6}(192n^2 + 24n + 20)m + \frac{1}{6}(-416n^3 - 132n^2 + 98n)$$

Figure 2 presents the areas where the complex Information filter or the complex Kalman filter is faster; subject to the model dimensions. The obtained Rule of Thumb for time-invariant models has as follows:

The ACIF is faster than ACKF, when $m/n > 0.75$

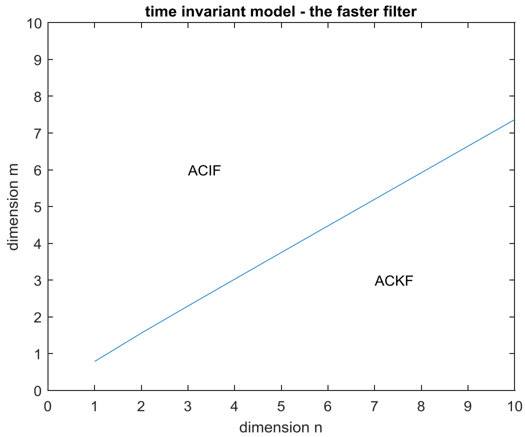

Fig. 2: The faster filter – time-invariant model

It becomes obvious that the knowledge of the model dimensions is sufficient in order to determine which filter is faster.

7 Conclusions

The augmented complex Kalman filter and the complex Information filter are equivalent with respect to a) the derivation of the state estimations and predictions and the corresponding error covariances, b) their stability, since the stability of the Kalman filter is classically ensured by the controllability and the observability of linear time-varying models.

In this paper, a comparison study between the (discrete time) augmented complex Kalman filter and complex Information filter was obtained. The computational requirements of both these complex filters were derived. It was established that the computational burdens of the filters are functions of the state dimension n and the measurement dimension m. A method was derived and described to select, before the implementation of the filters, the faster complex filter. The basic result is:

- in the general case of time-varying models, the complex Information filter is faster than the complex Kalman filter when $m > 1.5n$,
- in the special case of time-invariant models, the complex Information filter is faster than the complex Kalman filter when $m > 0.75n$.

The impact of this result on Kalman filtering combined with AI techniques to determine the fastest filter, can be to reduce processing time in complex signal processing applications.

References:

[1] R. E. Kalman, A new approach to linear filtering and prediction problems, *J. Bas. Eng., Trans. ASME*, Ser. D, vol. 8(1), 1960, pp. 34-45, [Online]. https://courses.cs.duke.edu/compsci527/cps274/fall11/papers/Kalman60.pdf (Accessed Date: September 30, 2024).

[2] B. D. O. Anderson, J. B. Moore, *Optimal Filtering*, Dover Publications, New York, 2005, [Online]. https://researchportalplus.anu.edu.au/en/publications/optimal-filtering (Accessed Date: September 30, 2024).

[3] N. Assimakis, M. Adam, A. Douladiris, Information filter and Kalman filter comparison: Selection of the faster filter, *Information Engineering*, vol. 2(1), 2012, pp. 1-5, [Online]. https://www.researchgate.net/publication/292145072_Information_Filter_and_Kalman_Filter_Comparison_Selection_of_the_Faster_Filter (Accessed Date: September 30, 2024).

[4] B. Ristic, S. Arulampalam, N. Gordon, *Beyond The Kalman Filter*, Boston London, Artech House, 2004, [Online]. https://us.artechhouse.com/Beyond-the-Kalman-Filter-Particle-Filters-for-Tracking-

Applications-P1376.aspx (Accessed Date: September 30, 2024).

[5] U. Arumugam, N. M. Nor and M. F. Abdullah, A Brief Review on Advances of Harmonic State Estimation Techniques in Power Systems, *International Journal of Information and Electronics Engineering*, vol. 1(3), pp. 217-222, 2011, DOI: http://dx.doi.org/10.7763/IJIEE.2011.V1.34.

[6] G. Wang, S. S. Ge, R. Xue, J. Zhao, C. Li, Complex-valued Kalman filters based on Gaussian entropy, *Signal Processing* 160, 178–189, 2019, DOI: http://dx.doi.org/10.1016/j.sigpro.2019.02.024

[7] M. H. Terra, J. Y. Ishihara, A. C. Padoan, Information filtering and array algorithms for descriptor systems subject to parameter uncertainties. *IEEE Transactions on Signal Processing*, 55(1), pp. 1-9, 2007, https://doi.org/10.1109/TSP.2006.885747.

[8] D. W. Casbeer, R. Beard, R., Distributed information filtering using consensus filters, *2009 American Control Conference*, St. Louis, MO, USA, pp. 1882-1887, 2009, DOI: http://dx.doi.org/10.1109%2FACC.2009.5160531.

[9] C. Cadena, J. Neira, J., SLAM in O (log n) with the Combined Kalman-Information Filter, *Robotics and Autonomous Systems*, 58(11), 1207-1219, 2010, DOI: 10.1109/IROS.2009.5354521.

[10] R. Verma, L. Shrinivasan and K. Shreedarshan, GPS/INS integration during GPS outages using machine learning augmented with Kalman filter, *WSEAS Transactions on Systems and Control* (16), pp. 294-301, 2021, DOI: https://doi.org/10.37394/23203.2021.16.25.

[11] Y. Li, Q. Gui, S. Han, Y. Gu, Tikhonov Regularized Kalman Filter and its Applications in Autonomous Orbit Determination of BDS, *WSEAS Transactions on Mathematics*, (16), pp. 187-196, 2017.

[12] A. Mohammadi, K.N. Plataniotis, Structure-induced complex Kalman filter for decentralized sequential Bayesian estimation, *IEEE Signal Process. Lett.* 22 (9), 1419–1423, 2015, DOI: http://dx.doi.org/10.1109/LSP.2015.2407196.

[13] D. Borio, M. Susi, M., Bicomplex Kalman Filter Tracking for GNSS Meta-Signals, In Proceedings of the *36th International Technical Meeting of the Satellite Division of The Institute of Navigation* (ION GNSS+ 2023), Denver, Colorado, pp. 3353-3373, 2023, DOI: http://dx.doi.org/10.33012/2023.19233.

[14] P. K. Dash, R. K. Jena, G. Panda, A. Routray, An extended complex Kalman filter for frequency measurement of distorted signals, *IEEE Transactions on instrumentation and measurement*, 49(4), 746-753, 2000, DOI: http://dx.doi.org/10.1109/19.863918.

[15] M. El-Nagar, K. Ahmed, E. Hamdan, A. S. Abdel-Khalik, M. S. Hamad, S. Ahmed, Modified extended complex Kalman filter for DC offset and distortion rejection in grid-tie transformerless converters, *Applied Sciences*, 13(15), 2023, https://doi.org/10.3390/app13159023.

[16] E. T. Andrew, K. H. Ahmed, D. Holliday, A new model predictive current controller for grid-connected converters in unbalanced grids, *IEEE Transactions on Power Electronics*, 37(8), pp. 9175-9186, 2022, DOI: 10.1109/TPEL.2022.3158016.

[17] X. Zhang, Y. Xia, C. Li and L. Yang, Unscented Kalman Filter With General Complex-Valued Signals, *IEEE Signal Processing Letters*, 29, pp. 2023-2027, 2022, DOI: http://dx.doi.org/10.1109/LSP.2022.3207414.

[18] N. Petukhov, V. Zamolodchikov, E. Zakharova and A. Shamina, Synthesis and Comparative Analysis of Characteristics of Complex Kalman Filter and Particle Filter in Two-dimensional Local Navigation System, *2019 Ural Symposium on Biomedical Engineering, Radioelectronics and Information Technology (USBEREIT)*, Yekaterinburg, Russia, pp. 225-228, 2019, DOI: 10.1109/USBEREIT.2019.8736595.

[19] W. Dang, L. L. Scharf, Extensions to the theory of widely linear complex Kalman filtering. *IEEE Transactions on Signal Processing*, 60(12), pp. 6669-6674, 2011, DOI: http://dx.doi.org/10.1109/TSP.2012.2214213.

APPENDIX

Calculation Burdens of Complex Matrix Operations

In the following, $r, r1, r2$ are real numbers and $c, c1, c2$ are complex numbers.

The calculation burdens (CB) of real and complex scalar operations are summarized in Table 4 and Table 5, respectively.

Table 4. Real scalar operations

code	real scalar operation	real scalar adds	real scalar mults	real scalar divs	CB
R1	$r1 + r2 = r$	1	0	0	1
R2	$r1 \cdot r2 = r$	0	1	0	1
R3	$r1/r2 = r$	0	0	1	1

Table 5. Complex scalar operations

code	complex scalar operation	real scalar adds	real scalar mults	real scalar divs	CB
C1	$c1 + c2 = c$	2	0	0	2
C2	$c1 \cdot c2 = c$	2	4	0	6
C3	$c1 + c2 = r$	1	0	0	1
C4	$r + c1 = c$	1	0	0	1
C5	$r \cdot c1 = c$	0	2	0	2
C6	$c1 \cdot c2 = r$	1	2	0	3
C7	$c \cdot \bar{c} = r$	1	2	0	3
C8	$c1/r = c$	0	0	2	2
C9	$c1/c2 = c$	3	6	2	11

In the following, $x, x1, x2$ are complex vectors; C, C1, C2 are general complex matrices; H, H1, H2 are complex Hermitian matrices; S, S1, S2 are complex symmetric matrices; I is the identity matrix of dimension n; $x^a, x1^a, x2^a$ are augmented complex vectors of the form $\begin{bmatrix} x \\ \bar{x} \end{bmatrix}$; $A^a, A1^a, A2^a$ are augmented complex matrices of the form $\begin{bmatrix} C1 & C2 \\ \overline{C2} & \overline{C1} \end{bmatrix}$; $A_s^a, A1_s^a, A2_s^a$ are special augmented complex matrix of the form $\begin{bmatrix} H & S \\ \bar{S} & \bar{H} \end{bmatrix}$, I^a is the identity matrix of dimension 2n.

The calculation burdens (CB) of complex matrices addition and complex augmented matrices addition operations are summarized in Table 6 and Table 7, respectively.

Table 6. Complex matrices addition

code	Complex Matrices Addition	oper	CB	total CB
A1	$x1 + x2 = x$ $(n \times 1) + (n \times 1)$	C1	n	$2n$
A2	$x1 + x2 = x$ $(m \times 1) + (m \times 1)$	C1	m	$2m$
A3	$I + C1 = C$ $(n \times n) + (n \times n)$	R1	n	n
A4	$C1 + C2 = C$ $(n \times m) + (n \times m)$	C1	nm	$2nm$
A5	$C1 + C2 = C$ $(n \times m) + (n \times m)$	C1	nm	$2nm$
A6	$C1 + C2 = C$ $(n \times n) + (n \times n)$	C1	n^2	$2n^2$
A7	$C1 + C2 = H$ $(n \times n) + (n \times n)$	C1 C3	$\frac{n^2 - n}{2}$ n	n^2
A8	$C1 + C2 = H$ $(m \times m) + (m \times m)$	C1 C3	$\frac{m^2 - m}{2}$ m	m^2
A9	$H1 + H2 = H$ $(n \times n) + (n \times n)$	R1 C1	n $\frac{n^2 - n}{2}$	n^2
A10	$H1 + H2 = H$ $(m \times m) + (m \times m)$	R1 C1	m $\frac{m^2 - m}{2}$	m^2
A11	$S1 + S2 = S$ $(n \times n) + (n \times n)$	C1	$\frac{n^2 + n}{2}$	$n^2 + n$
A12	$S1 + S2 = S$ $(m \times m) + (m \times m)$	C1	$\frac{m^2 + m}{2}$	$m^2 + m$
A13	$C1 + C2 = S$ $(m \times m) + (m \times m)$	C1	$\frac{n^2 + n}{2}$	$n^2 + n$

Table 7. Complex augmented matrices addition

code	Complex Augmented Matrices Addition	oper	CB	total CB
A14	$x1^a + x2^a = x^a$ $(2n \times 1) + (2n \times 1)$	A1	$2n$	$2n$
A15	$x1^a + x2^a = x^a$ $(2m \times 1) + (2m \times 1)$	A2	$2m$	$2m$
A16	$I^a + A1^a = A^a$ $(2n \times 2n) + (2n \times 2n)$	A3	n	n
A17	$A1_s^a + A2_s^a = A_s^a$ $(2n \times 2n) + (2n \times 2n)$	A9 A11	n^2 $n^2 + n$	$2n^2 + n$
A18	$A1_s^a + A2_s^a = A_s^a$ $(2m \times 2m) + (2m \times 2m)$	A10 A12	m^2 $m^2 + m$	$2m^2 + m$

The calculation burdens (CB) of complex matrices multiplication and complex augmented matrices multiplication operations are summarized in Table 8 and Table 9, respectively.

Table 8. Complex matrices multiplication

code	Complex Matrices Multiplication	oper	CB	total CB
M1	$C1 \cdot x1 = x$ $(n \times n) \cdot (n \times 1)$	C1 C2	$n(n-1)$ n^2	$8n^2 - 2n$
M2	$S \cdot x1 = x$ $(n \times n) \cdot (n \times 1)$	C1 C2	$n(n-1)$ n^2	$8n^2 - 2n$
M3	$C1 \cdot x1 = x$ $(n \times m) \cdot (m \times 1)$	C1 C2	$n(m-1)$ nm	$8nm - 2n$
M4	$C1 \cdot x1 = x$ $(m \times n) \cdot (n \times 1)$	C1 C2	$m(n-1)$ nm	$8nm - 2m$
M5	$H \cdot x1 = x$ $(n \times n) \cdot (n \times 1)$	C1 C2 C5	$n(n-1)$ $n(n-1)$ n	$8n^2 - 6n$
M6	$C1 \cdot C2 = C$ $(n \times n) \cdot (n \times n)$	C1 C2	$n^2(n-1)$ n^3	$8n^3 - 2n^2$
M7	$C1 \cdot C2 = C$ $(n \times n) \cdot (n \times m)$	C1 C2	$nm(n-1)$ n^2m	$8n^2m - 2nm$
M8	$C1 \cdot C2 = C$ $(n \times m) \cdot (m \times n)$	C1 C2	$n^2(m-1)$ n^2m	$8n^2m - 2n^2$
M9	$C1 \cdot C2 = C$ $(n \times m) \cdot (m \times m)$	C1 C2	$nm(m-1)$ nm^2	$8nm^2 - 2nm$
M10	$C1 \cdot C2 = C$ $(m \times n) \cdot (n \times n)$	C1 C2	$nm(n-1)$ n^2m	$8n^2m - 2nm$
M11	$C1 \cdot C2 = C$ $(m \times n) \cdot (n \times m)$	C1 C2	$m^2(n-1)$ nm^2	$8nm^2 - 2m^2$
M12	$C1 \cdot H = C$ $(m \times n) \cdot (n \times n)$	C1 C2 C5	$nm(n-1)$ $nm(n-1)$ nm	$8n^2m - 6nm$
M13	$H \cdot C1 = C$ $(n \times n) \cdot (n \times m)$	C1 C2 C5	$nm(n-1)$ $nm(n-1)$ nm	$8n^2m - 6nm$
M14	$C1 \cdot H = C$ $(n \times m) \cdot (m \times m)$	C1 C2 C5	$nm(m-1)$ $nm(m-1)$ nm	$8nm^2 - 6nm$
M15	$C1 \cdot H = C$ $(n \times n) \cdot (n \times n)$	C1 C2 C5	$n^2(n-1)$ $n^2(n-1)$ n^2	$8n^3 - 6n^2$
M16	$C1 \cdot C2 = H$ $(m \times n) \cdot (n \times m)$	R1 C1 C2 C6	$m(n-1)$ $\left(\frac{m^2-m}{2}\right)(n-1)$ $\left(\frac{m^2-m}{2}\right)n$ nm	$4nm^2 - m^2$
M17	$C1 \cdot C2 = H$ $(n \times m) \cdot (m \times n)$	R1 C1 C2 C6	$n(m-1)$ $\left(\frac{n^2-n}{2}\right)(m-1)$ $\left(\frac{n^2-n}{2}\right)m$ nm	$4n^2m - n^2$
M18	$C1 \cdot C2 = H$ $(n \times n) \cdot (n \times n)$	R1 C1 C2 C6	$n(n-1)$ $\left(\frac{n^2-n}{2}\right)(n-1)$ $\left(\frac{n^2-n}{2}\right)n$ n^2	$4n^3 - n^2$
M19	$C1 \cdot H1 = H$ $(n \times n) \cdot (n \times n)$	C1 C2 C5	$\left(\frac{n^2+n}{2}\right)(n-1)$ $\left(\frac{n^2+n}{2}\right)(n-1)$ $\frac{n^2+n}{2}$	$4n^3 + n^2 - 3n$
M20	$C1 \cdot C2 = S$ $(n \times m) \cdot (m \times n)$	C1 C2	$\left(\frac{n^2+n}{2}\right)(m-1)$ $\left(\frac{n^2+n}{2}\right)m$	$4n^2m + 4nm$ $-n^2 - n$
M21	$S \cdot H = C$ $(n \times n) \cdot (n \times n)$	C1 C2 C5	$n^2(n-1)$ $n^2(n-1)$ n^2	$8n^3 - 6n^2$
M22	$H \cdot C = S$ $(n \times n) \cdot (n \times n)$	C1 C2 C5	$\left(\frac{n^2+n}{2}\right)(n-1)$ $\left(\frac{n^2+n}{2}\right)(n-1)$ $\left(\frac{n^2+n}{2}\right)$	$4n^3 + n^2 - 3n$
M23	$C1 \cdot S = C$ $(m \times n) \cdot (n \times n)$	C1 C2	$nm(n-1)$ n^2m	$8n^2m - 2nm$
M24	$S \cdot C1 = C$ $(n \times n) \cdot (n \times m)$	C1 C2	$nm(n-1)$ n^2m	$8n^2m - 2nm$
M25	$C1 \cdot S = C$ $(n \times n) \cdot (n \times n)$	C1 C2	$n^2(n-1)$ n^3	$8n^3 - 2n^2$
M26	$C1 \cdot S = C$ $(n \times m) \cdot (m \times m)$	C1 C2	$nm(m-1)$ nm^2	$8nm^2 - 2nm$

Table 9. Complex augmented matrices multiplication

code	Complex Augmented Matrices Multiplication	oper (times)	CB	total CB
M27	$A^a \cdot x1^a = x^a$ $(2n \times 2n) \cdot (2n \times 1)$	M1(2) A1(1)	$16n^2 - 4n$ $2n$	$16n^2 - 2n$
M28	$A^a \cdot x1^a = x^a$ $(2n \times 2m) \cdot (2m \times 1)$	M3(2) A1(1)	$16nm - 4n$ $2n$	$16nm - 2n$
M29	$A^a \cdot x1^a = x^a$ $(2m \times 2n) \cdot (2n \times 1)$	M4(2) A2(1)	$16nm - 4m$ $2m$	$16nm - 2m$
M30	$A_s^a \cdot x1^a = x^a$ $(2n \times 2n) \cdot (2n \times 1)$	M5(1) M2(1) A1(1)	$8n^2 - 6n$ $8n^2 - 2n$ $2n$	$16n^2 - 6n$
M31	$A1^a \cdot A2^a = A^a$ $(2n \times 2n) \cdot (2n \times 2n)$	M6(4) A6(2)	$32n^3 - 8n^2$ $4n^2$	$32n^3 - 4n^2$
M32	$A1^a \cdot A2^a = A^a$ $(2n \times 2m) \cdot (2m \times 2n)$	M8(4) A6(2)	$32n^2m - 8n^2$ $4n^2$	$32n^2m - 4n^2$
M33	$A1^a \cdot A_s^a = A^a$ $(2m \times 2n) \cdot (2n \times 2n)$	M12(2) M23(2) A5(2)	$16n^2m - 12nm$ $16n^2m - 4nm$ $4nm$	$32n^2m - 12nm$
M34	$A_s^a \cdot A1^a = A^a$ $(2n \times 2n) \cdot (2n \times 2m)$	M13(2) M24(2)	$16n^2m - 12nm$ $16n^2m - 4nm$	$32n^2m - 12nm$

		A4(2)	$4nm$	
M35	$A1^a \cdot A_s^a = A^a$ $(2n \times 2n) \cdot (2n \times 2n)$	M15(2) M25(2) A6(2)	$16n^3 - 12n^2$ $16n^3 - 4n^2$ $4n^2$	$32n^3 - 12n^2$
M36	$A1^a \cdot A2^a = A_s^a$ $(2n \times 2n) \cdot (2n \times 2n)$	M6(4) A7(1) A13(1)	$32n^3 - 8n^2$ n^2 $n^2 + n$	$32n^3 - 6n^2 + n$
M37	$A1^a \cdot A2^a = A_s^a$ $(2m \times 2n) \cdot (2n \times 2m)$	M11(4) A10(1) A12(1)	$32nm^2 - 8m^2$ m^2 $m^2 + m$	$32nm^2 - 6m^2 + m$
M38	$A1^a \cdot A2^a = A_s^a$ $(2n \times 2m) \cdot (2m \times 2n)$	M8(4) A9(1) A11(1)	$32n^2m - 8n^2$ n^2 $n^2 + n$	$32n^2m - 6n^2 + n$
M39	$A1^a \cdot A_s^a = A^a$ $(2n \times 2m) \cdot (2m \times 2m)$	M14(2) M26(2) A4(2)	$16nm^2 - 12nm$ $16nm^2 - 4nm$ $4nm$	$32nm^2 - 12nm$
M40	$A1^a \cdot A1_s^a = A_s^a$ $(2n \times 2n) \cdot (2n \times 2n)$	M15(2) M25(2) A7(1) A13(1)	$16n^3 - 12n^2$ $16n^3 - 4n^2$ n^2 $n^2 + n$	$32n^3 - 14n^2 + n$

The following recursive algorithm is used for the complex Hermitian matrix inversion:

$$M = \begin{bmatrix} A & b \\ b^H & d \end{bmatrix}, M = M^H, A = A^H$$

Dimensions:
M nxn, A $(n-1)x(n-1)$,
b $(n-1)x1$, b^H $1x(n-1)$, d 1x1

$$M^{-1} = \begin{bmatrix} A^{-1} + s\left(\frac{1}{s}A^{-1}b\right)\left(\frac{1}{s}b^H A^{-1}\right) & -\frac{1}{s}A^{-1}b \\ -\frac{1}{s}b^H A^{-1} & \frac{1}{s} \end{bmatrix}$$

$$s = d - b^H A^{-1} b$$

The calculation burden (CB) of the recursive algorithm for the complex Hermitian matrix inversion is summarized in Table 10.

Table 10. Complex Hermitian matrix inversion recursive algorithm

Matrix operation	oper	times	total CB
A^{-1} (NxN)			$T(N)$
$A^{-1}b$ (NxN) · (Nx1)	C1	$(N-1)N$	$8N^2 - 2N$
	C2	N^2	
$b^H A^{-1} = (A^{-1}b)^H$ (1xN) · (NxN)			
$b^H(A^{-1}b)$ Real (1xN) · (Nx1)	C3	$N-1$	$4N-1$
	C6	N	
$s = d - b^H A^{-1} b$ real (1x1) + (1x1)	R1	1	1
$\frac{1}{s}$ Real (1x1)/(1x1)	R3	1	1
$\frac{1}{s}(A^{-1}b)$ $-\frac{1}{s}(A^{-1}b)$ (1x1) · (Nx1)	C5	N	2N
$\frac{1}{s}(b^H A^{-1}) = \left[\frac{1}{s}(A^{-1}b)\right]^H$ $-\frac{1}{s}(b^H A^{-1}) = -\left[\frac{1}{s}(A^{-1}b)\right]^H$ (1x1) · (1xN)			
$\left(\frac{1}{s}A^{-1}b\right)\left(\frac{1}{s}b^H A^{-1}\right)$ Hermitian (Nx1) · (1xN)	C2 C6	$\frac{N^2 - N}{2}$ N	$3N^2$
$s\left(\frac{1}{s}A^{-1}b\right)\left(\frac{1}{s}b^H A^{-1}\right)$ Hermitian (1x1) · (NxN)	C5	$\frac{N^2 + N}{2}$	$N^2 + N$
$A^{-1} + s\left(\frac{1}{s}A^{-1}b\right)\left(\frac{1}{s}b^T A^{-1}\right)$ Hermitian (NxN) + (NxN)	R1	N	N^2
$A^{-1} + s\left(\frac{1}{s}A^{-1}b\right)\left(\frac{1}{s}b^T A^{-1}\right)$ Hermitian (NxN) + (NxN)	C1	$\frac{N^2 - N}{2}$	N^2
			$13N^2 + 5N + 1$

Here $N = n - 1$

Then
$T(n) = T(n-1) + f(n)$
with
$$f(n) = 13(n-1)^2 + 5(n-1) + 1 = 13n^2 - 21n + 9$$
$f(1) = 1$
So
$T(n) = T(n-1) + 13n^2 - 21n + 9$
$T(1) = 1$

Thus
$T(1) = 1$
$T(2) = T(1) + f(2)$
$T(3) = T(2) + f(3) = T(1) + f(2) + f(3)$
...
$T(n) = T(1) + f(2) + f(3) + \cdots + f(n)$

Hence, the calculation burden (CB) of the recursive algorithm for the nxn complex Hermitian matrix inversion is:

$$\begin{aligned} T(n) &= \{T(1) - f(1)\} + \{f(1) + f(2) + f(3) + \cdots + f(n)\} \\ &= 13\sum_{i=1}^{n} i^2 - 21\sum_{i=1}^{n} i + 9\sum_{i=1}^{n} 1 \\ &= 13\frac{n(n+1)(2n+1)}{6} - 21\frac{n(n+1)}{2} + 9n \\ &= \frac{26n^3 - 24n^2 + 4n}{6} \end{aligned}$$

The calculation burdens (CB) of complex Hermitian matrices inversion operations are summarized in Table 11.

Table 11. Complex Hermitian matrices inversion

code	Complex Hermitian Matrix Inversion	method	CB
I1	H^{-1} $(n \times n)$	recursive algorithm	$\frac{1}{6}(26n^3 - 24n^2 + 4n)$
I2	H^{-1} $(m \times m)$	recursive algorithm	$\frac{1}{6}(26m^3 - 24m^2 + 4m)$
I3	H^{-1} $(2n \times 2n)$	computation for I1 with 2n	$\frac{1}{6}(208n^3 - 96n^2 + 8n)$
I4	H^{-1} $(2m \times 2m)$	computation for I2 with 2m	$\frac{1}{6}(208m^3 - 96m^2 + 8m)$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Athanasios Polyzos: Methodology, Investigation Writing - original draft preparation, Writing - review and editing and Visualization, Validation.
Christos Tsinos: Formal analysis, Investigation.
Maria Adam: Formal analysis, Investigation.
Nicholas Assimakis: Conceptualization, Software.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.