A Beneficial Numerical Approach to Solve Systems of Linear Integro-Differential Equations

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Abstract: The system of linear Fredholm-Volterra integro-differential equations (FVIDEs) has been solved in this paper by an improved approximation method. Generalised Bernstein polynomials and collocation points have been used to construct the theory of the method. The aim of the technique is to reduce systems of integro-differential equations into an algebraic matrix equation, which corresponds to a linear algebraic equation system, by means of Bernstein polynomials. In order to analyse the applicability of the method, some illustrative examples have also been considered. It has been shown that the proposed method is faster and more effective than the others when comparing the numerical results.

Key-Words: Bernstein polynomial approach; collocation method; system of integro-differential equations

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1 Introduction

Systems of linear integro-differential equations (IDEs) have a major role in the fields of natural science, engineering, chemistry, physics, biology, astronomy, potential theory, electrostatics, and financial mathematics, etc. Many problems, such as dynamic and genetic structures, risky businesses (e.g. assurance companies), and neural networks with time-varying delays, can be modelled by IDEs. Therefore, numerical solutions of IDEs have become a remarkable study both in the fields of mathematics and physical science.

To date, many papers have been published related to numerical methods for solutions of linear and nonlinear IDEs systems. Studies on linear systems of IDEs are collocation methods based on the Bernstein operational matrix, [1], Bessel polynomials, [2], [3], Euler polynomials, [4], Taylor polynomials, [5], [6], Chebyshev polynomials, [7], and Fibonacci polynomials, [8]. Apart from the collocation methods, a numerical method based on rationalized Haar functions, [9], has been presented to solve a system of linear Fredholm IDEs. In addition, a spectral method, [10], has been developed for the solution of a linear Volterra IDE system. Moreover, the Chebyshev collocation method has been used to solve a system of second order IDEs modeling the Markow-modulated jump-diffusion process, [11].

Considering the above promising studies associated with collocation method, in this study an alternative collocation method has been revealed regarding to derivability property of the generalized Bernstein polynomials to solve the system of linear Fredholm-Volterra integro-differential equations (FVIDEs). Basis of the developed method depends on the definitions and the matrix relations of Bernstein polynomials and their derivatives, [12], [13], [14].

Definition 1.1: The generalized Bernstein basis polynomials of Nth degree are defined by

$$p_{r,N}(x) = \frac{1}{(b-a)^N}\binom{N}{r}(x-a)^r(b-x)^{N-r}\,;\; r = 0,1,...,N$$

on the interval $[a,b]$. For convenience, $p_{r,N}(x) = 0$ is accepted for $r < 0$ and $r > N$. Besides, $p_{r,N}(a) = p_{r,N}(b) = 0$ are verified for $0 < r < N$, and $p_{0,N}(b) = p_{N,N}(a) = 0$, $p_{0,N}(a) = p_{N,N}(b) = 1$.

Definition 1.2: Let $y : [a,b] \rightarrow \mathbb{R}$ is continuous function. Then the generalized Bernstein polynomials of Nth degree are defined by

$$B_N(y;x) = \sum_{r=0}^{N} y\left(a + \frac{(b-a)r}{N}\right)p_{r,N}(x)$$

on the interval $[a, b]$.

Theorem 1.1: There is a relation between the generalized Bernstein basis polynomials matrix and their derivatives in the form

$$\mathbf{p}^{(k)}(x) = \mathbf{p}(x)\,\mathbf{d}^k;\ k = 0, 1, ..., m$$

such that $\mathbf{p}(x) = [\ p_{0,N}(x) \quad p_{1,N}(x) \quad \dots \quad p_{N,N}(x)\ ]$ and the elements of matrix $\mathbf{d} = (d_{rs});\ r, s = 0, 1, ..., N$ is as follows:

$$d_{rs} = \frac{1}{b-a}\begin{cases} N-r & ;\ s = r+1 \\ 2r-N & ;\ s = r \\ -r & ;\ s = r-1 \\ 0 & ;\ \text{otherwise} \end{cases}.$$

The rest of the paper is structured as follows: In Section 2, the theory of the method has been explained, and the solution algorithm has been given. In Section 3, the application of the method to the system of FVIDEs has been demonstrated on three problems. Besides, the numerical results of the proposed method have been compared with the other methods. Some conclusions have been given in the last section.

2 Description of the Method Based on Bernstein Polynomials

A projection method was given by [15], for mth-order linear FVIDE in the most general form. The main idea of this method is applied and improved to approximate the solution of the following FVIDEs system:

$$\sum_{k=0}^{m} \mathbf{q}_k(x)\,\mathbf{y}^{(k)}(x) = \mathbf{g}(x) + \int_a^b \mathbf{f}(x,t)\,\mathbf{y}(t)\,dt + \int_a^x \mathbf{v}(x,t)\,\mathbf{y}(t)\,dt;\ a \le x, t \le b \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}\left[\mathbf{A}_k\mathbf{y}^{(k)}(a) + \mathbf{B}_k\mathbf{y}^{(k)}(b) + \mathbf{C}_k\mathbf{y}^{(k)}(c)\right] = \lambda,\ a < c < b \tag{2}$$

by the generalized Bernstein polynomials as follows:

$$y_i^{(k)}(x) \cong B_N^{(k)}(y_i; x) = \sum_{r=0}^{N} y\left(a + \frac{(b-a)r}{N}\right) p_{r,N}^{(k)}(x);\ i = 1, 2, ..., n. \tag{3}$$

Here $\mathbf{q}_k(x) = \left[q_{ij}^k(x)\right]$, $\mathbf{f}(x,t) = [f_{ij}(x,t)]$, $\mathbf{v}(x,t) = [v_{ij}(x,t)]$ are $n \times n$ matrices; $\mathbf{g}(x) = [g_i(x)]$ and $\mathbf{y}(x) = [y_i(x)]^T$ are $n \times 1$ matrices for $i, j = 1, ..., n$. $\mathbf{A}_k = \left[\alpha_l^k\right]$, $\mathbf{B}_k = \left[\beta_l^k\right]$, $\mathbf{C}_k = \left[\gamma_l^k\right]$ are $m \times n$ matrices; and $\lambda = [\lambda_l]$ is $m \times 1$ matrix for $l = 1, ..., m$.

Theorem 2.1: Let $x_s \in [a, b]$ be collocation points. If system (1) has a generalized Bernstein polynomial solution (3), linear FVIDEs system with n unknowns and mixed conditions have following matrix relations:

$$\left[\sum_{k=0}^{m} \mathbf{Q}_k\mathbf{P}\mathbf{D}^k - \mathbf{F} - \mathbf{V}\right]\mathbf{Y} = \mathbf{G}, \tag{4}$$

$$\sum_{k=0}^{m-1}\left[\mathbf{A}_k\overline{\mathbf{p}}(a)\,\overline{\mathbf{d}}^k + \mathbf{B}_k\overline{\mathbf{p}}(b)\,\overline{\mathbf{d}}^k + \mathbf{C}_k\overline{\mathbf{p}}(c)\,\overline{\mathbf{d}}^k\right]\mathbf{Y} = \lambda. \tag{5}$$

Here $\overline{\mathbf{p}}(x)$ is $n \times n(N+1)$ matrix, $\overline{\mathbf{d}}^k$ is $n(N+1) \times n(N+1)$ matrix and $\overline{\mathbf{Y}}$ is $n(N+1) \times 1$ matrix, $\mathbf{Q}_k = diag[\mathbf{q}_k(x_s)]$, $\mathbf{P} = [\overline{\mathbf{p}}(x_s)]$, $\mathbf{D}^k = \left[\overline{\mathbf{d}}^k\right]$, $\mathbf{F} = [\mathbf{F}(x_s)]$ and $\mathbf{V} = [\mathbf{V}(x_s)]$ are $n(N+1) \times n(N+1)$ matrices. $\mathbf{G} = [\mathbf{g}(x_s)]$ and $\mathbf{Y} = [\overline{\mathbf{Y}}]$ are $n(N+1) \times 1$ matrices.

Proof. Since system (1) has a generalized Bernstein polynomial solution (3), unknown functions and their derivatives can be written as

$$y_i^{(k)}(x) \cong \mathbf{p}^{(k)}(x)\,Y_i = \mathbf{p}(x)\,\mathbf{d}^k Y_i;\ \ i = 1, ..., n.$$

Here $\mathbf{p}(x)$ is $1 \times (N+1)$ matrix, $\mathbf{d}$ is $(N+1) \times (N+1)$ matrix defined in Theorem 1.1, and

$$Y_i = \left[\ y_i(a) \quad y_i\left(a + \tfrac{b-a}{N}\right) \quad \dots \quad y_i(b)\ \right]^T;\ i = 1, ..., n$$

is $(N+1) \times 1$ matrix. Compactly, the unknow functions and their derivatives can be restated by

$$\mathbf{y}^{(k)}(x) \cong \overline{\mathbf{p}}(x)\,\overline{\mathbf{d}}^k\mathbf{Y};\ k = 0, 1, ..., m, \tag{6}$$

where the elements of matrices are defined as follows:

$$\mathbf{y}^{(k)}(x) = \begin{bmatrix} y_1^{(k)}(x) \\ y_2^{(k)}(x) \\ \vdots \\ y_n^{(k)}(x) \end{bmatrix},\ \overline{\mathbf{p}}(x) = \begin{bmatrix} \mathbf{p}(x) & 0 & \dots & 0 \\ 0 & \mathbf{p}(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{p}(x) \end{bmatrix}_{n\times n},$$

$$\overline{\mathbf{d}}^k = \begin{bmatrix} \mathbf{d}^k & 0 & \dots & 0 \\ 0 & \mathbf{d}^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{d}^k \end{bmatrix}_{n\times n},\ \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}.$$

Substituting relation (6) into Eq. (1) yields

$$\sum_{k=0}^{m} \mathbf{q}_k(x)\,\overline{\mathbf{p}}(x)\,\overline{\mathbf{d}}^k\mathbf{Y} \cong \mathbf{g}(x) + \mathbf{F}(x)\,\mathbf{Y} + \mathbf{V}(x)\,\mathbf{Y}. \tag{7}$$

Here, the explicit forms of the above matrices are as follows:

$$\mathbf{q}_k(x) = \begin{bmatrix} q_{11}^k(x) & q_{12}^k(x) & \dots & q_{1n}^k(x) \\ q_{21}^k(x) & q_{22}^k(x) & \dots & q_{2n}^k(x) \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}^k(x) & q_{n2}^k(x) & \dots & q_{nn}^k(x) \end{bmatrix}$$

$$\mathbf{g}(x)=\begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix},$$

$$\mathbf{F}(x)=\begin{bmatrix} \int_a^b f_{11}(x,t)\mathbf{p}(t)\,dt & \int_a^b f_{12}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^b f_{1n}(x,t)\mathbf{p}(t)\,dt \\ \int_a^b f_{21}(x,t)\mathbf{p}(t)\,dt & \int_a^b f_{22}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^b f_{2n}(x,t)\mathbf{p}(t)\,dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_{n1}(x,t)\mathbf{p}(t)\,dt & \int_a^b f_{n2}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^x f_{nn}(x,t)\mathbf{p}(t)\,dt \end{bmatrix},$$

$$\mathbf{V}(x)=\begin{bmatrix} \int_a^x v_{11}(x,t)\mathbf{p}(t)\,dt & \int_a^x v_{12}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^x v_{1n}(x,t)\mathbf{p}(t)\,dt \\ \int_a^x v_{21}(x,t)\mathbf{p}(t)\,dt & \int_a^x v_{22}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^x v_{2n}(x,t)\mathbf{p}(t)\,dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^x v_{n1}(x,t)\mathbf{p}(t)\,dt & \int_a^x v_{n2}(x,t)\mathbf{p}(t)\,dt & \dots & \int_a^x v_{nn}(x,t)\mathbf{p}(t)\,dt \end{bmatrix}.$$

Since $y_i^{(k)}(x_s)=B_N^{(k)}(y_i;x_s); i=1,...,n$ is valid on the collocation points $x_s\in[a,b]$ for $s=0,1,...,N$, Equation (7) becomes

$$\sum_{k=0}^{m}\mathbf{q}_k(x_s)\,\overline{\mathbf{p}}(x_s)\,\overline{\mathbf{d}}^k\mathbf{Y}-\mathbf{F}(x_s)\,\mathbf{Y}-\mathbf{V}(x_s)\,\mathbf{Y}=\mathbf{g}(x_s).$$

This system of equations can also be written compactly $\mathbf{WY}=\mathbf{G}$ such that $\mathbf{W}=\sum_{k=0}^{m}\mathbf{Q}_k\mathbf{P}\mathbf{D}^k-\mathbf{F}-\mathbf{V}$, where

$$\mathbf{Q}_k=\begin{bmatrix} \mathbf{q}_k(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{q}_k(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{q}_k(x_N) \end{bmatrix},\ \mathbf{P}=\begin{bmatrix} \overline{\mathbf{p}}(x_0) \\ \overline{\mathbf{p}}(x_1) \\ \vdots \\ \overline{\mathbf{p}}(x_N) \end{bmatrix},\ \mathbf{D}^k=\left[\overline{\mathbf{d}}^k\right],$$

$$\mathbf{F}=\begin{bmatrix} \mathbf{F}(x_0) \\ \mathbf{F}(x_1) \\ \vdots \\ \mathbf{F}(x_N) \end{bmatrix},\mathbf{V}=\begin{bmatrix} \mathbf{V}(x_0) \\ \mathbf{V}(x_1) \\ \vdots \\ \mathbf{V}(x_N) \end{bmatrix},\mathbf{G}=\begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}.$$

Similarly, substituting $x=a$, $x=b$, and $x=c$ into Eq. (6), given conditions is written in the form $\mathbf{UY}=\lambda$ such that

$$\mathbf{U}=\sum_{k=0}^{m-1}\mathbf{A}_k\overline{\mathbf{p}}(a)\,\overline{\mathbf{d}}^k+\mathbf{B}_k\overline{\mathbf{p}}(b)\,\overline{\mathbf{d}}^k+\mathbf{C}_k\overline{\mathbf{p}}(c)\,\overline{\mathbf{d}}^k. \quad (8)$$

Thus, the proof is completed.

The following steps are applied to solve the system of FVIDEs (1) under the mixed conditions (2):

Step 1. First, the matrices $\mathbf{Q}_k$, $\mathbf{P}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{V}$ defined in Theorem 2.1 are computed depending on the collocation points, and then the fundamental matrix relation belonging to (4) is obtained, it can be stated as

$$\mathbf{WY}=\mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}]. \quad (9)$$

This matrix equation corresponds to an $n(N+1)$ dimensional system of linear algebraic equations with unknown coefficients matrix $\mathbf{Y}$.

Step 2. By calculating the matrices in Eq. (8) at the given points, the augmented matrix form of the mixed conditions can be expressed as

$$[\mathbf{U};\lambda]. \quad (10)$$

Step 3. There are two techniques available for obtaining the solution of Eq. (9) under conditions (10). Initially, the arrays of the row matrices (10) can be added under the matrix (9). This gives the new augmented rectangular matrix $\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right]$, where $\widetilde{\mathbf{W}}$ is a matrix of dimensions $n(N+m+1)$-by-$n(N+1)$. This can be called an addition technique. On the other hand, the augmented square matrix $\left[\widehat{\mathbf{W}};\widehat{\mathbf{G}}\right]$ is produced by replacing some of the rows of matrix (9) with rows of matrix (10) by removing the first, middle or last rows of matrix (9) and writing the rows of matrix (10) in their place. This can also be called displacement technique. The number of collocation points, the given conditions and the order of the equations all affect these strategies.

Step 4. If $rank(\widetilde{\mathbf{W}})=rank\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right]=n(N+1)$ or $rank(\widehat{\mathbf{W}})=rank\left[\widehat{\mathbf{W}};\widehat{\mathbf{G}}\right]=n(N+1)$, the unknown coefficient matrix $\mathbf{Y}$ is uniquely determined by $\mathbf{Y}=\left(\widetilde{\mathbf{W}}\right)^{-1}\widetilde{\mathbf{G}}$ or $\mathbf{Y}=\left(\widehat{\mathbf{W}}\right)^{-1}\widehat{\mathbf{G}}$. This system can be solved easily by the standard methods.

3 Illustrations of the Method

In this section, three numerical examples are provided to support the suggested approach. The first example demonstrates how simple it is to implement the method. Moreover, the success of the method is shown by the numerical results. Furthermore, the numerical results of the proposed method have been compared with those obtained using different methods. Accordingly, the suggested method's outcomes demonstrated its importance, significant, remarkable, and effective in solving the system of FVIDEs. These numerical results have been calculated using the MATLAB programme. The absolute and maximum errors listed in the tables are defined by:

$$e_{i,N} = |y_{i,N}(x) - y_i(x)| \text{ and } \|e_{i,N}\|_\infty = \max_{x_s \in [a,b]} |e_{i,N}(x_s)|.$$

Here $y_{i,N}(x)$ is the Bernstein approximation and $y_i(x)$ is the exact solution of the system.

Example 3.1: The second-order system of FIDEs defined on $-1 \le x, t \le 1$ given in [5], is as follows:

$$\begin{aligned} -y_1'' + y_2' + xy_1 + y_2 &= -x^4 + 6x^2 + 7x - 2 \\ &+ \int_{-1}^{1} \left[ty_1(t) - t^3 y_2(t)\right] dt + \int_{-1}^{x} xy_2(t)\, dt \\ xy_1'' - y_2'' + y_1 - x^2 y_2 &= -3x^4 - x^3 - x^2 + 3x - 14 \\ &+ \int_{-1}^{1} \left[(t+2)y_1(t) - ty_2(t)\right] dt + \int_{-1}^{x} ty_1(t)\, dt \\ y_1(0) = 2, \quad y_1'(1) = 3, &\quad y_2(0) = 0, \quad y_2'(-1) = -6. \end{aligned}$$

Let us approach the solution of the system using Bernstein polynomials for $m = 2$ and $N = 2$, and then show step by step the application of the method to this problem. Then, the collocation points are of the form

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1.$$

According to the matrices provided in Section 2, the matrices in Eq. (1) for this system are as follows:

$$\mathbf{q}_0(x) = \begin{bmatrix} x & 1 \\ 1 & -x^2 \end{bmatrix}, \ \mathbf{q}_1(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{q}_2(x) = \begin{bmatrix} -1 & 0 \\ x & -1 \end{bmatrix}, \ \mathbf{v}(x,t) = \begin{bmatrix} 0 & x \\ t & 0 \end{bmatrix},$$

$$\mathbf{f}(x,t) = \begin{bmatrix} t & -t^3 \\ t+2 & -t \end{bmatrix},$$

$$\mathbf{g}(x) = \begin{bmatrix} -x^4 + 6x^2 + 7x - 2 \\ -3x^4 - x^3 - x^2 + 3x - 14 \end{bmatrix},$$

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \mathbf{y}'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix},$$

$$\mathbf{y}''(x) = \begin{bmatrix} y_1''(x) \\ y_2''(x) \end{bmatrix}, \ \lambda = \begin{bmatrix} 2 & 0 & 3 & -6 \end{bmatrix}^T.$$

Then the fundamental matrix of the system can be written as

$$(\mathbf{Q}_2\mathbf{P}\mathbf{D}^2 + \mathbf{Q}_1\mathbf{P}\mathbf{D} + \mathbf{Q}_0\mathbf{P} - \mathbf{V} - \mathbf{F})\mathbf{Y} = \mathbf{G}.$$

Here the entries of this matrices are as follows:

$$\mathbf{Q}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q}_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} -4 \\ -20 \\ -2 \\ -14 \\ 10 \\ -16 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -17/48 & -1/8 & -1/48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 2/3 & 2/3 \\ -1/3 & 0 & 1/3 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} -1/3 & 0 & 1/3 & 1/5 & 0 & -1/5 \\ 1 & 4/3 & 5/3 & 1/3 & 0 & -1/3 \\ -1/3 & 0 & 1/3 & 1/5 & 0 & -1/5 \\ 1 & 4/3 & 5/3 & 1/3 & 0 & -1/3 \\ -1/3 & 0 & 1/3 & 1/5 & 0 & -1/5 \\ 1 & 4/3 & 5/3 & 1/3 & 0 & -1/3 \end{bmatrix}.$$

Finally, by considering Equations (9) and (10), the augmented matrix and conditions become

$$[\mathbf{W};\mathbf{G}] = \begin{bmatrix} -7/6 & 1 & -5/6 & -1/5 & 1 & 1/5 & ; & -4 \\ -1/2 & -1/3 & -13/6 & -11/6 & 1 & -1/6 & ; & -20 \\ -1/6 & 1 & -5/6 & -9/20 & 1/2 & 19/20 & ; & -2 \\ -19/48 & -17/24 & -67/48 & -5/6 & 1 & -1/6 & ; & -14 \\ -1/6 & 1 & 1/6 & -13/15 & -5/3 & 23/15 & ; & 10 \\ -1/6 & -7/3 & -1/2 & -5/6 & 1 & -7/6 & ; & -16 \end{bmatrix}$$

and

$$[\mathbf{U};\lambda] = \begin{bmatrix} 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & ; & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & ; & 3 \\ 0 & 0 & 0 & -1 & 1 & 0 & ; & -6 \end{bmatrix}.$$

Utilizing the addition technique mentioned in Step 3, the augmented matrix for this system occurs as

$$\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right]=\begin{bmatrix} -7/6 & 1 & -5/6 & -1/5 & 1 & 1/5 & ; & -4\\ -1/2 & -1/3 & -13/6 & -11/6 & 1 & -1/6 & ; & -20\\ -1/6 & 1 & -5/6 & -9/20 & 1/2 & 19/20 & ; & -2\\ -19/48 & -17/24 & -67/48 & -5/6 & 1 & -1/6 & ; & -14\\ -1/6 & 1 & 1/6 & -13/15 & -5/3 & 23/15 & ; & 10\\ -1/6 & -7/3 & -1/2 & -5/6 & 1 & -7/6 & ; & -16\\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & ; & 2\\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & ; & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & ; & 3\\ 0 & 0 & 0 & -1 & 1 & 0 & ; & -6 \end{bmatrix}.$$

Solving this linear system gives the exact solution of the problem as

$$y_1(x)=3x+2 \quad \text{and} \quad y_2(x)=3x^2.$$

Although, [5], found exact solutions for $N=3$ in their study, we have found exact solutions for $N=2$. Thus, the proposed method is faster than the Taylor collocation method. It also shows that the exact solution can be found in cases where the solution is polynomial if the value N is taken as the degree of the polynomial.

Example 3.2: Let us consider a second-order system of VIDEs, [1], [10] :

$$y_1''(x)+2xy_1'(x)-y_1(x)-\int_0^x (y_1(t)-y_2(t))\,dt = 2+x-e^x+2xe^x-\cos x,$$

$$y_2''(x)+y_2'(x)-2xy_2(x)-\int_0^x (y_1(t)+y_2(t))\,dt = 2\cos x-3x-(1+2x)\sin x-e^x,$$

with the initial conditions

$$y_1(0)=1,\;\; y_1'(0)=1,\;\; y_2(0)=1,\;\; y_2'(0)=1,$$

and exact solutions $y_1(x)=e^x$, $y_2(x)=1+\sin x$.

By the proposed method, fundamental matrix equation and its conditionals become

$$\left(\mathbf{PD}^2+\mathbf{Q}_1\mathbf{PD}+\mathbf{Q}_0\mathbf{P}-\mathbf{V}\right)\mathbf{Y}=\mathbf{G},$$

$$\left[\overline{\mathbf{p}}(0)+\overline{\mathbf{p}}(0)\overline{\mathbf{d}}\right]\mathbf{Y}=\lambda.$$

In Table 1 and Table 2, the absolute errors are compared with those of the other methods, [1], [10]. The numerical results of the proposed method have been calculated using the displacement technique. In our method, collocation points have been considered as $x_s=s/N; s=0,1,...,N$, and Newton-Cotes points have been considered as $x_s=(2s-1)/(2(N+1)); s=1,2,...,2N-2$ for Bernstein operational matrix method. Although the tables indicate that the results of the proposed method are close to the other methods, the values of the proposed method are better than the others as move away from the initial point. Besides, the absolute errors of the proposed method approach zero with increasing N values.

Table 1. The Comparison of the Absolute Errors for $\mathbf{y}_1(x)$.

x_s	Proposed Method			Bernstein Operational Matrix Method [1]		Spectral Method [10]
	$N=5$	$N=10$	$N=15$	$N=5$	$N=10$	$N=5$
0.0	0	0	0	$8.9e-016$	$8.9e-016$	$0.0e-009$
0.1	$2.5e-007$	$6.7e-014$	$2.2e-015$	$6.7e-007$	$1.5e-013$	$1.0e-009$
0.2	$1.2e-006$	$1.7e-013$	$4.8e-015$	$1.4e-006$	$2.9e-013$	$9.1e-008$
0.3	$2.1e-006$	$2.6e-013$	$4.1e-015$	$1.8e-006$	$4.3e-013$	$1.1e-006$
0.4	$2.5e-006$	$3.5e-013$	$6.8e-015$	$2.4e-006$	$5.6e-013$	$6.0e-006$
0.5	$3.1e-006$	$4.4e-013$	$8.1e-015$	$3.0e-006$	$7.0e-013$	$2.3e-005$
0.6	$4.3e-006$	$5.2e-013$	$9.2e-015$	$3.3e-006$	$8.3e-013$	$7.0e-005$
0.7	$4.9e-006$	$6.1e-013$	$1.1e-014$	$4.3e-006$	$9.5e-013$	$1.8e-004$
0.8	$2.1e-006$	$6.6e-013$	$1.2e-014$	$1.2e-005$	$1.1e-012$	$4.1e-004$
0.9	$3.5e-005$	$1.7e-012$	$1.3e-014$	$4.2e-005$	$4.4e-013$	$8.5e-004$
1.0	$1.3e-004$	$2.8e-011$	$1.3e-013$	$1.3e-004$	$3.1e-011$	$1.6e-003$

Table 2. The Comparison of the Absolute Errors for $\mathbf{y}_2(x)$.

x_i	Proposed Method			Bernstein Operational Matrix Method [1]		Spectral Method [10]
	$N=5$	$N=10$	$N=15$	$N=5$	$N=10$	$N=5$
0.0	0	0	0	$7.8e-016$	$8.9e-016$	$0.0e-009$
0.1	$4.6e-008$	$4.2e-014$	$1.3e-015$	$1.4e-007$	$9.0e-014$	$0.0e-009$
0.2	$2.1e-007$	$9.9e-014$	$2.7e-015$	$2.9e-007$	$1.6e-013$	$2.0e-009$
0.3	$3.6e-007$	$1.5e-013$	$6.3e-015$	$3.6e-007$	$2.4e-013$	$4.3e-008$
0.4	$4.1e-007$	$1.9e-013$	$6.9e-015$	$4.5e-007$	$3.0e-013$	$3.3e-007$
0.5	$4.7e-007$	$2.4e-013$	$8.3e-015$	$5.5e-007$	$3.6e-013$	$1.5e-006$
0.6	$6.9e-007$	$2.8e-013$	$9.7e-015$	$5.6e-007$	$4.2e-013$	$5.5e-006$
0.7	$7.7e-007$	$3.2e-013$	$1.1e-014$	$7.7e-007$	$4.7e-013$	$1.6e-005$
0.8	$1.1e-006$	$3.4e-013$	$1.2e-014$	$2.7e-006$	$5.4e-013$	$4.1e-005$
0.9	$9.8e-006$	$9.5e-013$	$1.4e-014$	$1.1e-005$	$3.4e-013$	$9.4e-005$
1.0	$3.5e-005$	$1.6e-011$	$5.5e-013$	$3.3e-005$	$1.7e-011$	$2.0e-004$

Example 3.3: Let us consider following system of FIDEs given in $0\le x\le 1$:

$$y_1''-xy_2'-y_1=(x-2)\sin x+\int_0^1 (x\cos t\,y_1(t)-x\sin t\,y_2(t))\,dt$$

$$y_2''-2xy_1'+y_2=-2x\cos x+\int_0^1 (\sin x\cos t\,y_1(t)-\sin x\sin t\,y_2(t))\,dt$$

$$y_1(0)=0,\; y_1'(0)=1,\; y_2(0)=1,\; y_2'(0)=0.$$

Exact solution of this problem is $y_1(x)=\sin x$, $y_2(x)=\cos x$.

Table 3. The Comparison of the Maximum Errors for $\mathbf{y}_1(x)$.

N	Proposed Method	Bessel Collocation Method [2]	Fibonacci Collocation Method [8]
3	$4.4e-003$	$5.0e-003$	$5.3e-003$
7	$9.9e-008$	$5.0e-007$	$5.0e-007$
9	$1.9e-010$	$4.0e-009$	$4.0e-009$
10	$1.6e-011$	$2.7e-010$	$2.7e-010$
11	$2.3e-013$	$2.5e-011$	$2.5e-011$
12	$1.9e-014$	$1.2e-012$	$1.1e-012$

Table 4. The Comparison of the Maximum Errors for $\mathbf{y}_2(x)$.

N	Proposed Method	Bessel Collocation Method, [2]	Fibonacci Collocation Method, [8]
3	$1.3e-002$	$1.4e-002$	$1.4e-002$
7	$2.2e-007$	$6.3e-007$	$6.3e-007$
9	$4.0e-010$	$4.2e-009$	$4.2e-009$
10	$7.6e-010$	$3.0e-010$	$3.0e-010$
11	$4.9e-013$	$2.6e-011$	$2.6e-011$
12	$3.2e-014$	$1.6e-012$	$1.5e-012$

In Table 3 and Table 4, the maximum errors are compared with those of the others, [2], [8]. For all methods in the tables, the numerical results have been calculated on the collocation points $x_s = s/N$. The tables indicate that the proposed method yields better results than the others for increasing N values. Moreover, the numerical results that computed by replacing the last rows of the augmented matrix with the conditions are more effective than the other displacement and addition techniques. Thus, one reason for the effective results is that the augmented matrix has been obtained by using the replacement technique.

4 Conclusions and Inferences

In this study, a collocation method is improved by considering the fundamental properties of Bernstein polynomials to solve a system of linear IDEs. The proposed method transforms a system of linear FVIDEs into a system of linear algebraic equations due to the matrix forms of the Bernstein polynomials and their derivatives. To demonstrate the applicability and efficiency of the method, three examples are considered. Example 3.1 illustrates how this method is applied to the problem and shows that it is faster than the Taylor collocation method. In Examples 3.2 and 3.3, the results of the errors are compared with those of other methods, which are Bernstein operational matrix, Spectral, Bessel, and Fibonacci collocation methods. In both examples, better results are obtained as the value of N increases, by using the proposed method. Moreover, when the solution is polynomial, the exact solution can be found if N is chosen as the degree of the polynomial. The use of the displacement technique to obtain the augmented matrix proves advantageous for achieving more effective results. Considering all aspects of the study, the proposed method is a suitable and rigorous numerical approach for solving various linear systems, including differential, integral, and integro-differential equations. Therefore, this method could be applied to models like the Markow modulated jump diffusion process, [11], in future studies.

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