On Numerical Radius Inequalities for Hilbert Space Operators

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Abstract: - This paper aims to find special cases of some inequalities for numerical radii and spectral radii of a bounded linear operator on a Hil-bert space, we focus on numerical radii inequalities for restricted linear operators on complex Hil-bert spaces for the case of one and two operators, and study the numerical range of an operator K on a complex Hil-bert space H, after that we present some inequalities for numerical radii and spectral radii and studied it to find new results. At the end of this paper we find several inequalities for numerical radii by using the spectral norm, this study is necessary to find other bound for zeros of polynomials and this study is necessary to find new bounds for the zeros of polynomials by a playing the new results to the companion matrix.

Key-Words: - Spectral norm, Numerical radii, Spectral radii, Hil-bert space, Numerical range, Complex Hil-bert space.

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1 Introduction

The concepts of numerical radii and numerical range play an important role in various fields of contemporary mathematics, including operator trigonometry, operator theory, spectral analysis, and others. Since 1997, one of these objects has grown greatly. [1], however, the size of the areas of applications for numerical radii and ranges is very problem to estimate. The numerical range of an operator K on a complex Hil-bert space H. For the subset of the complex plane $\mathbb{C}$ given by:
$W(\mathrm{K})=\{\langle Kx,x\rangle : x\in H, \|x\|=1\}$. [2]

The numerical radii ω(K) of an operator K on H is given by :

$$\omega(K)=sup\{|T|, T\in W(K)\} \quad . [3]$$
$$= sup\{|\langle Kx,x\rangle|, \|x\|=1\}.$$

(1.1.1)

Obviously, by (1.1.1), for any x ∈ H someone has $|\langle Kx,x\rangle|\le \omega(\mathrm{K})\|x\|^2$.

It focuses on numerical radii inequalities for restricted linear operators on complex Hil-bert spaces for the case of one and two operators. Let H be a complex Hil-bert space with an inner product $\langle \ldots \rangle$ and let $B(\mathrm{H})$ denote the algebra of bounded linear operators on H. The algebra of all complex matrices will be denoted by $M_n(\mathbb{C})$ To get our goal, [4] we list some of the following Theorems.

Theorem 1.1, [5]

(1) If KD = DK, then
$\omega(KD)\le 2\omega(\mathrm{K})\omega(\mathrm{D})$.

(2) If K or D is normal such that KD = DK, then
$$\omega(KD)\le \omega(\mathrm{K})\omega(\mathrm{D}).$$

Lemma 1.1, [6], [7]

If *E, K,* and *D* are self-ad joint operators in *B(H),* then $\omega^r(\mathrm{E})\le \frac{1}{2}\left\| |KK^*|^{\alpha r} + |DD^*|^{(1-\alpha)r}\right\|$.

Theorem 1.2, [8], [9].

Suppose *K* and *D* be self-adjoint operators in *B(H),* and *r ≥ 1,* then
$\omega^r(K+D)\le 2^{r-1}\left\| |K|^r + |D|^r\right\|$.

Theorem 1.3, [10].

Suppose $\mathrm{K},\mathrm{D}\in B(H)$, then

$$\omega^r(K)\le \frac{1}{2}\left\| |K|^{2r\alpha} + |K^*|^{2r(1-\alpha)}\right\|,$$

$$\omega^r(K+D) \leq 2^{r-2}\left\| |K|^{2r\alpha} + |K^*|^{2r(1-\alpha)} + |D|^{2r\alpha} + |D^*|^{2r(1-\alpha)} \right\|$$

for 0 < α < 1 and r ≥ 1.

2 New Results for Numerical Radius and Spectral Radius Inequalities

We present some new inequalities for numerical radii and spectral radii.

In the following theorem, we obtain a special case for lemma 1.1.

Theorem 2.1
Let *E, K,* and *D* operators in *B(H)* such that *A* and *B* are positive and $\|Ex\| \leq \|Kx\|$, $\|E^*x\| \leq \|Dx\|$. *Then*

$$w(E) \leq \frac{1}{\sqrt[3]{2}}\left\| |KK^*| + |DD^*|^2 \right\|^{1/3}.$$

Proof:
Also, by Lemma 1.1, we have

$$\omega^r(E) \leq \frac{1}{2}\left\| |KK^*|^{\alpha r} |DD^*|^{(1-\alpha)r} \right\| \tag{1}$$

Letting $r = 3$ and $\alpha = \frac{1}{3}$. Then

$$\omega^3(E) \leq \frac{1}{2}\left\| |KK^*| + |DD^*|^2 \right\|. \tag{2}$$

$$w^3(E) \leq \frac{1}{2}\left\| |KK^*| + |DD^*|^2 \right\|. \tag{3}$$

By taking the cube root of both sides of (3) we obtain the result.

In the following theorem, we have a special result for Theorem 1.1.

Theorem 2.2
Let *K* and *D* be self-ad joint operators in *B(H).* Then

$$w(K+D) \leq \left\| |K| + |D| \right\| \tag{4}$$

Also, $$\omega(K+D) \leq \sqrt[3]{2}\left\| |K|^{\frac{3}{2}} + |D|^{\frac{3}{2}} \right\|^{2/3}$$

Proof:
By Theorem 1.1, we have

$$\omega^r(K+D) \leq 2^{r-1}\left\| |K|^r + |D|^r \right\| \quad \text{for} \quad r \geq 1$$

If r =1.5, we have

$$\omega^{\frac{3}{2}}(K+D) \leq \sqrt{2}\left\| |K|^{\frac{3}{2}} + |D|^{\frac{3}{2}} \right\|$$

So, we get the result.
and by letting $r = 1$, we have
$\omega(K+D) \leq \left\| |K| + |D| \right\|$, and since. Hence, we get the result.

Now we prove the famous inequality $\omega(K) \leq \|K\|$ as the following Theorem

Theorem 2.3
Let K be self-adjoint operator in B(H). Then

$$\omega(K) \leq \|K\|$$

Proof:
Taking D = *K and r = 1.5* in Theorem 1.2, we have

$$\omega^{\frac{3}{2}}(2K) \leq \sqrt{2}\left\| 2|K|^{\frac{3}{2}} \right\|$$

$$\omega(2K) \leq \sqrt[3]{2}\sqrt[3]{4}\left\| |K| \right\|.$$

$$\omega(2K) \leq 2\left\| |K| \right\|.$$

$$\omega(K) \leq \left\| |K| \right\|.$$

Theorem 2.4
Suppose K, D ∈ B(H), then

$$\omega(\mathrm{K}) \leq \frac{1}{\sqrt[3]{4}}\left\| |K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right\|^{2/3}$$

Proof:
By Theorem 1.3, we have

$\omega^r(\mathrm{K}) \leq \frac{1}{2}\left\| |K|^{2r\alpha} + |K^*|^{2r(1-\alpha)} \right\|$, for $0 < \alpha < 1$, $r \geq 1$

If $r = 1.5$, $\alpha = \frac{1}{2}$, we have

$$\omega^{\frac{3}{2}}(\mathrm{K}) \leq \frac{1}{2}\left\| |K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right\|$$

$$\omega(\mathrm{K}) \leq \frac{1}{\sqrt[3]{4}}\left\| |K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right\|^{2/3}$$

In the following theorem, we get a special result for Theorem 1.3

Theorem 2.5 *If K∈ B(H), then*

$$\omega(\mathrm{K}) \leq \frac{1}{\sqrt[3]{4}}\left\| \left(|K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

Proof:
By Theorem 1.3, we have

$$\omega^r(K+D) \le 2^{r-2}\left\| |K|^{2r\alpha} + |K^*|^{2r(1-\alpha)} + |D|^{2r\alpha} + |D^*|^{2r(1-\alpha)} \right\|$$

for $0 < \alpha < 1$, $r \ge 1$

If D = *K*, *r* = 1.5, and $\alpha = \frac{1}{2}$, we have

$$\omega^{\frac{3}{2}}(K+K) \le \frac{1}{\sqrt{2}}\left\| |K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} + |K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right\|$$

$$\omega^{\frac{3}{2}}(2K) \le \frac{1}{\sqrt{2}}\left\| 2\left(|K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right) \right\|$$

$$\omega(\mathrm{K}) \le \frac{\sqrt[3]{4}}{2\sqrt[3]{2}}\left\| \left(|K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

$$\omega(\mathrm{K}) \le \frac{1}{\sqrt[3]{4}}\left\| \left(|K|^{\frac{3}{2}} + |K^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

Theorem 2.6
Suppose K∈B(H), then

$$\omega(K^2) \le 2\|K\|$$

Proof:
By Theorem 1.1, we have

$$\omega(KD) \le 2\omega(\mathrm{K})\omega(\mathrm{D}).$$

put *D = 2K*, we have

$$\omega(2K^2) \le 2\omega(\mathrm{K})\omega(2K)$$

$$\omega(2K^2) \le 4\omega^2(\mathrm{K}) \le 4\|K\|^2$$

Hence, $\omega(2K^2) \le 4\|K\|^2$

$$\omega(K^2) \le 2\|K\|^2$$

3 Conclusions

We provided some inequalities for the numerical radii and operator norm of the bounded linear operator on a complete Hil-bert space. We have also achieved some new results in this area of research. In our graduate work, we will also have to derive particular cases of some inequalities for numerical radii and spectral norms of a bounded linear operator on a Hil-bert space. At the end of this thesis we will have to find several inequalities for numerical radii by using the spectral norm.

4 Future Works

We know that, if *K ∈ B(H), then*

$$r(\mathrm{K}) \le w(K) \le \|K\|$$

Also, if $\omega(\mathrm{K}) = \|\mathrm{K}\|$, then $r(\mathrm{K}) = \|K\|$.

From the inequalities in this paper, we can introduce similar inequalities of the spectral radius of a matrix. Hence, we can find new bound for the zeros of polynomials by applying these inequalities to the companion matrix.

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