Pole Assignment for Symmetric Quadratic Dynamical Systems: An Algorithmic Method

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Abstract: In this article an algorithmic method is proposed for the solution of the pole assignment problem which is associated with a symmetric quadratic dynamical system, when it is completely controllable. The problem is shown to be equivalent to two subproblems, one linear and the other multi-linear. Solutions of the linear problem must be decomposable vectors, i.e. they must lie in an appropriate Grassmann variety. The proposed method computes a reduced set of quadratic Plucker relations, with only three terms each, which describe completely the specific Grassmann variety. Using these relations one can solve the multi-linear problem and consequently calculate the feedback matrices which give a solution to the pole assignment problem. An illustrative example of the proposed algorithmic procedure is given. The main advantage of our approach is that the complete set of feedback solutions is obtained, over which further optimisation can be carried out, if desired. This is important for problems with structural constraints (e.g. decentralization) or norm-constraints on the feedback gain-matrix.

Key-Words: Control Theory, Pole assignment, Quadratic matrix pencils, Grassmann variety, Plucker relations, numerical algorithm

Received: May 29, 2023. Revised: May 26, 2024. Accepted: June 27, 2024. Published: July 29, 2024.

1 Introduction

The fundamental criterion of controllability is useful in deciding whether a symmetric quadratic system is state-controllable. This condition is intimately linked to state-feedback control which allows the designer to modify the behavior of the closed-loop dynamics in order to relocate troublesome eigenvalues in the complex plane, thus avoiding long transients, resonance phenomena and potential instability under model uncertainty. The paper provides a systematic approach for designing state-feedback controllers of symmetric quadratic dynamical systems, by decomposing the problem into two separate sub-problems, one linear and the other multi-linear, [1], [2], [3], [4]. To avoid excessive computational complexity, care has been taken to reduce algorithmic redundancy, via the computation of a reduced set of quadratic Plucker relations. The paper describes the two design steps of the approach which are illustrated with a numerical example. A background on aspects of algebraic geometry, on which many of the results are based, is also provided. The main advantage of our method is that it can be used to parametrize the complete set of state-feedback solutions in the multivariable case which is useful when additional optimization constraints are imposed, e.g. structural constraints in the form of decentralization patterns, or norm-constraints on the state-feedback matrix which may address robust-stability issues under model uncertainty. A further advantage of the proposed method is that it can be extended in a relatively straightforward manner to wider classes of models, e.g. higher-order matrix models or implicit (descriptor) systems described by a set of differential and algebraic equations, [5], [6], [7], [8]. Several results obtained in these two directions will be reported in future publications.

2 Problem definition and Methodology

The equation of motion for a matrix second-order system, e.g. a structural system with viscus damping and without externally applying forces is expressed as follows:

$$Mq''(t) + Dq'(t) + Kq(t) = 0 \qquad (1)$$

Here, M, D, K are the $n \times n$ mass, damping and stiffness matrices respectively. Also, $q(t)$ is the displacement vector, $q'(t)$ the velocity vector and $q''(t)$ the acceleration vector. In most applications M, D and K are symmetric matrices. Furthermore, M is typically positive definite and D, K are positive semi-definite or positive definite.

Separation of variables $q(t) = e^{\lambda t}c$ where c is a constant vector gives us the quadratic eigenvalue problem:

$$P(\lambda_j)c_j = 0, \quad j = 1, 2, \ldots, 2n$$

in which $P(\lambda)$ denotes the quadratic matrix pencil: $P(\lambda) = \lambda^2 M + \lambda D + K$. The eigenvalues and eigenvectors of the pencil govern the free response of the system. The model can be used to identify poorly damped oscillations and resonance phenomena, which we may be able to avoid by the appropriate selection of the matrix parameters.

When an actuating force $F(t)$ is applied, the model described by the above equations becomes:

$$Mq''(t) + Dq'(t) + Kq(t) = F(t) \tag{2}$$

Often the force can be selected by the designer, in which case

$$F(t) = Bu(t) \tag{3}$$

where $u(t)$ is the $m \times 1$ control input vector and B is the $n \times m$ input matrix.

One of the major concerns for the control engineer is to ensure stability of the control system. Thus, the behaviour of the system is usually modified by applying state-feedback control to relocate the troublesome eigenvalues in the complex plane. In the single-input case, under the assumption of controllability, the problem has a unique solution. This is no longer the case in the multivariable case, in which typically an infinite set of state-feedback matrices can be used to assign the closed-loop eigenvalues at specified locations. Existing algorithms [9], [10], [11] do not in general parameterize completely the solution set, or give an implicit parametrization which is difficult to apply in practice. In contrast, our approach gives a complete solution to the problem, although intermediate steps of the algorithm may be computationally demanding, especially for high-dimensional problems, if certain redundant relations are not removed. The analysis in the present work involves quadratic symmetric models, which arise naturally in mechanical vibration and electrical circuit systems, however the techniques can be extended with minor modifications to matrix higher order systems and implicit (descriptor) systems. Deriving parametric solutions to a particular design objectives is important in order to select specific solutions which optimize additional objectives or constraints. In the present context, in addition to placing the eigenvalues of the closed-loop system to specific locations, a designer may need to consider further issues, such as sparsity in the state-feedback matrix corresponding to specific decentralization objectives, minimizing the norm of the state-feedback matrix to improve the sensitivity properties of the design, or more general problems of eigen-structure assignment. If the system is (open-loop) unstable, we aim, as a minimum requirement, to stabilize it. If the system is (open-loop) stable, it is desirable to maintain some degree of relative stability and robust-stability margins, i.e. ensure that stability is maintained under realistic uncertainty conditions. In general, the notion of stabilization is connected to the problem of relocation of troublesome eigenvalues of the system in equation (2), also called pole placement. In theory, eigenvalue relocation to arbitrary locations of the complex plane can always be achieved if the system is state-controllable, although in practice this is limited by constraints on the amplitude or energy of the control signal, system bandwidth constraints, and robust stability issues. For the model described in equations (2) and (3), for any choice of a conjugate-symmetric set of poles, suitable real $m \times n$ matrices F_1 and F_2 can be found such that under the state-feedback control law:

$$u(t) = -F_1 q(t) - F_2 q'(t) \tag{4}$$

the corresponding closed-loop system described by the equation

$$Mq''(t) + (D + BF_2)q'(t) + (K + BF_1)q(t) = 0 \tag{5}$$

has the chosen set of poles. Equivalently, suppose that the required closed-loop polynomial is:

$$f(s) = a_0 + a_1 s + \ldots + a_{2n-1}s^{2n-1} + s^{2n} \tag{6}$$

This can be written more compactly in the form

$$f(s) = e^t_{2n}(s)\tilde{a} \tag{7}$$

where

$e^t_{2n}(s) = [1 \ s \ \ldots \ s^{2n}]$ and $\tilde{a}^T = [a_0 \ a_1 \ \ldots \ a_{2n-1} \ 1]$

We wish to compute the matrices F_1 and F_2 such that the characteristic polynomial

$$\tilde{\varphi}(s) = \det(s^2 M + s(D + BF_2) + (K + BF_1)) \tag{8}$$

of the quadratic pencil

$$P_c := s^2 M + s(D + B\,F_2) + (K + B\,F_1) \tag{9}$$

associated with the closed-loop system (5) is equal to:

$$\tilde{\varphi}(s) = (\det M) f(s) \tag{10}$$

Define:

$$Q(s) := [s^2 M + sD + K \mid sB \mid B] \tag{11}$$

and

$$V = \begin{bmatrix} I_n \\ F_2 \\ F_1 \end{bmatrix} \tag{12}$$

Then we can rewrite equation (9) as:

$$P_c(s) = Q(s)V \tag{13}$$

Using compound matrices and applying the Binet-Cauchy theorem we can write,

$$\tilde{\varphi}(s) = \det(P_c(s)) \equiv C_n(P_c(s)) \tag{14}$$
$$= C_n(Q(s)V) = C_n(Q(s))C_n(V) \tag{15}$$

We now define:

$$P := P(M, D, K, B) \in \mathbb{R}^{(2n+1)\times\ell} \tag{16}$$

Since $C_n(Q(s)) := e^t_{2n}(s)P$, the columns of P are the coefficient vectors of the elements of $C_n(Q(s))$, which are polynomials of maximum degree $2n$ and

$$g := C_n(V) = C_n\left(\begin{bmatrix} I_n \\ F_2 \\ F_1 \end{bmatrix}\right) \in \mathbb{R}^\ell \tag{17}$$

where $\ell := \binom{n+2m}{n}$. Combining equations (6), (10) and (15) we get

$$e^t_{2n}(s)Pg = e^t_{2n}(s)(\det M)\tilde{a} := e^t_{2n}(s)a$$

which in turn implies that:

$$Pg = a \tag{18}$$

3 Techniques from algebraic geometry and algorithmic solution

In this section we construct the quadratic Plucker relations of the problem which are algebraically independent. In order to achieve this, a criterion based on the correspondence between vectors of the Grassmann variety and the lexicographical ordering is applied. Let us assume that $\Omega(m, n)$ is the Grassmann Variety of the projective space $P^k(\mathbb{F})$ with $k = \binom{n}{m} - 1$ whose dimension is given by the equation

$$\text{dim}\Omega(m, n) = m(n - m) \tag{19}$$

It can be proved, [12], that the total number of Plucker relations which define the Grassmann Variety is:

$$\binom{n}{m-1}\binom{n}{m+1} \tag{20}$$

However these are dependent, in the sense that some of these relations are linear combinations of the others. Hence, if all these relations are used as equality constraints in the remaining part of the algorithm, this would raise the complexity of the solution unnecessarily. The minimum number of independent equations n_{eq} that describe a variety, of dimension equal to dim$\mathcal{V}$, is

$$n_{eq} = \text{dim}P - \text{dim}\mathcal{V} \tag{21}$$

and so

$$n_{eq} = \binom{n}{m} - 1 - m(n - m) \tag{22}$$

In this section, we propose an algorithm which reduces the full set of equations that describes the Grassmann variety to an exact minimal set of independent equations whose number is given in equation (22). To give an illustration of the reduction in complexity that this entails, note that, for example if $m = 2$ and $n = 4$, the full number of Plucker relations according to the equation (20) is 16. This is reduced to only 1 equation which consists the reduced set (see (22)).In case where $m = 3$ and $n = 5$ the total number of Plucker relations according to equation (20) is 50 equations while the reduced set from the equation (22) has only 3 , i.e. in this case 47 of the total number of equations are redundant. It should be clear that the reduction in complexity is highly significant and grows with the dimensionality of the problem, [12].

Before stating the Theorems on which the algorithm is based, we introduce the following notation and definitions, [12], [13], [14]. Let the binomial coefficient be

$$k := \binom{n}{m}, n, m \in \mathbb{N},\ n \geq m$$

Denote the set of the first n natural numbers as:

$$T_n := \{\kappa \in \mathbb{N}^* : 1 \leq \kappa \leq n\}$$

with $\mathbb{N}^*$ the set of naturals (excluding zero). Let

$$T^n_m := \{(a_1, \ldots, a_m) :\ a_j \in \mathcal{T}_n\ , 1 \leq j \leq m\}$$

be the set of sequences of length m with elements a_j, $1 \leq j \leq m$, not necessarily distinct. Also, let

$$\tilde{T}^n_m := \{(a_1, \ldots, a_m)\ \in \mathcal{T}^n_m\ : a_\kappa \neq a_\lambda,\ \text{if}\ \kappa \neq \lambda\}$$

be the set of sequences, where the elements of each sequence are distinct. Denote by D^n_m all m-tuples $< a_1, a_2, \ldots, a_m >$ such that $(a_1, \ldots, a_m)\ \in \tilde{T}^n_m$ and $a_\kappa < a_\lambda$ if $\kappa < \lambda$. We can order the elements of the set D^n_m as follows:

$$< a_1, a_2, \ldots, a_m > \prec < j_1, j_2, \ldots, j_m >$$

if and only if there exists $\kappa \in \mathcal{T}_n$ such that $a_\kappa < j_\kappa$ and $a_\lambda = j_\lambda$ for every $\lambda < \kappa$. The above relation is a total ordering called lexicographical ordering. Using the lexicographical ordering, we can now characterize the k elements of D_m^n by

$$\begin{aligned} a_0 &= <1,2,\ldots,m-1,m> \\ a_1 &= <1,2,\ldots,m-1,m+1> \\ &\vdots \\ a_{k-1} &= <n-m+1,\ldots,n-1,n> \end{aligned}$$

Note that $a_0 \prec a_1 \prec \ldots \prec a_{k-1}$. Next, we define the map

$$\delta : \tilde{T}_m^n \to N \times D_m^n$$

such that:

$$(a_1, a_2, \ldots, a_m) \xrightarrow{\delta} (\lambda, < a_1', a_2', \ldots, a_m' >)$$

where λ is the number of permutations needed to order the elements of $(a_1, a_2, \ldots, a_m)$ in normal ordering $< a_1', a_2', \ldots, a_m' >$.

If $[x_0, x_1, \ldots, x_{k-1}]^T \in \mathcal{P}^{k-1}(\mathbb{R})$, we define the following maps: First,

$$\phi : D_m^n \to [x_0, x_1, \ldots, x_{k-1}]^T$$

by $a_\kappa \xrightarrow{\phi} x_\kappa$, $0 \le \kappa \le k-1$, where it is clear from the above definition of ϕ that $\phi^{-1}(x_\kappa) = a_\kappa$. Secondly we define:

$$\tilde{\phi} : N \times D_m^n \to \mathbb{R}$$

by

$$(\lambda, a_\kappa) \xrightarrow{\tilde{\phi}} (-1)^\lambda \phi(a_\kappa) = (-1) x_\kappa$$

Finally, if we denote $\beta := (a_1, a_2, \ldots, a_m) \in \mathcal{T}_m^n$, we define the map

$$g : T_m^n \to \mathbb{R}$$

by

$$g(\beta) = \begin{cases} (\tilde{\phi} \circ \delta)(\beta), & \text{if } \beta \in \tilde{T}_m^n \\ 0, & \text{if } \beta \in T_m^n - \tilde{T}_m^n \end{cases}$$

We can now construct the Plucker relations: For every group of $m-1$ indices

$$< t_1, t_2, \ldots, t_{m-1} > \in \tilde{T}_{m-1}^n$$

and from every group of $m+1$ indices

$$< p_1, p_2, \ldots, p_{m+1} > \in \tilde{T}_{m+1}^n$$

we define for $\kappa = 1, \ldots, m+1$,

$$\beta_\kappa := (t_1, t_2, \ldots, t_{m-1}, p_\kappa)$$

and

$$\gamma_\kappa := (p_1, p_2, \ldots, p_{\kappa-1}, p_{\kappa+1}, \ldots, p_{m+1}),$$

for $\kappa = 1, \ldots, m+1$ and the corresponding Plucker relation is as follows

$$\sum_{\kappa=1}^{m+1} (-1)^\kappa g(\beta) g(\gamma_\kappa) = 0$$

Next we state the following theorems which extract from the whole set of quadratic Plucker relations a reduced set of relations which have simple form and describe completely the Grassmann variety of the corresponding projective space.

Theorem 1: Assume that

$$x = [x_0, x_1, \ldots, x_{k-1}]^t \in \Omega(m, n)$$

and

$$\phi^{-1}(x_\kappa) = < a_1, a_2, \ldots, a_m > \in D_m^n$$

with $a_2 < n-m+1$. Then, there exists a three-term Plucker relation

$$\sigma(x_\kappa, x_{\kappa 1}, x_{\kappa 2}, x_{\kappa 3}, x_{\kappa 4}, x_{\kappa 5}) = 0$$

where $\kappa < \kappa_i,\ i = 1, 2, \ldots, 5$.

Theorem 2: Assume that $x = [x_0, x_1, \ldots, x_{k-1}]^t \in \Omega(m, n)$. The full number of the coordinates x_κ having the property $\phi^{-1}(x_\kappa) = < a_1, a_2, \ldots, a_m >$ with $a_2 < n-m+1$ is

$$r := \binom{n}{m} - (n-m)m - 1$$

Corollary 1: For every $x = [x_0, x_1, \ldots, x_{k-1}]^t \in \Omega(m, n)$, there exists a set S of three-term quadratic Plucker relations, of the form

$$\sigma_i(x_{\kappa i}, x_{\lambda i}, x_{\mu i}, x_{\nu i}, x_{\xi i}, x_{\rho i}) = 0$$

where $1 \le i \le r$ with

$$\kappa_i \le \min(\lambda_i, \mu_i, \nu_i, \xi_i, \rho_i), \text{ for } 1 \le i \le r,$$

and

$$\kappa_j < \kappa_{j+1},\ 1 \le i \le r-1.$$

Theorem 3 [14]: The three-term quadratic Plucker relations given by the set S of Corollary 1 describe completely the Grassmann variety $\Omega(m, n)$ of the projective space $P^{k-1}(\mathbb{R})$.

Next, we propose the following algorithm which computes the Reduced Set of Quadratic Plucker Relations (RSQPR).

Algorithm RSQPR

- Step 1: Read the dimensions n, m.
- Step 2: Compute $k = \binom{n}{m}$.
- Step 3: Repeat for $k = 0, 1, \ldots, k-1$.
  - a. Find the κ-th order multi-index $< a_1, a_2, \ldots, a_m >$.
  - b. If $\lambda_2 < n - m + 1$ then
  - b1. Find indices $j_1, j_2 \in T_n - \{a_1, a_2, \ldots, a_m\}$ such that $j_1 > a_2$ and $j_2 > a_2$.
  - b2. Define $a_{m+1} := j_2$, $\beta_\rho := (a_3, \ldots, a_m, j_1, a_\rho)$ and $\gamma_\rho := (a_1, \ldots, a_{\rho-1}, a_{\rho+1}, \ldots, a_{m+1})$.
  - b3. Type Plucker relation

$$\sum(-1)^\rho \cdot g(\beta_\rho) \cdot g(\gamma_\rho) = 0$$

- Step 4: End.

Finally, suppose we have computed the coordinates of a decomposable vector as shown below

$$x = [x_0, x_1, \ldots, x_{k-1}]^t \ \in \Omega(m, n)$$

using the quadratic Plucker relations generated by the above algorithm. Then, we would like to construct $H \in \mathbb{R}^{n\times m}$ with the property:

$$C_m(H) = x \tag{23}$$

The following Proposition is helpful for this purpose.

Proposition 1 [14]: Let

$$x = [x_0, x_1, \ldots, x_{k-1}]^t \ \in \mathbb{R}^k$$

be a decomposable vector and $x_p \neq 0$ for an index p, $1 \leq p \leq k-1$, such that

$$\phi^{-1}(x_p) =< a_1, a_2, \ldots, a_m >$$

Then, the elements h_{ij} of matrix $H \ \in \mathbb{R}^{n\times m}$ which satisfies equation (23) are given by

$$h_{ij} = g((a_1, a_2, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_m)) \tag{24}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

4 Numerical Example

The benefits of the proposed algorithmic procedure are clarified in the following example. We consider the system of the form (2), with

$$M = 10I_3, \ K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}$$

and

$$D = 0_{3\times3}, \ B = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

The open-loop e-values are

$$\{\pm 3.6039i, \pm 2.4940i, \pm 0.8901i\}$$

. Suppose we want to shift these to

$$\{-1, -2, -3, -4, -5, -6\}$$

using the state feedback law (4) where $F_1, \ F_2 \ \in \mathbb{R}^{1\times3}$. In this case the problem consists of finding matrices $F_1, \ F_2$ such that the closed-loop system (5) has a charactristic polynomial

$$\tilde{\phi}(s) = (\det M) f(s)$$

where

$$f(s) = \prod_{i=1}^{6}(s - \lambda_i) = \prod_{i=1}^{6}(s + i)$$

.

After several computations, we arrive at a linear system of the form (18) which is $Pg = a$, the solution of which is given by:

$$g_0 = 1, \ g_1 = 2932.5, \ g_2 = -4318.571, \ g_3 = 3345$$
$$g_4 = -5737.143, \ g_6 = 1447.5, \ g_7 = -2705.714$$

Variables $g_5, \ g_8, \ g_9$ can be chosen arbitrarily since they do not appear in equations of the linear system. Now we must solve a multinear subproblem, which consists of the following three equations, the so-called reduced set of quadratic Plucker relations:

$$\begin{aligned} g_4g_6 - g_3g_7 + g_0g_9 &= 0 \\ -g_5g_6 + g_3g_8 - g_1g_9 &= 0 \\ g_5g_7 - g_4g_8 + g_2g_9 &= 0 \end{aligned}$$

These equations are generated by the proposed algorithm. Using a symbolic package, i.e. Mathematica, and substituting the solutions of the linear system to the above set of quadratic relations the following solutions are obtained:

$$g_5 = -2378500, \ g_8 = -1683375, \ g_9 = -746100$$

Now, using equation (18) together with Proposition 1, we obtain:

$$F_1 = [g_7, -g_4, g_2]$$

$$= [-2705.714 \ , \ 5737.143 \ , \ -4318.571]$$

and

$$F_2 = [g_6, -g_3, g_1] = [1447.5\ ,\ -3345\ ,\ 2932.5]$$

In order to avoid cumbersome calculations by hand, a numerical software package can be used to verify that the closed-loop system (5) has the following poles:

$$\begin{aligned}
\tilde{\lambda}_1 &= -1.000000000047105 \\
\tilde{\lambda}_2 &= -1.999999999594935 \\
\tilde{\lambda}_3 &= -3.000000000111104 \\
\tilde{\lambda}_4 &= -3.999999999929748 \\
\tilde{\lambda}_5 &= -5.000000000132230 \\
\tilde{\lambda}_6 &= -6.000000000120886
\end{aligned}$$

Note that whatever the starting guess for the approximation of the eigenvalues, the error $|\lambda_i - \tilde{\lambda}_i|$, for $i = 1, \ldots, 6$, is always smaller than 10^{-9}.

5 Conclusion

An efficient algorithmic framework has been proposed for the solution of the pole-assignment problem of symmetric quadratic systems. This involves the computation of a reduced set of quadratic Plucker relations describing completely the Grassmann variety of the corresponding projective space. The extraction of this reduced set has been achieved by the use of a simple criterion based on the correspondence between the coordinates of a decomposable vector and lexicographical orderings. The minimum number of the linear independent quadratic Plucker relations which describe completely the Grassmann variety is given in equation (22). Each equation of this reduced set is homogeneous and contains only three terms. We have to emphasize that the proposed algorithm is free from numerical errors because it does not contain any numerical calculation.This fact has beneficial influence on the overall complexity of the solution of the problem which is mainly due to its non-linearity. The redundant Plucker relations can be expressed as linear combinations of the remaining minimal set, and hence can be completely ignored in the subsequent steps of the algorithm, which reduces the complexity of the solution significantly. An advantage of the proposed method relative to existing ones is that it generates the whole family of feedback matrices with the specified pole-placement properties; this family can be subsequently used for further optimization purposes if additional design objectives are desirable or further constraints are imposed. Another advantage of the approach is that it can be extended to more general classes of models, including descriptor (implicit) systems and matrix higher-order models (only quadratic matrix models have been considered in this paper). Descriptor systems are widely employed in systems and control engineering to model and simulate dynamical systems with algebraic constraints, i.e. systems described by mixed differential and algebraic equations. They have a wide range of applications, such as in mechanical multi-body systems, electrical circuits, chemical (process) engineering, fluid dynamics and many others. The methods can be extended also to non-linear systems and can be used to solve a wide range of related initial-value and boundary value problems. The method can also be extended to the solution of other problems of control theory having a similar multi-linear nature, [15], [16], [17], [18].

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.