QUEST Aided EKF for Attitude and Rate Estimation Using Modified Rodrigues Parameters

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Abstract: - A conventional attitude estimation system for a nanosatellite involves direct input of the attitude sensor measurements to a Kalman filter. However, in case of using an extended Kalman filter (EKF) for the attitude filtering, frequent calculations of the Jacobian matrices bring an excessive computational burden which may not be practical for a nanosatellite on-board computer. In order to deal with this problem, in this study, a QUEST aided EKF attitude and attitude rate estimation system is proposed. QUEST algorithm is used to obtain an initial coarse attitude estimation and then, this estimation is filtered via an EKF. The proposed integrated system reduces the computational burden that an EKF brings since the direct input of the attitude measurements to the filter makes the measurement model linear. For the attitude representation, modified Rodrigues parameters (MRPs) are used unlike widely used quaternions due to the advantages they provide. MRP representation has a singularity at only at the multiples of 2π, therefore, any rotation can be represented by MRPs, except a complete $360°$ rotation. This singularity can be easily avoided switching between alternate MRP sets which is also discussed in this study. The performance of the proposed system is tested with several simulations and the results are presented together with the estimation errors and variances.

Key-Words: - Nanosatellite, attitude estimation, modified Rodrigues parameters, QUEST, magnetometer, sun sensor

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1 Introduction

The contribution of small satellites to advances in space technologies is increasing day after day. Their low costs and short development time have led more people and private companies to do research and produce products in this field. New mission concepts have emerged and more elegant solutions to existing problems have been developed. However, these satellites, which have strict constraints in terms of cost, mass, and size, have also brought new challenges. Especially, due to the cost constraint, the use of expensive high-capability components may not be possible. Considering that the selection of the on-board computer is also affected by this constraint, the developed software for the satellite must be computationally light.

The attitude determination and control (ADCS) sub-system is one of the most important sub-systems in a satellite. It is very important to meet the desired pointing accuracy in order to perform the control actions properly. An inaccurate attitude estimation may cause wrong control actions to be taken and lead dire consequences, up to loss of mission. For nanosatellites which consist of 10 cm × 10 cm × 10 cm units and have masses ranging from 1 to 10 kg [1], conventional attitude estimation algorithms are usually not applicable due to the high demand of computational power. Thus, it is crucial to design an elegant attitude estimation system. The designed system should be light, fast, and give good enough estimations.

The attitude of a nanosatellite (or any spacecraft) can be represented using different attitude parameters such as quaternions, Euler angles, and Rodrigues parameters [2]. Each representation has advantages and disadvantages over each; however, up to date, quaternions are one of the most commonly used parameters [3-5]. Although quaternions are non-singular attitude parameters with attractive properties, the quaternion unit norm constraint makes the attitude estimation process complicated. In this regard, utilizing from other attitude parameters can be beneficial, provided that singularities are avoided. A good suggestion might be the modified Rodrigues parameters (MRPs), which have become increasingly popular in recent years [6]. MRP representation has a singularity only at the multiples of 2π, therefore, any rotation can be represented by the MRPs, except a complete $360°$ rotation. This singularity can be easily avoided switching between alternate MRP sets which will be discussed in Section 2.

The attitude determination methods for a nanosatellite, on the other hand, can be divided into two main categories as “static attitude determination methods” and “attitude filtering methods”. To the author’s best knowledge, the “algebraic method” developed by Harold Black in 1964 [7] is the first published static attitude determination method. This method was also presented as TRIAD which stands for “Tri-Axial Attitude Determination” by Malcolm Shuster in his 1981’s paper [8]. The TRIAD (or algebraic) method aims to find the transformation matrix between the spacecraft body frame and the reference frame of interest using only two vector observations and cannot accommodate more than two vectors. One year after the Black’s method, Grace Wahba published her famous problem [9] that removes the two-vector constraint and can be used for any number of vector observations. The Wahba’s problem contains minimization of a loss function and the first practical solution to the problem was presented by Davenport known as the “q-method” [10]. Q-method requires an eigenvalue/eigenvector decomposition of a 4×4 matrix in order to minimize the Wahba’s loss function which inherently demands a considerable amount of computational power. Following the q-method, “QUaternion ESTimator” (QUEST) was developed by Shuster [8] which does not require an eigenvalue/eigenvector decomposition and is computationally more efficient than the q-method. Later in 1988, Markley presented a new method to minimize the loss function [11]. This method is based on the singular value decomposition (SVD) of a 3×3 matrix and known as SVD method. All the mentioned methods have advantages and disadvantages compared to each other. Therefore, it is important to choose the most appropriate method, taking into account the purpose and requirements of the mission. One can gain more insight by examining the studies comparing these methods [5, 12].

One of the biggest disadvantages of static attitude determination methods is that they are highly dependent on the quality of attitude sensors. Any malfunction in these sensors can make the attitude estimation system unreliable or insomuch that completely losing a sensor can make the estimation impossible. To cope with this problem, filtering techniques, especially the use of Kalman filters, has become an important part of the spacecraft attitude estimation problem [13]. Unlike static methods, attitude estimation algorithms with Kalman filtering take advantage of the satellite’s mathematical model in addition to sensor measurements. Thus, the estimation system continues to give attitude estimates even if there is no available attitude sensor measurement. Since the satellite’s dynamics and kinematics equations are nonlinear, attitude estimation algorithms require nonlinear filtering, two of the most popular ones being extended Kalman filter (EKF) [13] and unscented Kalman filter (UKF) [14]. The survey paper [3] presents a comprehensive study of nonlinear attitude estimation methods for spacecrafts including two-step estimator, particle filters and orthogonal attitude filter.

For nanosatellites, attitude estimation algorithms with Kalman filtering can also be divided into two sub-categories as traditional and non-traditional (or integrated) approaches. Traditional approaches are approaches where attitude sensor measurements are directly given as input to the filter [15, 16]. Since the measurement models of some basic attitude sensors are nonlinear (e.g., magnetometers), the computational load is increased in this approach, especially if EKF is used. On the other hand, in integrated approaches, sensor measurements are first pre-processed by one of the static attitude determination methods to obtain an initial coarse attitude estimation. Then, this attitude estimation is given as input to the Kalman filter to filter the result [17-21]. In these approaches, where the first phase attitude estimation is given directly to the filter, the measurement model becomes linear unlike the traditional approaches. As a result, compared to traditional approaches, integrated approaches reduce the computational load relatively. However, overall accuracy of the attitude estimation system in integrated approaches depends also on the chosen first phase static attitude determination method as well as the accuracy of measurements and filter properties. In [17, 18], authors integrate the TRIAD method with an EKF to estimate the attitude angles and the angular velocity vector. In [19], SVD method is integrated with an EKF whereas in [20] SVD is integrated with an UKF. And in [21], a study integrating the q-method with an EKF is presented by the authors.

In this study, an integrated QUEST/EKF attitude and angular rate estimation system is presented. The QUEST algorithm is chosen because of its accurate and efficient algorithm. For the attitude representation and the satellite mathematical model, MRPs are used due to the advantages they provide.

The remainder of this paper is organized as follows: MRPs are introduced in Section 2. Mathematical model of satellite attitude motion is given in Section 3, and the measurement models for the attitude sensors are given in Section 4. In Section 5, the QUEST algorithm and its covariance analysis is presented. Kalman filter formulation using the MRPs are given in Section 6, and simulation results for the proposed system is presented in Section 7. Lastly, in Section 8 conclusion is drawn and the study is summarized.

2 Modified Rodrigues Parameters for Attitude Representation

The modified Rodrigues parameters (MRPs) are a minimal set for attitude representation with useful properties. The relationship between the MRP vector $\boldsymbol{p}$ and the quaternion vector $\boldsymbol{q}$ is given by [6]

$$p_i = \frac{q_i}{1+q_4} \quad i = 1,2,3 \tag{1}$$

where the quaternion vector $\boldsymbol{q}$ is defined in terms of the principal rotation elements as

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} e_1\, sin(\theta/2) \\ e_2\, sin(\theta/2) \\ e_3\, sin(\theta/2) \\ cos(\theta/2) \end{bmatrix} \tag{2}$$

The inverse transformation is given by

$$q_4 = \frac{1-p^2}{1+p^2} \tag{3}$$

$$q_i = \frac{2p_i}{1+p^2} \quad i = 1,2,3 \tag{4}$$

where $p^2 = \boldsymbol{p}^T\boldsymbol{p}$. Using Eq. (1) and (2), principal rotation elements can be directly related to the MRP vector as

$$\boldsymbol{p} = tan\frac{\theta}{4}\hat{\boldsymbol{e}} \tag{5}$$

The attitude matrix A in terms of MRPs, on the other hand, is given in compact form by

$$A = I_{3\times3} + \frac{8[\boldsymbol{p}^\times]^2 - 4(1-p^2)[\boldsymbol{p}^\times]^2}{(1+p^2)^2} \tag{6}$$

where $\boldsymbol{p}^\times$ is the skew-symmetric matrix of $\boldsymbol{p}$ and given by

$$\boldsymbol{p}^\times = \begin{bmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{bmatrix} \tag{7}$$

Looking at Eq. (5), one can easily see that the MRP vector has a singularity at $\theta = \pm 360°$, meaning that any rotation can be represented, except a complete $360°$ rotation, using the MRP vector without encountering any singularity problem. Also, it is obvious that

$$|\boldsymbol{p}| \le 1 \quad \text{if} \quad \theta \le 180° \tag{8}$$

$$|\boldsymbol{p}| \ge 1 \quad \text{if} \quad \theta \ge 180° \tag{9}$$

$$|\boldsymbol{p}| = 1 \quad \text{if} \quad \theta = 180° \tag{10}$$

Another important feature of MRPs is that, similar to quaternions, there are two unique MRP sets representing the same attitude. The alternate set is known as the "shadow set" and defined as

$$p_i^s = \frac{-q_i}{1-q_4} \quad i = 1,2,3 \tag{11}$$

The shadow set can also be given in terms of the principal rotation elements by

$$\boldsymbol{p}^S = tan\left(\frac{\theta - \pi}{4}\right)\hat{\boldsymbol{e}} \tag{12}$$

which is singular at $\theta = 0°$, unlike the original MRP set which is singular at $\theta = \pm 360°$. Since both sets represent the same attitude and also satisfy the same kinematic differential equation, the singularity problem can be avoided by switching between these two sets. The relationship between the original set and the shadow set is given by

$$p_i^S = \frac{-p_i}{p^2} \quad i = 1,2,3 \tag{13}$$

There are no strict restrictions about when to switch between two sets; however, by looking at Eq.

(5), (12), and (13) one can notice that one set of MRP always corresponds to a $\theta \leq 180°$ principal rotation and the other corresponds to a $\theta \geq 180°$ rotation. Therefore, choosing $\boldsymbol{p}^T\boldsymbol{p} = 1$ as the switching surface is convenient for control purposes because now, attitude will always be represented by the short rotation. Also, at $\boldsymbol{p}^T\boldsymbol{p} = 1$, Eq. (13) simplifies to $\boldsymbol{p}^S = -\boldsymbol{p}$ and length of the attitude representation becomes bounded above by 1.

3 Mathematical Model of Satellite Attitude Motion

The attitude dynamics for a satellite is derived using the relationship between the time derivative of the angular momentum vector $d\boldsymbol{L}/dt$ and the applied torque vector $\boldsymbol{N}$ as [22]

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N} - \boldsymbol{\omega}_{bi} \times \boldsymbol{L} = J\frac{d\boldsymbol{\omega}_{bi}}{dt} \tag{14}$$

where J is the satellite moment of inertia matrix and $\boldsymbol{\omega}_{bi}$ is the body angular velocity vector with respect to the inertial frame. Since $\boldsymbol{L} = J\boldsymbol{\omega}_{bi}$, Eq. (14) can be rewritten as

$$\frac{d\boldsymbol{\omega}_{bi}}{dt} = J^{-1}[\boldsymbol{N} - \boldsymbol{\omega}_{bi} \times (J\boldsymbol{\omega}_{bi})] \tag{15}$$

The net torque applied to a satellite consists of two parts as disturbance torques and control torques. Disturbance torques can be caused by different effects such as gravity gradient, aerodynamic drag, and solar radiation pressure. For a nanosatellite orbiting in a low Earth orbit (LEO), the most dominant disturbance torque is known to be the gravity gradient torque and other disturbances can be neglected. Therefore, assuming zero control torque input, the attitude dynamics for a nanosatellite can be simplified as

$$\frac{d\boldsymbol{\omega}_{bi}}{dt} = J^{-1}\left[\boldsymbol{N_{gg}} - \boldsymbol{\omega}_{bi} \times (J\boldsymbol{\omega}_{bi})\right] \tag{16}$$

where $\boldsymbol{N}_{gg}$ is gravity gradient torque given by [23]

$$\boldsymbol{N}_{gg} = -3\frac{\mu}{r_0^3}\begin{bmatrix}(J_{yy} - J_{zz})A_{23}A_{33} \\ (J_{zz} - J_{xx})A_{13}A_{33} \\ (J_{xx} - J_{yy})A_{13}A_{23}\end{bmatrix} \tag{17}$$

along the orbit. Here, μ is the gravitational parameter for the Earth and r_0 is the distance between the center of mass of the Earth and the satellite.

On the other hand, the kinematic equations of motion of the satellite via MRP attitude representation is given as follows [24]

$$\dot{\boldsymbol{p}} = \frac{1}{2}\left\{\frac{1}{2}(1 - \boldsymbol{p}^T\boldsymbol{p})I_{3\times3} + [\boldsymbol{p}^\times] + \boldsymbol{p}\boldsymbol{p}^T\right\}\boldsymbol{\omega}_{br} \tag{18}$$

where $\boldsymbol{\omega}_{br}$ is the body angular velocity vector with respect to the reference frame.

Here, one should note that Eq. (18) is highly dependent on the chosen reference frame. If the reference frame is chosen as Earth-centered inertial (ECI) frame, then following equity can be written

$$\boldsymbol{\omega}_{br} = \boldsymbol{\omega}_{bi} \tag{19}$$

However, if the chosen reference frame is not an inertial frame, then $\boldsymbol{\omega}_{bi}$ and $\boldsymbol{\omega}_{br}$ should be related accordingly. In case it is chosen as the orbital frame, they can be related as

$$\boldsymbol{\omega}_{br} = \boldsymbol{\omega}_{bo} = \boldsymbol{\omega}_{bi} - A[0 \quad -\omega_o \quad o]^T \tag{20}$$

where $\boldsymbol{\omega}_{bo}$ is the body angular velocity vector with respect to the orbital frame and ω_o is the angular velocity of the orbit given by

$$\omega_o = \sqrt{\frac{\mu}{r_0^3}} \tag{21}$$

4 Attitude Sensors and Sensor Measurement Models

The proposed QUEST aided EKF nanosatellite attitude estimation system uses a three-axis magnetometer and sun sensor as the attitude sensors. Also, it has a gyroscope that provides body angular velocity measurements. In this section, measurement models related to these sensors are given where all the sensors are assumed to be in-flight calibrated.

4.1 Magnetometer Measurement Model

The magnetometer measurement model including noise can be given as follows [23]

$$\widetilde{\boldsymbol{B}}_b = A\boldsymbol{B_r} + \boldsymbol{\eta}_m \tag{22}$$

where $\widetilde{\boldsymbol{B}}_b$ is the measured magnetic field vector in the body frame, $\boldsymbol{B_r}$ is the magnetic field vector in the reference frame, and $\boldsymbol{\eta}_m$ is the magnetometer noise vector which is assumed to be a zero-mean Gaussian white noise with the characteristics of

$$E\{\boldsymbol{\eta}_{m_k}\boldsymbol{\eta}_{m_j}^T\} = I_{3\times3}\sigma_m^2\delta_{kj} \quad (23)$$

where σ_m is the standard deviation of the magnetometer error and δ_{kj} is the Kronecker delta symbol. The Earth magnetic field vector $\boldsymbol{B_r}$ can be modeled in the orbital frame using the International Geomagnetic Reference Field (IGRF) as [23]

$$B_{r_x} = \frac{M_e}{r_0^3}\{c(\omega_o t)[c(\varepsilon)s(i) - s(\varepsilon)c(i)c(\omega_e t)] - s(\omega_o t)s(\varepsilon)s(\omega_e t)\} \quad (24)$$

$$B_{r_y} = -\frac{M_e}{r_0^3}\{c(\varepsilon)s(i) + s(\varepsilon)s(i)c(\omega_e t)\} \quad (25)$$

$$B_{r_z} = \frac{2M_e}{r_0^3}s(\omega_o t)[c(\varepsilon)s(i) - s(\varepsilon)c(i)c(\omega_e t)] - 2s(\omega_o t)s(\varepsilon)s(\omega_e t) \quad (26)$$

where $M_e = 7.71\times10^{15}$ Wb.m is the magnetic dipole moment of the Earth, $\varepsilon = 9.3°$ is the magnetic dipole tilt angle, $\omega_e = 7.29\times10^{-5}$ rad/s is the spin rate of the Earth, and i is the orbit inclination. $c(\cdot)$ and $s(\cdot)$ are abbreviations for cosine and sine, respectively.

4.2 Sun Sensor Measurement Model

The sun sensor measurement model including noise can be given as follows [23]

$$\tilde{\boldsymbol{S}}_b = A\boldsymbol{S_r} + \boldsymbol{\eta}_s \quad (27)$$

where $\tilde{\boldsymbol{S}}_b$ is the measured sun direction vector in the body frame, $\boldsymbol{S_r}$ is the sun direction vector in the reference frame, and $\boldsymbol{\eta}_s$ is the sun sensor noise vector which is assumed to be a zero-mean Gaussian white noise with

$$E\{\boldsymbol{\eta}_{s_k}\boldsymbol{\eta}_{s_j}^T\} = I_{3\times3}\sigma_s^2\delta_{kj} \quad (28)$$

where σ_s is the standard deviation of the sun sensor error. The sun direction vector $\boldsymbol{S_r}$ can be calculated via well-known algorithms. One example of a standard sun direction vector calculation algorithm is well-documented in [25] and will not be repeated here for the sake of brevity.

4.3 Gyroscope Measurement Model

The gyroscope measurement model including noise can be given as follows [23]

$$\tilde{\boldsymbol{\omega}}_{br} = \boldsymbol{\omega}_{br} + \boldsymbol{\eta}_\omega \quad (29)$$

where $\tilde{\boldsymbol{\omega}}_{br}$ is the measured body angular velocity, $\boldsymbol{\omega}_{br}$ is the true body angular velocity vector, and $\boldsymbol{\eta}_\omega$ is the gyroscope noise vector which is assumed to be a zero-mean Gaussian white noise with the characteristics of

$$E\{\boldsymbol{\eta}_{\omega_k}\boldsymbol{\eta}_{\omega_k}^T\} = I_{3\times3}\sigma_\omega^2\delta_{kj} \quad (30)$$

where σ_ω is the standard deviation of the gyroscope error.

5 Minimization of Wahba's Problem and the QUEST Algorithm

QUEST algorithm is one of the most popular and efficient static attitude determination methods developed to minimize Wahba's loss function. This section is devoted to explaining the QUEST algorithm with details and follows the original work [8] of Malcolm D. Shuster, creator of the QUEST.

The original loss function proposed by Wahba is given as [9]

$$L(A) = \frac{1}{2}\sum_{i=1}^{n} a_i\left|\widehat{\boldsymbol{W}}_i - A\widehat{\boldsymbol{V}}_i\right|^2 \quad (31)$$

where $\widehat{\boldsymbol{W}}_i$, $i = 1, \dots, n$ are observation unit vectors, $\widehat{\boldsymbol{V}}_i$ are reference unit vectors, and a_i are nonnegative weights assigned to each particular observation.

The aim of the QUEST algorithm is to find the optimal attitude matrix A_{opt} that minimizes Eq. (31) or that maximizes the corresponding gain function $g(A)$ given by

$$g(A) = 1 - L(A) = \sum_{i=1}^{n} a_i\widehat{\boldsymbol{W}}_i^T A\widehat{\boldsymbol{V}}_i \quad (32)$$

Considering that the attitude matrix A is an orthogonal matrix and using the cyclic invariance property of the trace operation, gain function can be rewritten as

$$g(A) = tr[AB^T] \quad (33)$$

where $tr[\cdot]$ denotes the trace operation and the matrix B is known as the "attitude profile matrix" defined as

$$B = \sum_{i=1}^{n} a_i\widehat{\boldsymbol{W}}_i\widehat{\boldsymbol{V}}_i^T \quad (34)$$

It can be seen easily from Eq. (33) that $L(A)$ is minimized when $tr[AB^T]$ is maximized.

Since A is a 3×3 orthogonal matrix subject to six constraints, it is expressed in terms of its related quaternions in order to reduce the complexity during the maximization of Eq. (33). The relationship between the attitude matrix A and its related quaternion vector is given by

$$A(\boldsymbol{q}) = ({q_4}^2 - \boldsymbol{Q} \cdot \boldsymbol{Q})I_{3\times3} + 2\boldsymbol{Q}\boldsymbol{Q}^T + 2q_4\boldsymbol{Q}^\times \quad (35)$$

where $\boldsymbol{Q} = [q_1 \quad q_2 \quad q_3]$ is the vector component of the quaternion vector $\boldsymbol{q}$ and $\boldsymbol{Q}^\times$ is the skew-symmetric matrix of $\boldsymbol{Q}$ given by

$$\boldsymbol{Q}^\times = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} \quad (36)$$

Using Eq. (35), gain function can be written in terms of quaternions as follows

$$g(\boldsymbol{q}) = ({q_4}^2 - \boldsymbol{Q} \cdot \boldsymbol{Q})tr[B^T] + 2tr[\boldsymbol{Q}\boldsymbol{Q}^T B^T] + 2q_4 tr[\boldsymbol{Q}^\times B^T] \quad (37)$$

In order to obtain the bilinear form of Eq. (37), following quantities are introduced

$$\sigma = tr[B] \quad (38)$$

$$S = B + B^T \quad (39)$$

$$\boldsymbol{Z} = \sum_{i=1}^{n} a_i(\widehat{W}_i \times \widehat{V}_i) \quad (40)$$

These quantities are used to create the 4×4 K matrix as

$$\mathrm{K} = \begin{bmatrix} \mathrm{S} - \sigma\mathrm{I} & \boldsymbol{Z} \\ \boldsymbol{Z}^T & \sigma \end{bmatrix} \quad (41)$$

And using the matrix K, gain function can be written in the bilinear form as

$$g(\boldsymbol{q}) = \boldsymbol{q}^T K \boldsymbol{q} \quad (42)$$

The attitude determination problem can now be thought of as finding the quaternion vector $\boldsymbol{q}_{opt}$ that maximizes the Eq. (42). However, it is known that the quaternion vector $\boldsymbol{q}$ must satisfy the unit-length constraint given by

$$\boldsymbol{q}^T \boldsymbol{q} = 1 \quad (43)$$

Using the method of Lagrange multipliers, this constraint can be taken into account and a new gain function $g'(\boldsymbol{q})$ can be defined as

$$g'(\boldsymbol{q}) = \boldsymbol{q}^T K \boldsymbol{q} - \lambda \boldsymbol{q}^T \boldsymbol{q} \quad (44)$$

where λ is chosen to satisfy the constraint.

Differentiating Eq. (44), one can see that $g'(\boldsymbol{q})$ is maximized when

$$K\boldsymbol{q} = \lambda\boldsymbol{q} \quad (45)$$

Therefore, it can be said that $\boldsymbol{q}_{opt}$ must be an eigenvector and λ must be an eigenvalue of K. Gain function $g(\boldsymbol{q})$ is maximized when $\boldsymbol{q}_{opt}$ is chosen to be the eigenvector corresponding to the largest eigenvalue of K. This statement can be expressed mathematically as

$$K\boldsymbol{q}_{opt} = \lambda_{max}\boldsymbol{q}_{opt} \quad (46)$$

5.1 Finding the Optimal Quaternion Set

Finding the exact largest eigenvalue and its corresponding eigenvector of the 4×4 K matrix, as in Davenport's q-method, was expensive to compute for an on-board computer in the early of 1980s. The QUEST algorithm is a numerical approximation of the Davenport's q-method which was developed to ease this computational burden. Shuster derived an analytic formula of the characteristic equation of the K matrix given as [8]

$$\lambda^4 - (a + b)\lambda^2 - c\lambda + (ab + c\sigma - d) = 0 \quad (47)$$

where the coefficients a, b, c, and d are defined as

$$a = \sigma^2 - \mathrm{tr}[\mathrm{adj}(S)] \quad (48)$$

$$b = \sigma^2 + \boldsymbol{Z}^T\boldsymbol{Z} \quad (49)$$

$$c = det(S) + \boldsymbol{Z}^T S \boldsymbol{Z} \quad (50)$$

$$d = \boldsymbol{Z}^T S^2 \boldsymbol{Z} \quad (51)$$

Here $adj(\cdot)$ denotes the adjoint and $det(\cdot)$ denotes the determinant operation. Any iterative technique can be used to find the root of Eq. (47) and a starting value of $\lambda = 1$ can be used for this process. After finding the largest eigenvalue λ_{max}, following quantities are introduced

$$\alpha = \lambda_{max}^2 - \sigma^2 + tr[adj(S)] \quad (52)$$

$$\beta = \lambda_{max} - \sigma \quad (53)$$

$$\gamma = (\lambda_{max} + \sigma)\alpha - det(S) \quad (54)$$

$$\boldsymbol{X} = (\alpha I + \beta S + S^2)\boldsymbol{Z} \quad (55)$$

and the optimal quaternion vector is constructed as

$$\boldsymbol{q}_{opt} = \frac{1}{\sqrt{\gamma^2 + |\boldsymbol{X}|^2}} \begin{bmatrix} \boldsymbol{X} \\ \gamma \end{bmatrix} \quad (56)$$

One should note that solution given by Eq. (56) is an approximation and inherently less accurate than the exact solution given by Davenport's q-method.

5.2 QUEST Algorithm Covariance Analysis

The variances of the attitude parameters are needed to compose the measurement noise covariance matrix R for the Kalman filter. The 3×3 quaternion covariance matrix is given by [8]

$$P_{QQ} = E[\delta \boldsymbol{Q} \delta \boldsymbol{Q}^T] \quad (57)$$

where $E[\cdot]$ denotes the expectation value. Assuming that for a very small rotation, reference and observation vectors are identical, that is

$$\widehat{\boldsymbol{W}}_i = \widehat{\boldsymbol{V}}_i \quad (\mathrm{i} = 1, \ldots, \mathrm{n}) \quad (58)$$

Then, $\delta\boldsymbol{Q}$ can be written as

$$\delta \boldsymbol{Q} = M^{-1} \delta \boldsymbol{Z} \quad (59)$$

where M and $\delta\boldsymbol{Z}$ is given by

$$M = 2I_{3\times3} - 2\sum_{i=1}^{n} a_i \widehat{\boldsymbol{W}}_i \widehat{\boldsymbol{W}}_i^T \quad (60)$$

$$\delta \boldsymbol{Z} = \sum_{i=1}^{n} a_i \left(\delta \widehat{\boldsymbol{W}}_i \times \widehat{\boldsymbol{V}}_i + \widehat{\boldsymbol{W}}_i \times \delta \widehat{\boldsymbol{V}}_i\right) \quad (61)$$

The 3×3 quaternion covariance matrix given in Eq. (57) can now be rewritten as

$$P_{QQ} = M^{-1} E[\delta \boldsymbol{Z} \delta \boldsymbol{Z}^T] M^{-1} \quad (62)$$

where $E[\delta\boldsymbol{Z}\delta\boldsymbol{Z}^T]$ can be calculated as

$$E[\delta \boldsymbol{Z} \delta \boldsymbol{Z}^T] = \sum_{i=1}^{n} a_i^2 \sigma_i^2 \left[I - \widehat{\boldsymbol{W}}_i \widehat{\boldsymbol{W}}_i^T\right] \quad (63)$$

Finally, substituting Eq. (60) and (63) into Eq. (62) yields

$$P_{QQ} = \frac{1}{4} \sigma_{tot}^2 \left[I - \sum_{i=1}^{n} a_i \widehat{\boldsymbol{W}}_i \widehat{\boldsymbol{W}}_i^T\right]^{-1} \quad (64)$$

$$(\sigma_{tot}^2)^{-1} = \sum_{i=1}^{n} (\sigma_i^2)^{-1} = \sum_{i=1}^{n} \left(\sigma_{\boldsymbol{V}_i}^2 + \sigma_{\boldsymbol{W}_i}^2\right)^{-1} \quad (65)$$

Eq. (64) gives the final form of the 3×3 quaternion covariance matrix; however, it is more convenient to define the quaternion covariance matrix as a 4×4 matrix by including also the covariance of the scalar component q_4. The relationship between the 4×4 quaternion covariance matrix P_{qq} and 3×3 quaternion covariance matrix P_{QQ} is given by

$$P_{qq} = [\boldsymbol{q}_{opt}] \begin{bmatrix} P_{QQ} & \boldsymbol{0} \\ \boldsymbol{0}^T & \boldsymbol{0} \end{bmatrix} [\boldsymbol{q}_{opt}]^T \quad (66)$$

where the matrix $[\boldsymbol{q}]$ is defined as

$$[q] = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \quad (67)$$

5.2.1 Computation of the Covariance Matrix for the Modified Rodrigues Parameters

The proposed QUEST aided EKF attitude estimation system provides the attitude information in terms of MRPs. The 3×3 MRP covariance matrix can be obtained from P_{qq} using the well-known covariance law [26] as

$$P_{pp} = H P_{qq} H^T \quad (68)$$

where the matrix H is calculated as

$$H = \begin{bmatrix} \frac{\partial p_1}{\partial q_1} & \frac{\partial p_1}{\partial q_2} & \frac{\partial p_1}{\partial q_3} & \frac{\partial p_1}{\partial q_4} \\ \frac{\partial p_2}{\partial q_1} & \frac{\partial p_2}{\partial q_2} & \frac{\partial p_2}{\partial q_3} & \frac{\partial p_2}{\partial q_4} \\ \frac{\partial p_3}{\partial q_1} & \frac{\partial p_3}{\partial q_2} & \frac{\partial p_3}{\partial q_3} & \frac{\partial p_3}{\partial q_4} \end{bmatrix} \quad (69)$$

Since the relationship between MRPs and quaternions change when the original MRP set $\boldsymbol{p}$ is switched to shadow set $\boldsymbol{p^S}$ as stated in Eq. (1) and (11), one should be careful while creating the H matrix and calculate the derivatives accordingly.

6 Kalman Filter Formulation Using Modified Rodrigues Parameters

The 6×1 state vector for the proposed QUEST aided EKF consists of MRPs and body angular velocity vector components, given by

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{\omega}_{bi} \end{bmatrix} \tag{70}$$

The state vector can be propagated using the following discrete model [27]

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, k) + \boldsymbol{w}_k \tag{71}$$

where the function $\boldsymbol{f}(\boldsymbol{x}_k, k)$ is the nonlinear state transition function and $\boldsymbol{w}_k$ is the process noise. The state transition function $\boldsymbol{f}(\boldsymbol{x}_k, k)$ maps the previous state to the current state and formed by integrating Eq. (16) and (18). The measurement vector, on the other hand, is modeled as

$$\boldsymbol{z}_k = \boldsymbol{h}(\boldsymbol{x}_k, k) + \boldsymbol{v}_k \tag{72}$$

where $\boldsymbol{h}(\boldsymbol{x}_k, k)$ is the nonlinear measurement model and $\boldsymbol{v}_k$ is the measurement noise. The nonlinear measurement model $\boldsymbol{h}(\boldsymbol{x}_k, k)$ maps the current state to measurements.

The process noise $\boldsymbol{w}_k$ and measurement noise $\boldsymbol{v}_k$ are both assumed to be zero-mean white Gaussian with the covariances

$$E\left[w_k w_j^T\right] = Q_k \delta(kj) \tag{73}$$

$$E\left[v_k v_j^T\right] = R_k \delta(kj) \tag{74}$$

It is also assumed that both noises are uncorrelated, that is

$$E\left[w_k v_j^T\right] = 0 \tag{75}$$

The propagation of the state error covariance matrix P in discrete time is given by

$$P_{k+1} = \Phi_k P_k \Phi_k^T + Q_k \tag{76}$$

where Φ is the state transition matrix and calculated by using the first-order Taylor series expansion as follows

$$\Phi(t) = e^{Ft} \approx I + Ft \tag{77}$$

where the matrix F is given by

$$F = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \tag{78}$$

The discrete state transition matrix Φ_k can be found evaluating Eq. (77) at the sampling time T_s as

$$\Phi_k = e^{F_k T_s} \approx \mathrm{I} + \mathrm{F_k T_s} \tag{79}$$

where the matrix F_k is obtained evaluating the Eq. (78) at $\boldsymbol{x}_k$. Eq. (71), and Eq. (76) constitutes the prediction part of the EKF. Hereafter, the uncorrected predictions obtained via Eq. (71) and (76) will be denoted by $\hat{\cdot}^-$ symbol.

The correction or update part of the filter starts with calculating the Kalman gain matrix K_{k+1} as

$$K_{k+1} = \frac{\hat{P}_{k+1}^- H_{k+1}^T}{H_{k+1} \hat{P}_{k+1}^- H_{k+1}^T + R_{k+1}} \tag{80}$$

where H_{k+1} is known as the state observation matrix calculated as

$$H_{k+1} = \left. \frac{\partial \boldsymbol{h}(\hat{\boldsymbol{x}}_{k+1}^-, k+1)}{\partial \boldsymbol{x}} \right|_{x = \hat{x}_{k+1}^-} \tag{81}$$

At this point, the advantage of the proposed QUEST aided EKF attitude estimation system comes to the fore. In the system, the attitude and angular velocity measurements are obtained from the QUEST algorithm and gyroscopes, respectively. Looking at Eq. (72), one can realized that for the proposed system the states are directly observable, that is

$$\boldsymbol{z}_k = \boldsymbol{x}_k + \boldsymbol{v}_k \tag{82}$$

Using the QUEST algorithm before the filtering and obtaining an initial coarse attitude estimation makes the nonlinear measurement model $\boldsymbol{h}(\boldsymbol{x}_k, k)$ linear and reduces the computational load that an EKF brings. The matrix H_{k+1} becomes just an identity matrix independent of the sampling index $k+1$.

After the calculation of the Kalman gain matrix K_{k+1}, the update or correction is made for the predicted state vector and state error covariance matrix as

$$P_{k+1}^+ = (I - K_{k+1} H) \hat{P}_{k+1}^- \tag{83}$$

$$\boldsymbol{x}_{k+1}^+ = \hat{\boldsymbol{x}}_{k+1}^- + K_{k+1} y_{k+1} \tag{84}$$

where y_{k+1} is the measurement residual defined as

$$\boldsymbol{y}_{k+1} = \boldsymbol{z}_{k+1} - H\widehat{\boldsymbol{x}}_{k+1}^{-} \tag{85}$$

Here one should note that for the H matrix sampling index $k+1$ is omitted since it is just an identity matrix independent of the sampling index.

6.1 MRP Shadow Set Consideration for Kalman Filtering

If after the state prediction with Eq. (71) or update with Eq. (84), the MRP set exceeds the switching surface, that is $\boldsymbol{p}^T\boldsymbol{p} \geq 1$ or $|\boldsymbol{p}| \geq 1$, it is switched to the shadow set using Eq. (13). Then, the state vector $\boldsymbol{x}$ becomes

$$\boldsymbol{x}^S = \begin{bmatrix} \boldsymbol{p}^S \\ \boldsymbol{\omega}_{bi} \end{bmatrix} \tag{86}$$

Also, the state error covariance matrix shadow set transformation is given by [28]

$$P^S = \begin{bmatrix} SP_{pp}S^T & SP_{p\omega_{bi}} \\ P_{p\omega_{bi}}^T S^T & P_{\omega_{bi}\omega_{bi}} \end{bmatrix} \tag{87}$$

where P_{aa} is the covariance matrix of a, P_{ab} is the cross-correlation matrix between a and b, and matrix S is calculated as

$$S = 2p^{-4}\boldsymbol{p}\boldsymbol{p}^T - p^{-2}I_{3\times 3} \tag{88}$$

One other important consideration is related to the MRP part of the measurement residual $\boldsymbol{y}_{k+1}$ which is given by

$$\boldsymbol{y}_{\boldsymbol{p}_{k+1}} = \widetilde{\boldsymbol{p}}_{k+1} - H_{k+1}\widehat{\boldsymbol{p}}_{k+1} \tag{89}$$

Near the switching surface where the original MRP set measurement $\widetilde{\boldsymbol{p}}_{k+1}$ or estimate $\widehat{\boldsymbol{p}}_{k+1}$ is close to one, the calculation of the measurement residual can be problematic. To illustrate, consider the situation where $\widetilde{\boldsymbol{p}}_{k+1} = [1,0,0]$, $\widetilde{\boldsymbol{p}}_{k+1}^S = [-1,0,0]$, and $\widehat{\boldsymbol{p}}_{k+1} = [-1,0,0]$. Calculating the measurement residual using the original set yields

$$\boldsymbol{y}_{\boldsymbol{p}_{k+1}} = [1,0,0] - [-1,0,0] = [2,0,0] \tag{90}$$

On the other hand, calculating the measurement residual using the shadow set yields

$$\boldsymbol{y}_{\boldsymbol{p}_{k+1}} = [-1,0,0] - [-1,0,0] = [0,0,0] \tag{91}$$

As can be seen easily, Eq. (90) and (91) yields different measurement residuals and Eq. (90) causes a correction to be applied when it is not needed. In order to deal with this problem, an elegant algorithm is proposed by the authors in [29] which the pseudocode for it is given as follows

Algorithm 1: Measurement residual algorithm for MRP based EKF	
1	$\boldsymbol{y}_{\boldsymbol{p}_{k+1}} = \widetilde{\boldsymbol{p}}_{k+1} - H_{k+1}\widehat{\boldsymbol{p}}_{k+1}$
2	**if** $|\widetilde{\boldsymbol{p}}_{k+1}| > 1/3$ **then**
3	$\boldsymbol{y}'_{\boldsymbol{p}_{k+1}} = \widetilde{\boldsymbol{p}}_{k+1}^S - \boldsymbol{H}_{k+1}\widehat{\boldsymbol{p}}_{k+1}$
4	**if** $|\boldsymbol{y}'_{\boldsymbol{p}_{k+1}}| < |\boldsymbol{y}_{\boldsymbol{p}_{k+1}}|$
5	$\boldsymbol{y}_{\boldsymbol{p}_{k+1}} = \boldsymbol{y}'_{\boldsymbol{p}_{k+1}}$
6	**end if**
7	**end if**

7 Simulations and Results

In the proposed QUEST aided EKF attitude and angular rate estimation system, the QUEST algorithm estimates the MRP set at each time step using magnetometer and sun sensor measurements. Then, this estimate is given to the EKF as an input together with the gyroscope measurement which is simulated separately. The EKF filters the QUEST solution and the gyroscope measurement with the help of the mathematical model of the satellite dynamics and kinematics. In order to verify the performance of the proposed estimation system, simulations are performed for 583.4 seconds with a sampling time $T_s = 0.1$ second in a circular LEO, assuming an environment where only the gravity gradient torque exists as the disturbance torque. Simulated orbit has an altitude of $626\ km$ and its inclination and longitude of the ascending node are chosen to be $111.5°$ and $15°$, respectively. It is also assumed that the satellite never enters eclipse throughout the orbit so that measurements are available from the sun sensor continuously.

Figure 1, 2, and 3 show the estimation results for each individual MRP (p_1, p_2, p_3) including the absolute values of estimation errors and variances, respectively. Sudden drops in the parameters indicate the switching between alternate MRP sets.

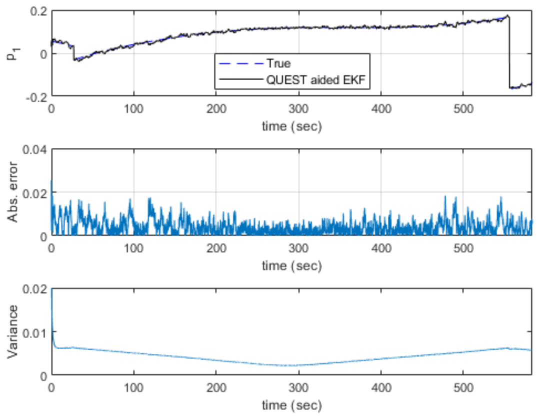

Fig. 1 QUEST aided EKF estimation results for p_1

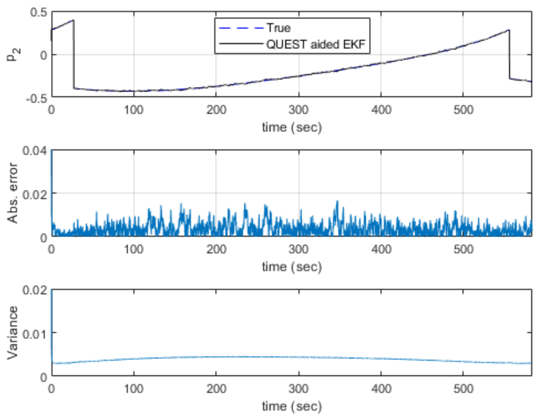

Fig. 2 QUEST aided EKF estimation results for p_2

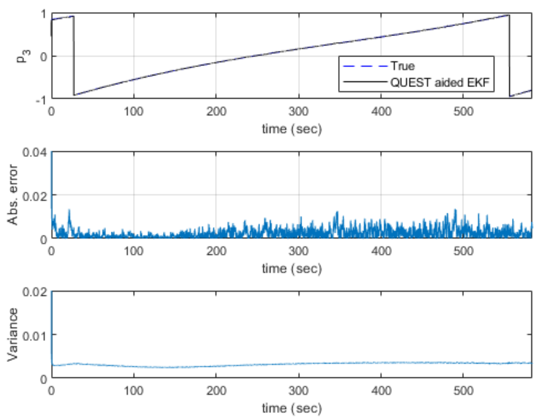

Fig. 3 QUEST aided EKF estimation results for p_3

In addition to the figures, Table 1 shows the average root mean square error (RMSE) value for each individual MRP after 500 simulation runs are executed and Table 2 shows the overall attitude estimation error in terms of the principal rotation angle (PRA).

MRP	**QUEST aided EKF estimation error**
$\boldsymbol{p_1}$	5.7572×10^{-3}
$\boldsymbol{p_2}$	5.4996×10^{-3}
$\boldsymbol{p_3}$	4.8912×10^{-3}

Table 1 RMSE errors for each MRP
(500 simulation average)

PRA	**QUEST aided EKF estimation error**
$\boldsymbol{\theta}$	$0.78298°$

Table 2 RMSE error for PRA
(500 simulation average)

The proposed system also estimates the body angular velocity ω_{bi}. Figure 4, 5, and 6 show the estimation results for each component of the body angular velocity $(\omega_x, \omega_y, \omega_z)$ including the estimation error and variance, respectively.

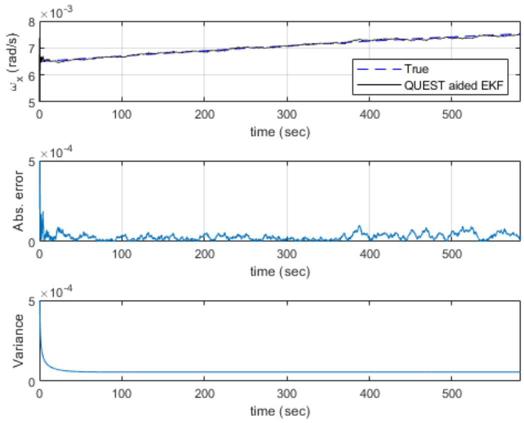

Fig. 4 QUEST aided EKF estimation results for ω_x

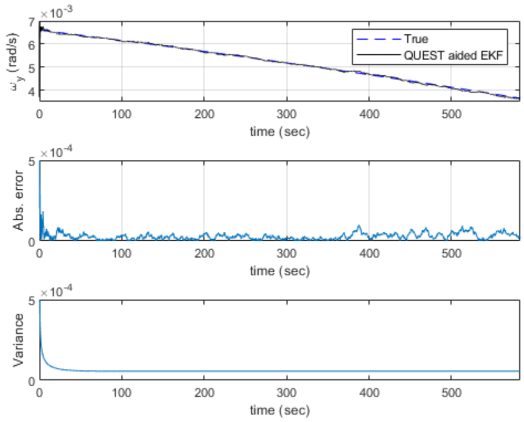

Fig. 5 QUEST aided EKF estimation results for ω_y

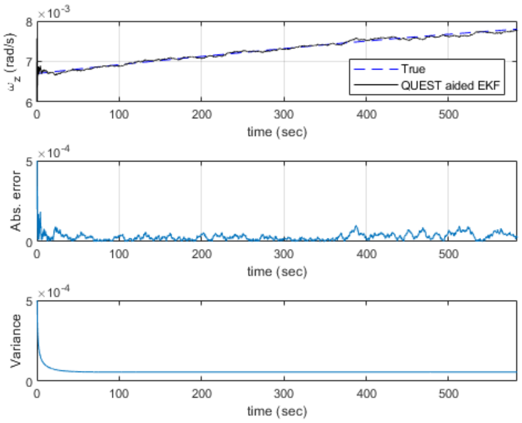

Fig. 6 QUEST aided EKF estimation results for ω_z

In addition to the figures, Table 3 shows the average root mean square error (RMSE) value for each angular velocity component after 500 simulation runs are executed.

MRP	**QUEST aided EKF estimation error**
ω_x	4.6287×10^{-5} rad/s
ω_y	4.4945×10^{-5} rad/s
ω_z	4.6353×10^{-5} rad/s

Table 3 RMSE errors for each angular velocity component (500 simulation average)

8 Conclusion

In this study, an integrated QUEST/Extended Kalman Filter (EKF) attitude and angular rate estimation system is proposed for a nanosatellite. QUEST algorithm is used to obtain an initial coarse attitude estimation and then, this estimation is filtered via an EKF. The proposed integrated system reduces the computational burden that an EKF brings since the direct input of the attitude to the filter makes the measurement model linear. For the attitude representation and the satellite mathematical model, modified Rodrigues parameters (MRPs) are used due to the advantages they provide.

In order to verify the performance of the proposed system, several simulations are performed and the attitude and angular rate estimations are obtained. As a result of the simulations, the attitude estimation errors for each individual MRP are obtained on the order of 10^{-3} whereas for the angular rate estimations, errors are obtained on the order of 10^{-5}. In addition, the overall attitude estimation error in terms of the principal rotation angle is obtained as $0.78298°$ which is considered to be sufficient for a nanosatellite estimation accuracy requirement.

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