Some lower bounds for a double integral depending on six adaptable functions

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Abstract: This paper is devoted to sharp lower bounds of a special double integral depending on six adaptable functions. The bounds obtained depend on simple integrals. Some connections with the famous Hilbert integral inequality and its variants are made. Several numerical examples illustrate the results.

Key-Words: Integral inequalities, Bernoulli inequality, Integral calculus.

Received: April 26, 2024. Revised: September 17, 2024. Accepted: October 18, 2024. Published: November 14, 2024.

1 Introduction

Obtaining lower bounds for double integrals gives information about the minimum values they can reach. This is of particular interest in fields such as probability theory and analysis. Indeed, sharp lower bounds allow us to optimize inequalities and improve the accuracy of estimates in bivariate distributions, integral equations and differential equations.

In this paper, we investigate the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)\pm w(x)z(y)}f(x)g(y)dxdy, \tag{1}$$

where $f, g, u, v, w, z : [0, +\infty) \to [0, +\infty)$ are adaptable functions satisfying certain assumptions, including some integral convergence assumptions. From a probabilistic point of view, it corresponds to the following mathematical quantity:

$$\mathbb{E}\left[\frac{1}{u(X)+v(Y)\pm w(X)z(Y)}\right],$$

where $\mathbb{E}$ denotes the expectation operator, and X and Y are independent lifetime random variables with probability density functions f and g, respectively. This quantity can appear in many situations, such as the study of reliability systems and actuarial science. For example, in reliability theory, the variables X and Y may represent the lifetimes of two components in a system, and the function within the expectation may model a particular performance of this system. Similarly, in actuarial science, X and Y may represent the times to two independent claims, with the expression quantifying the expected value of some measure of risk.

Some special cases of the double integral in Equation (1) have attracted attention in the literature. The most notable example is the case $u(x) = x, v(y) = y$, and $w(x) = 0$ or $z(x) = 0$, where the double integral becomes

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy,$$

which is the central term of the famous Hilbert integral inequality. In particular, this inequality gives a sharp upper bound for this double integral, as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \leq \pi\sqrt{\int_0^{+\infty}[f(x)]^2dx\int_0^{+\infty}[g(x)]^2dx}, \tag{2}$$

under the assumption of quadratic integrability on f and g. In fact, the universal constant π is the best possible one in this setting. The reference on this topic is [1]. Other variants of the Hilbert integral inequality have attracted much attention. See, for example, [2], [3], [4], [5], [6], [7], [8], [9] and [10], and the references therein. In addition to upper bounds, lower bounds have been established for some of these variants, sometimes called "inverse Hilbert integral inequality types". See [11], [12], [13], [14] and [15]. However, to our knowledge, the study of lower bounds of a general double integral term depending on six adaptable functions as in Equation (1) has not been the subject of a study. Therefore, this paper aims to fill this gap.

To obtain sharp lower bounds for this particular double integral, we distinguish the case "-"$w(x)z(y)$ and the case "+"$w(x)z(y)$ in the denominator of the

DOI: 10.37394/232020.2024.4.10

main integrated term, denoted Case I and Case II, respectively. For Case I, we use various sharp integral inequalities in combination with the Bernoulli inequality to determine a lower bound that depends on the sum of the squares of two simple integrals. An alternative result based on the Cauchy-Schwarz integral inequality will also be presented. In the case "+"$w(x)z(y)$, we will take a more direct approach, also using the previous main result under a special configuration. Again, an alternative result based on the Cauchy-Schwarz integral inequality will be presented. Several numerical examples based on specific functions f, g, u, v, w and z will illustrate the main lower bounds obtained. Thus, our results contribute to a deeper understanding and behavior of general double integrals.

The rest of the paper consists of the following sections: Section 2 is devoted to the lower bounds of the double integral for Case I, together with some numerical examples. Section 3 does the same for Case II. A conclusion is given in Section 4.

2 Lower bounds for Case I

2.1 First lower bound

The proposition below is about a lower bound for the double integral in Equation (1) for Case I.

Proposition 2.1 *Let* $f, g, u, v, w, z : [0, +\infty) \to [0, +\infty)$ *be functions such that the future integrals which will depend on them converge (this is a minimum condition), and* u, v, w *and* z *satisfy, for any* $x \geq 0$ *and* $y \geq 0$,

$$u(x) + v(y) - w(x)z(y) \geq 0.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)-w(x)z(y)} f(x)g(y)dxdy$$
$$\geq \left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2$$
$$+\frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2,$$

where $\Gamma(a)$ *denotes the standard gamma function defined by* $\Gamma(a) = \int_0^{+\infty} t^{a-1}e^{-t}dt$ *with* $a > 0$.

Proof. Using an integral calculus, the assumption $u(x) + v(y) - w(x)z(y) \geq 0$ and the Fubini-Tonelli theorem, we can write

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)-w(x)z(y)} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \left[\int_0^1 t^{u(x)+v(y)-w(x)z(y)-1}dt\right] f(x)g(y)dxdy$$
$$= \int_0^1 \left[\int_0^{+\infty}\int_0^{+\infty} t^{u(x)+v(y)-1}t^{-w(x)z(y)} f(x)g(y)dxdy\right] dt.$$

With the aim of working with separable functions in a new integrated term with respect to x and y, we want to lower bound $t^{-w(x)z(y)}$. To this end, we recall that the Bernoulli inequality states that, for any $s \in \mathbb{R}/(0,1)$ and $x \geq -1$, we have $(1+x)^s \geq 1+sx$. Since $w(x) \geq 0$, $z(y) \geq 0$ and $t \in [0,1]$, this inequality applied to $s = -w(x)z(y) < 0$ and $x = t-1 \geq -1$ gives

$$t^{-w(x)z(y)} = [1+(t-1)]^{-w(x)z(y)}$$
$$\geq 1 - w(x)z(y)(t-1) = 1 + (1-t)w(x)z(y),$$

which is positive. Using this inequality and the positivity of the functions involved, we get

$$\int_0^1 \left[\int_0^{+\infty}\int_0^{+\infty} t^{u(x)+v(y)-1}t^{-w(x)z(y)} f(x)g(y)dxdy\right] dt$$
$$\geq \int_0^1 \left\{\int_0^{+\infty}\int_0^{+\infty} t^{u(x)+v(y)-1}\left[1+(1-t)w(x)z(y)\right] f(x)g(y)dxdy\right\} dt$$
$$= \int_0^1 \left\{\int_0^{+\infty}\int_0^{+\infty} t^{u(x)+v(y)-1} f(x)g(y)dxdy\right\} dt$$
$$+ \int_0^1 \left\{\int_0^{+\infty}\int_0^{+\infty} t^{u(x)+v(y)-1}(1-t)w(x)z(y)f(x)g(y)dxdy\right\} dt$$
$$= \int_0^1 \left[\int_0^{+\infty} t^{u(x)-1/2}f(x)dx\right]\left[\int_0^{+\infty} t^{v(y)-1/2}g(y)dy\right] dt$$
$$+ \int_0^1 \left[\int_0^{+\infty} t^{u(x)-1/2}\sqrt{1-t}w(x)f(x)dx\right] \times$$
$$\left[\int_0^{+\infty} t^{v(y)-1/2}\sqrt{1-t}z(y)g(y)dy\right] dt$$
$$= A + B,$$

where

$$A = \int_0^1 \left[\int_0^{+\infty} t^{u(x)-1/2}f(x)dx\right]\left[\int_0^{+\infty} t^{v(x)-1/2}g(x)dx\right] dt$$

and

$$B = \int_0^1 \left[\int_0^{+\infty} t^{u(x)-1/2}\sqrt{1-t}w(x)f(x)dx\right] \times$$
$$\left[\int_0^{+\infty} t^{v(x)-1/2}\sqrt{1-t}z(x)g(x)dx\right] dt.$$

Let us now lower bound A and B one by one.

Lower bound for A. A possible statement of the Cauchy-Schwarz integral inequality for positive functions defined on $[0, +\infty)$ is as follows: For any functions $h : [0,+\infty) \mapsto [0,+\infty)$ and $k : [0,+\infty) \mapsto [0,+\infty)$, we have

$$\int_0^{+\infty} h(x)k(x)dx \leq \sqrt{\int_0^{+\infty}[h(x)]^2dx \int_0^{+\infty}[k(x)]^2dx},$$

or in an equivalent way (even if the functions are not exactly the same),

$$\left[\int_0^{+\infty} h(x)dx\right]\left[\int_0^{+\infty} k(x)dx\right] \geq \left[\int_0^{+\infty}\sqrt{h(x)k(x)}dx\right]^2.$$

Using this last inequality with $h(x) = t^{u(x)-1/2}f(x)$ and $k(x) = t^{v(x)-1/2}g(x)$, we obtain

$$A \geq \int_0^1 \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{f(x)g(x)}dx\right]^2 dt.$$

Applying again the Cauchy-Schwarz integral inequality (first form, as described above) with respect to the variable t, $h(t) = \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{f(x)g(x)}dx\right]^2$ for $t \in [0,1)$ and $h(t) = 0$ for $t \geq 1$, and $k(t) = 1$ for $t \in [0,1)$ and $k(t) = 0$ for $t \geq 1$, the Fubini-Tonelli theorem and an integral calculus, we find that

$$\int_0^1 \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{f(x)g(x)}dx\right]^2 dt$$
$$\geq \left\{\int_0^1 \int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{f(x)g(x)}dxdt\right\}^2$$
$$= \left\{\int_0^{+\infty} \left[\int_0^1 t^{[u(x)+v(x)]/2-1/2}dt\right]\sqrt{f(x)g(x)}dx\right\}^2$$
$$= \left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2.$$

As a result, we have

$$A \geq \left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2.$$

Lower bound for B. Using the Cauchy-Schwarz integral inequality (second form) with $h(x) = t^{u(x)-1/2}\sqrt{1-t}w(x)f(x)$ and $k(x) = t^{v(x)-1/2}\sqrt{1-t}z(x)g(x)$, we obtain

$$B \geq \int_0^1 \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}\sqrt{w(x)f(x)z(x)g(x)}dx\right]^2 dt.$$

Applying again the Cauchy-Schwarz integral inequality (first form) with respect to the variable t,

$$h(t) = \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}\sqrt{w(x)f(x)z(x)g(x)}dx\right]^2$$

for $t \in [0,1)$ and $h(t) = 0$ for $t \geq 1$, and $k(t) = 1$ for $t \in [0,1)$ and $k(t) = 0$ for $t \geq 1$, and the Fubini-Tonelli theorem, we find that

$$\int_0^1 \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}\sqrt{w(x)f(x)z(x)g(x)}dx\right]^2 dt$$
$$\geq \left\{\int_0^1 \left[\int_0^{+\infty} t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}\sqrt{w(x)f(x)z(x)g(x)}dx\right] dt\right\}^2$$
$$= \left\{\int_0^{+\infty} \left[\int_0^1 t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}dt\right]\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2.$$

Let us now compute the integral with respect to t. Using the following well-known result:

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

for $a > 0$ and $b > 0$, and $\Gamma(3/2) = \sqrt{\pi}/2$ (see [16]), we get

$$\int_0^1 t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}dt$$
$$= \frac{\Gamma[u(x)+v(x)+1/2]\Gamma(3/2)}{\Gamma[u(x)+v(x)+1/2+3/2]}$$
$$= \frac{\sqrt{\pi}}{2} \times \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}.$$

Therefore, we have

$$\left\{\int_0^{+\infty} \left[\int_0^1 t^{[u(x)+v(x)]/2-1/2}\sqrt{1-t}dt\right]\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2$$
$$= \frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2.$$

As a result, we obtain

$$B \geq \frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2.$$

Combining the lower bounds found for A and B, we establish that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)-w(x)z(y)}f(x)g(y)dxdy \geq A+B$$
$$\geq \left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2$$
$$+\frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2.$$

This concludes the proof. □

A few examples are described below.

- For the case $w(x) = 0$ or $z(y) = 0$, Proposition 2.1 is reduced to

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)}f(x)g(y)dxdy$$
$$\geq \left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2.$$

  In particular, by choosing $u(x) = x$ and $v(y) = y$ satisfying $u(x)+v(y)-w(x)z(y) = x+y \geq 0$, the following inverse Hilbert integral inequality is obtained:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}f(x)g(y)dxdy$$
$$\geq \left\{\int_0^{+\infty} \frac{2}{1+2x}\sqrt{f(x)g(x)}dx\right\}^2.$$

  Some simple numerical examples illustrating this inequality are now presented.

  – Considering $f(x) = 1/(1+2x)$ and $g(y) = 1/(1+2y)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)(1+2x)(1+2y)}dxdy$$
$$= \frac{\pi^2}{8} \approx 1.2337$$

and

$$\left\{\int_0^{+\infty} \frac{2}{1+2x}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty} \frac{2}{(1+2x)^2}dx\right\}^2 = 1.$$

Obviously, we have $1.2337 > 1$. This illustrates the found lower bound.

– Considering $f(x) = 1/(1+x^2)$ and $g(y) = 1/(1+y^2)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)(1+x^2)(1+y^2)}dxdy = \frac{\pi}{2} \approx 1.5708$$

and

$$\left\{\int_0^{+\infty} \frac{2}{1+2x}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty} \frac{2}{(1+2x)(1+x^2)}dx\right\}^2 = \frac{1}{5^2}[\pi + \log(16)]^2 \approx 1.3990.$$

Since $1.5708 > 1.3990$, this is consistent with the result given in Proposition 2.1, and also shows the accuracy of the lower bound.

– Considering $f(x) = e^{-x}$ and $g(y) = e^{-y}$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} e^{-x-y}dxdy = 1$$

and

$$\left\{\int_0^{+\infty} \frac{2}{1+2x}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty} \frac{2}{1+2x}e^{-x}dx\right\}^2 \approx e\left[\mathrm{Ei}\left(-\frac{1}{2}\right)\right]^2 \approx 0.8517,$$

where $\mathrm{Ei}(x)$ is the exponential integral function defined by $\mathrm{Ei}(x) = -\int_{-x}^{+\infty}(e^{-t}/t)dt$. As expected, we have $1 > 0.8517$, illustrating the obtained lower bound.

• Let us now concentrate on the general case, where $u(x)$ is not necessarily x, $v(y)$ is not necessarily y, $w(x) \neq 0$ and $z(y) \neq 0$.

– Considering $u(x) = e^{-x}$, $v(y) = e^{-y}$, $w(x) = e^{-x}$, $z(y) = e^{-y}$, $f(x) = e^{-x}$ and $g(y) = e^{-y}$, and noticing that

$$u(x) + v(y) - w(x)z(y) = e^{-x} + e^{-y} - e^{-x-y} = e^{-x} + e^{-y}(1 - e^{-x}) \geq 0,$$

we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)-w(x)z(y)} f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{1}{e^{-x}+e^{-y}-e^{-x-y}} e^{-x-y}dxdy = \frac{\pi^2}{6} \approx 1.6449,$$

$$\left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty} \frac{2}{1+2e^{-x}}e^{-x}dx\right\}^2 = [\log(3)]^2 \approx 1.2069$$

and

$$\frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2 = \frac{\pi}{4}\left\{\int_0^{+\infty} \frac{\Gamma(2e^{-x}+1/2)}{\Gamma(2e^{-x}+2)}e^{-2x}dx\right\}^2 \approx 0.0298.$$

Since $1.6449 > 1.2069 + 0.0298 = 1.2367$, the demonstrated inequality is thus illustrated.

– Considering $u(x) = e^{-x}$, $v(y) = y/(1+y)$, $w(x) = e^{-x}$, $z(y) = y/(1+y)$, $f(x) = e^{-x}$ and $g(y) = e^{-y}$, noticing that

$$u(x) + v(y) - w(x)z(y) = e^{-x} + \frac{y}{1+y} - e^{-x}\frac{y}{1+y} = e^{-x} + \frac{y}{1+y}(1 - e^{-x}) \geq 0,$$

we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)-w(x)z(y)} f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{1}{e^{-x}+y/(1+y)-e^{-x}y/(1+y)} e^{-x-y}dxdy \approx 1.7507,$$

$$\left\{\int_0^{+\infty} \frac{2}{1+u(x)+v(x)}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty} \frac{2}{1+e^{-x}+x/(1+x)}e^{-x}dx\right\}^2 \approx 1.1068$$

and

$$\frac{\pi}{4}\left\{\int_0^{+\infty}\frac{\Gamma[u(x)+v(x)+1/2]}{\Gamma[u(x)+v(x)+2]}\sqrt{w(x)f(x)z(x)g(x)}dx\right\}^2$$
$$=\frac{\pi}{4}\left\{\int_0^{+\infty}\frac{\Gamma[e^{-x}+x/(1+x)+1/2]}{\Gamma[e^{-x}+x/(1+x)+2]}e^{-3x/2}\sqrt{\frac{x}{1+x}}dx\right\}^2$$
$$\approx 0.0237.$$

Since $1.7507 > 1.1068+0.0237 = 1.1305$, Proposition 2.1 is thus illustrated.

These are just some numerical and illustrative examples, so much others can be presented in a similar way, with different choices for the functions f, g, u, v, w and z.

2.2 Second lower bound

In the result below, we offer an alternative to Proposition 2.1, but with a ratio-type lower bound, which still depends on simple integrals.

Proposition 2.2 *Under the exact setting of Proposition 2.1, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)-w(x)z(y)}f(x)g(y)dxdy \geq \frac{1}{\Xi}\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(x)dx\right]^2,$$

where

$$\Xi=\left[\int_0^{+\infty}u(x)f(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]+\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}v(x)g(x)dx\right]-\left[\int_0^{+\infty}w(x)f(x)dx\right]\left[\int_0^{+\infty}z(x)g(x)dx\right].$$

Proof. We can write

$$\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]=\int_0^{+\infty}\int_0^{+\infty}f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{u(x)+v(y)-w(x)z(y)}}{\sqrt{u(x)+v(y)-w(x)z(y)}}\sqrt{f(x)g(y)}\sqrt{f(x)g(y)}dxdy.$$

Applying the Cauchy-Schwarz (double) integral inequality for the double integral, we get

$$\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right] \leq \sqrt{\int_0^{+\infty}\int_0^{+\infty}[u(x)+v(y)-w(x)z(y)]f(x)g(y)dxdy}\times\sqrt{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)-w(x)z(y)}f(x)g(y)dxdy}.$$

This implies that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)-w(x)z(y)}f(x)g(y)dxdy \geq \frac{1}{\Omega}\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(x)dx\right]^2,$$

where

$$\Omega=\int_0^{+\infty}\int_0^{+\infty}[u(x)+v(y)-w(x)z(y)]f(x)g(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}u(x)f(x)g(y)dxdy+\int_0^{+\infty}\int_0^{+\infty}v(y)f(x)g(y)dxdy-\int_0^{+\infty}\int_0^{+\infty}w(x)z(y)f(x)g(y)dxdy$$
$$=\left[\int_0^{+\infty}u(x)f(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]+\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}v(x)g(x)dx\right]-\left[\int_0^{+\infty}w(x)f(x)dx\right]\left[\int_0^{+\infty}z(x)g(x)dx\right]=\Xi.$$

This concludes the proof. □

For the case $w(x)=0$ or $z(y)=0$, Proposition 2.2 is reduced to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)}f(x)g(y)dxdy \geq \frac{\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(x)dx\right]^2}{\left[\int_0^{+\infty}u(x)f(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]+\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}v(x)g(x)dx\right]}.$$

In particular, by choosing $u(x)=x$ and $v(y)=y$ satisfying $u(x)+v(y)-w(x)z(y)=x+y\geq 0$, the following inverse Hilbert integral inequality is obtained:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \geq \frac{\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(x)dx\right]^2}{\left[\int_0^{+\infty}xf(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]+\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}xg(x)dx\right]}.$$

Based on this and the Hilbert integral inequality in Equation (2), this implies the following norm inequality:

$$\pi\sqrt{\int_0^{+\infty}[f(x)]^2dx\int_0^{+\infty}[g(x)]^2dx} \geq \frac{\left[\int_0^{+\infty}f(x)dx\right]^2\left[\int_0^{+\infty}g(x)dx\right]^2}{\left[\int_0^{+\infty}xf(x)dx\right]\left[\int_0^{+\infty}g(x)dx\right]+\left[\int_0^{+\infty}f(x)dx\right]\left[\int_0^{+\infty}xg(x)dx\right]}.$$

Furthermore, under the assumptions $\int_0^{+\infty}f(x)dx=1$ and $\int_0^{+\infty}g(x)dx=1$ (which means that f and g are probability density functions, since they are positive as the first assumption), we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \geq \frac{1}{\int_0^{+\infty}xf(x)dx+\int_0^{+\infty}xg(x)dx}.$$

For example, taking $f(x) = e^{-x}$ and $g(y) = e^{-y}$, corresponding to the probability density functions of the exponential distribution with parameter 1, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}e^{-x-y}dxdy = 1,$$

$\int_0^{+\infty} xf(x)dx = 1$ and $\int_0^{+\infty} xg(x)dx = 1$, implying that

$$\frac{1}{\int_0^{+\infty} xf(x)dx + \int_0^{+\infty} xg(x)dx} = \frac{1}{2} = 0.5.$$

Obviously, since $1 > 0.5$, this ends the example.

The second main lower bound is presented in the next section.

3 Lower bounds for Case II

3.1 First lower bound

The proposition below is about a lower bound for the double integral in Equation (1) for Case II.

Proposition 3.1 *Let $f, g, u, v, w, z : [0, +\infty) \to [0, +\infty)$ be functions such that the future integrals which will depend on them converge (this is a minimum condition). Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy \geq \left\{\int_0^{+\infty}\frac{2}{1+u(x)+v(x)+[w(x)]^2/2+[z(x)]^2/2}\sqrt{f(x)g(x)}dx\right\}^2.$$

Proof. Using the basic inequality $(a-b)^2 \geq 0$, i.e., $a^2+b^2 \geq 2ab$, with $a = w(x)$ and $b = z(y)$, we get

$$w(x)z(y) \leq \frac{1}{2}[w(x)]^2 + \frac{1}{2}[z(y)]^2.$$

Therefore, we have

$$\frac{1}{u(x)+v(y)+w(x)z(y)} \geq \frac{1}{u(x)+v(y)+[w(x)]^2/2+[z(y)]^2/2} = \frac{1}{u_*(x)+v_*(y)},$$

where $u_*(x) = u(x) + [w(x)]^2/2$ and $v_*(y) = v(y) + [z(y)]^2/2$. It follows from Proposition 2.1 applied with $u_*(x)$ instead of $u(x)$ and $v_*(y)$ instead of $v(y)$ and $w(x) = 0$ (or $z(y) = 0$), noticing that $u_*(x) + v_*(x) \geq 0$, that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy \geq \int_0^{+\infty}\int_0^{+\infty}\frac{1}{u_*(x)+v_*(y)-0}f(x)g(y)dxdy \geq \left\{\int_0^{+\infty}\frac{2}{1+u_*(x)+v_*(x)}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty}\frac{2}{1+u(x)+v(x)+[w(x)]^2/2+[z(x)]^2/2}\sqrt{f(x)g(x)}dx\right\}^2.$$

This concludes the proof. □

Let us now illustrate this theoretical result with some numerical examples.

- Considering $u(x) = e^{-x}$, $v(y) = e^{-y}$, $w(x) = e^{-x}$, $z(y) = e^{-y}$, $f(x) = e^{-x}$ and $g(y) = e^{-y}$, we have
$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{e^{-x}+e^{-y}+e^{-x-y}}e^{-x-y}dxdy \approx 1.2285$$
and
$$\left\{\int_0^{+\infty}\frac{2}{1+u(x)+v(x)+[w(x)]^2/2+[z(x)]^2/2}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty}\frac{2}{1+2e^{-x}+e^{-2x}}e^{-x}dx\right\}^2 = 1.$$
Since $1.2285 > 1$, Proposition 3.1 is thus illustrated.
- Considering $u(x) = e^{-x}$, $v(y) = y/(1+y)$, $w(x) = e^{-x}$, $z(y) = y/(1+y)$, $f(x) = e^{-x}$ and $g(y) = e^{-y}$, we have
$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{e^{-x}+y/(1+y)+e^{-x}y/(1+y)}e^{-x-y}dxdy \approx 1.3380,$$
and
$$\left\{\int_0^{+\infty}\frac{2}{1+u(x)+v(x)+[w(x)]^2/2+[z(x)]^2/2}\sqrt{f(x)g(x)}dx\right\}^2 = \left\{\int_0^{+\infty}\frac{2}{1+e^{-x}+x/(1+x)+e^{-2x}/2+x^2/[2(1+x)^2]}e^{-x}dx\right\}^2 \approx 0.8517.$$
We obviously have $1.3380 > 0.8517$, ending the numerical example.

3.2 Second lower bound

Similar to what Proposition 2.2 is for Proposition 2.1, we offer an alternative to Proposition 3.1, with the use of the Cauchy-Schwarz inequality.

Proposition 3.2 *Under the exact setting of Proposition 3.1, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy \geq \frac{1}{\Upsilon}\left[\int_0^{+\infty} f(x)dx\right]^2\left[\int_0^{+\infty} g(x)dx\right]^2,$$

where

$$\Upsilon = \left[\int_0^{+\infty} u(x)f(x)dx\right]\left[\int_0^{+\infty} g(x)dx\right] + \left[\int_0^{+\infty} f(x)dx\right]\left[\int_0^{+\infty} v(x)g(x)dx\right] + \left[\int_0^{+\infty} w(x)f(x)dx\right]\left[\int_0^{+\infty} z(x)g(x)dx\right].$$

Proof. The proof is almost identical to that of Proposition 2.2, except that one sign has to be changed in the right places. So we can write

$$\left[\int_0^{+\infty} f(x)dx\right]\left[\int_0^{+\infty} g(x)dx\right] = \int_0^{+\infty}\int_0^{+\infty} f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{\sqrt{u(x)+v(y)+w(x)z(y)}}{\sqrt{u(x)+v(y)+w(x)z(y)}}\sqrt{f(x)g(y)}\sqrt{f(x)g(y)}dxdy.$$

Applying the Cauchy-Schwarz (double) integral inequality for the double integral, we get

$$\left[\int_0^{+\infty} f(x)dx\right]\left[\int_0^{+\infty} g(x)dx\right] \le \sqrt{\int_0^{+\infty}\int_0^{+\infty} [u(x)+v(y)+w(x)z(y)]f(x)g(y)dxdy}\times \sqrt{\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy}.$$

This implies that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)+w(x)z(y)}f(x)g(y)dxdy \ge \frac{1}{\aleph}\left[\int_0^{+\infty} f(x)dx\right]^2\left[\int_0^{+\infty} g(x)dx\right]^2,$$

where

$$\aleph = \int_0^{+\infty}\int_0^{+\infty} [u(x)+v(y)+w(x)z(y)]f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} u(x)f(x)g(y)dxdy + \int_0^{+\infty}\int_0^{+\infty} v(y)f(x)g(y)dxdy + \int_0^{+\infty}\int_0^{+\infty} w(x)z(y)f(x)g(y)dxdy$$
$$= \left[\int_0^{+\infty} u(x)f(x)dx\right]\left[\int_0^{+\infty} g(x)dx\right] + \left[\int_0^{+\infty} f(x)dx\right]\left[\int_0^{+\infty} v(x)g(x)dx\right] + \left[\int_0^{+\infty} w(x)f(x)dx\right]\left[\int_0^{+\infty} z(x)g(x)dx\right] = \Upsilon.$$

This concludes the proof. □

To our knowledge, this is a new integral result in the literature.

4 Conclusion

In conclusion, this paper fills a gap in the literature by investigating an original sharp lower bound on a general double integral term that depends on six adaptable functions. It has the following form:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u(x)+v(y)\pm w(x)z(y)}f(x)g(y)dxdy.$$

By distinguishing between the cases of subtraction and addition of the product term in the denominator, we have developed new methods using sharp integral and classical inequalities. Several numerical examples are given to demonstrate the effectiveness of the proposed bounds. Our results contribute to a better understanding and analysis of complex double integrals with broad implications for future research.

Acknowledgment:
The author thanks the three reviewers for their thorough and constructive comments on the paper.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.