Twisted dynamical systems, Schrodinger representations

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Abstract: - Assume G is a locally compact Hausdorff group, A is a C^*-algebra, and (A, G, ω) is a dynamical system, we consider a Takai theorem that states the isomorphism $\Phi : (\mathrm{A} \times_\omega G) \times_{\hat{\omega}} \hat{G} \to \mathrm{A} \otimes LK(L^2(G))$ is equivariant for $\hat{\hat{\omega}} : G \to (\mathrm{A} \times_\omega G) \times_{\hat{\omega}} \hat{G}$ and for $\hat{\hat{\omega}} \otimes Ad(\rho) : G \to \mathrm{A} \otimes LK(L^2(G))$. Also, we show that $*$-surjective mapping $\Upsilon : C_C(G, \mathrm{A}) \to C_C(G, \mathrm{A}, \tau)$ can be extended to quotient mapping $\tilde{\Upsilon} : \mathrm{A} \times_\omega G \to \mathrm{A} \times_\omega^\tau G = \mathrm{A} \times_\omega G / (I \bigcap \mathrm{A} \times_\omega G)$ for the twisted dynamical system $(\mathrm{A}, G, \omega, \tau)$.

We establish that there exists an isomorphism of the Schrodinger C^*-algebra $Sch_\Lambda^\tau(\mathrm{A} \times_\omega^\tau G)$ to the reduced crossed product $\lfloor \mathrm{A} \times_\omega^\tau G \rfloor_{red}$; and show the representation $\mathrm{B}^\tau(\mathrm{A}) \mapsto Sch_\Lambda^\tau(\mathrm{A} \times_\omega^\tau G) \subset LB(L^2(G))$ is faithful for each amenable group G.

Key-Words: - Takai Duality, γ-duality, Wigner function, C^*-algebra, Pontryagin duality, induced representation, cross product.

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1 Introduction (dynamic systems)

Let G be a locally compact Hausdorff group and let μ be a Radon measure on G. Assume that N is a locally compact subgroup of G then G/N is locally compact Hausdorff group.

Definition. ***The group G is called an extension of the group N by the group H if the short group sequence***

$$e \to N \xrightarrow{j} G \xrightarrow{k} H \to e \qquad (1)$$

is exact, where j is a continuous homeomorphism onto the range of j and k is a continuous open surjective mapping.

If we assume that N a normal subgroup of G then H is a quotient group G/N. So, symbolically we can write a short exact sequence of groups

$$e \to N \xrightarrow{j} N \times G/N \xrightarrow{k} G/N \to e. \qquad (2)$$

Definition 2. ***Let A be a C^*-algebra and G locally compact Hausdorff group then the triplet (A, G, ω) is called a dynamical system where $\omega : G \to Aut(\mathrm{A})$ is a strongly continuous representation.***

Definition 3. ***Let (A, G, ω) be a dynamic system on the Hilbert space H. Let ρ be unitary representation $\rho : G \to U(H)$ and $\pi : A \to LB(H)$ representation on the Hilbert space H, such that***

$$\pi(\omega(g, a)) = \rho(g)\pi(a)\rho^*(g). \quad (3)$$

Then, the pair (π, ρ) is called a covariant representation of (A, G, ω).

Definition 4. ***Let (π, ρ) be a covariant representation of the dynamic system (A, G, ω) on the Hilbert space H. The L^1-norm-decreasing $*$-representation $C_C(G, A)$ on H is given by***

$$(\pi \times \rho)(\psi) = \int_G \pi(\psi(h))\rho(h)d\mu(h) \quad (4)$$

for all $\psi \in C_C(G)$.

We denote a continuous homomorphism $\Delta : G \to R_+$ such that the equality

$$\Delta(g)\int_G \psi(hg)d\mu(h) = \int_G \psi(h)d\mu(h) \quad (5)$$

holds for all $\psi \in C_C(G)$.

To show that mapping $\pi \times \rho$ is $*$-homomorphism, we compute

$$\begin{aligned}
&\pi \times \rho(\psi)^* = \int_G \left(\pi(\psi(h))\rho(h)\right)^* d\mu(h) = \\
&= \int_G \rho(h^{-1})\pi\left(\psi(h)^*\right)d\mu(h) = \\
&= \int_G \pi\left(\omega\left(h, \psi(h^{-1})^* \Delta(h^{-1})\right)\right)\rho(h)d\mu(h) = \\
&= \pi \times \rho(\psi^*).
\end{aligned} \quad (6)$$

Applying the Fubini theorem, we write

$$\begin{aligned}
&\pi \times \rho(\psi * \varphi) = \\
&= \int_G\int_G \frac{\pi\left(\psi(h)\omega\left(h, \varphi(h^{-1}g)\right)\right)}{\rho(g)d\mu(h)d\mu(g)} = \\
&= \int_G\int_G \frac{\pi(\psi(h))\rho(h)\pi\left(\varphi(h^{-1}g)\right)}{\rho(h^{-1}g)d\mu(g)d\mu(h)} = \\
&= \pi \times \rho(\psi) \circ \pi \times \rho(\varphi),
\end{aligned} \quad (7)$$

so $\pi \times \rho$ is $*$-homomorphism.

Definition 5. ***Let (A, G, ω) be a dynamic system (A, G, ω). The norm on $C_C(G, A)$, of the function $\psi \in C_C(G, A)$ given by***

$$\|\psi\| = \sup\left\{ \begin{array}{l} \|\pi \times \rho(\psi)\| \ : \ (\pi, \rho) \ \text{is} \\ \text{covariant representation of } (A, G, \omega) \end{array} \right\} \quad (8)$$

is called the universal norm.

Definition 6. ***The completion of the set $C_C(G, A)$ in the universal norm is called the cross product $A \times_\omega G$ of A by G and present a Banach C^*-algebra. The cross-product $A \times_\omega G$ is said to be associated with the dynamic system (A, G, ω).***

Definition 7. ***Let N and H be a pair of locally compact groups and let mapping $\gamma : H \to Aut(N)$ be continuous homomorphism maps as $(h, n) \mapsto \gamma(h)(n)$ for all $n \in N, \quad h \in H$, and let the short sequence***

$$e \to N \xrightarrow{\alpha} G \xrightarrow{\beta} H \to e \quad (9)$$

be exact, where $\gamma(h)(n) = \alpha^{-1}\left(\tau(h)\alpha(n)\tau(h^{-1})\right)$, homomorphism $\tau : H \to G$ satisfies $\beta \circ \tau = id(H)$ (identity map on H), then the

group G is called a semi-direct product $N\times_{\gamma} H$ of pair of groups N and H.

2. The Takai dynamical system

Let G be an Abelian locally compact group and let (A, G, ω) be a dynamical system. A homomorphism $\hat{\omega}:\hat{G}\to Aut(\mathsf{A}\times_{\omega} G)$ is given by extending of the mapping $\hat{\omega}(\chi):C_C(G,\mathsf{A})\to C_C(G,\mathsf{A})$, $\hat{\omega}(\chi)(\psi)(g)\mapsto \overline{\chi(g)}\,\psi(g)$.

We denote the space of all linear compact operators on the Hilbert space $L^2(G)$ by $LK(L^2(G))$.

The Takai duality theorem states ***that if assume (A, G, ω) is a dynamical system then the isomorphism $\Phi:(\mathsf{A}\times_{\omega} G)\times_{\hat{\omega}}\hat{G}\to \mathsf{A}\otimes LK(L^2(G))$ is equivariant for $\hat{\hat{\omega}}:G\to(\mathsf{A}\times_{\omega} G)\times_{\hat{\omega}}\hat{G}$ and for $\hat{\hat{\omega}}\otimes Ad(\rho):G\to \mathsf{A}\otimes LK(L^2(G))$, where the mapping $\rho:G\to\mathsf{A}\otimes LK(L^2(G))$ is the right regular representation.***

We remind our reader two statements of the Peter-Weyl theorem: the first statement, the collection of matrix coefficients of the group G is dense in $C(G)$ relevant to the uniform topology; the second statement, assume $\varpi:G\to H$ is a unitary representation of G in Hilbert space $H=L^2(G)$, then $H=L^2(G)$ can be presented in the form of the direct sum of irreducible finite-dimensional unitary representation of G. Let G be compact, the Peter-Weyl theorem implies that each $\hat{\chi}\in\hat{G}$ equals a subrepresentation of the left-regular representation $\lambda : G\to U(L^2(G))$.

The proof of the Takai theorem is based on the following sequence of isomorphisms

$$\begin{aligned}&(\mathsf{A}\times_{\omega} G)\times_{\hat{\omega}}\hat{G}\xrightarrow{\Phi_1}(\mathsf{A}\times_{id}\hat{G})\times_{\lambda^{-1}\otimes\omega} G\xrightarrow{\Phi_2}\\&\xrightarrow{\Phi_2}C_0(G,\mathsf{A})\times_{\lambda\otimes\omega} G\xrightarrow{\Phi_3}\\&\xrightarrow{\Phi_3}C_0(G,\mathsf{A})\times_{\lambda\otimes id} G\xrightarrow{\Phi_4}C_0(G)\times_{\lambda} G\otimes\mathsf{A}\\&\xrightarrow{\Phi_5}\mathsf{A}\otimes LK(L^2(G)).\end{aligned} \tag{10}$$

The subalgebras $C_C(\hat{G}\times G,\mathsf{A})$ and $C_C(G\times\hat{G},\mathsf{A})$ are dense in the $(\mathsf{A}\times_{\omega} G)\times_{\hat{\omega}}\hat{G}$ and $(\mathsf{A}\times_{id}\hat{G})\times_{\lambda^{-1}\otimes\omega} G$, respectively. The isomorphism $\Phi_1:C_C(\hat{G}\times G,\mathsf{A})\to C_C(G\times\hat{G},\mathsf{A})$ is given by $\Phi_1(f)(g,\chi)=\chi(g)\,f(\chi,g)$ for all $f\in C_C(\hat{G}\times G,\mathsf{A})$ next extends to $\Phi_1:(\mathsf{A}\times_{\omega} G)\times_{\hat{\omega}}\hat{G}\to(\mathsf{A}\times_{id}\hat{G})\times_{\lambda^{-1}\otimes\omega} G$. The mapping Φ_1 is continuous in the inductive limit topology.

The second isomorphism $\Phi_2:C_C(G\times\hat{G},\mathsf{A})\to C_C(G,C_0(G,\mathsf{A}))$ is a Fourier transform given by

$$\Phi_2(f)(g,h)=\int_{\hat{G}} f(g,\chi)\,\overline{\chi(h)}\,d\hat{\mu}(\chi) \tag{11}$$

for all $f\in C_C(G\times\hat{G},\mathsf{A})$.

The third isomorphism Φ_3 is defined as $\Phi_3(f)(g,h)=\omega^{-1}(h)(f(g,h))$ for all $f\in C_C(G,C_0(G,\mathsf{A}))$. Next, we need the following lemma.

Lemma (Raeburn) 1. ***Let (Λ,G,α) be a dynamical system and let Θ be a C^*-algebra, then***

$$(\Lambda\otimes_{\max}\Theta)\times_{\alpha\otimes id} G\cong\Lambda\times_{\alpha} G\otimes_{\max}\Theta \tag{12}$$

is equal in the isomorphic sense.

The proof is based on the Raeburn theorem.

We define an isomorphism $\breve{\Phi} : C_0(G)\times_\lambda G \to LK(L^2(G))$ by

$$\breve{\Phi}(f)(g,h)=\int_G f(k,h)\phi(k^{-1}g)\,d\mu(k) \quad (13)$$

for $f \in C_C(G\times G)C_0(G)\times_\lambda G$ and $\phi\in C_C(G)\subset L^2(G)$. Thus, by Raeburn lemma, there exists an equivariant isomorphism $C_0(G,\mathrm{A})\times_{\lambda\otimes id} G \xrightarrow{\tilde{\Phi}_4} \mathrm{A}\otimes LK(L^2(G))$.

The necessary isomorphism $\Phi:(\mathrm{A}\times_\omega G)\times_{\hat{\omega}}\hat{G}\to \mathrm{A}\otimes LK(L^2(G))$ can be written as a combination $\Phi=\tilde{\Phi}_4\circ\Phi_3\circ\Phi_2\circ\Phi_1$.

3. Twisted dynamical system

Let (A, G, ω) be a dynamic system. Let N be a normal subgroup of G. Let $UM(\mathrm{A})$ be a unitary group of multiplier algebra of A.

Definition 8. ***A continuous homomorphism*** $\tau: N\to UM(\mathrm{A})$ ***such that*** $\tau(n)a\tau(n)^*=\omega(n,a)$ ***and*** $\overline{\omega}(g,\tau(n))=\tau(gng^{-1})$ ***for all*** $n\in N$, $g\in G$, $a\in\mathrm{A}$***, is called a twisting map. The covariant representation*** (π,ρ) ***of the dynamic system*** (A, G, ω) ***is called preserving*** $\tau: N\to UM(\mathrm{A})$ ***if equality*** $\overline{\pi}(\tau(n))=\rho(n)$ ***holds for all*** $n\in N$***. The quartet*** $(\mathrm{A}, G,\omega,\tau)$ ***is called a twisted dynamical system.***

Definition 9. Let (i_{A}, i_G) be a canonical covariant homomorphism from (A, G, ω) to $M(\mathrm{A}\times_\omega G)$. Let ***$I$ be the ideal of multiplier algebra $M(\mathrm{A}\times_\omega G)$ of the cross product $\mathrm{A}\times_\omega G$ generated by a set consisting of*** $\{i_G(n)-\overline{i_{\mathrm{A}}}(\tau(n)):\ n\in N\}$.

We define the twisted crossed product by

$$\mathrm{A}\times_\omega^\tau G \overset{def}{=} \mathrm{A}\times_\omega G/(I\ \cap \mathrm{A}\times_\omega G). \quad (14)$$

Definition 10. ***We denote by $C_C(G,\mathrm{A},\tau)$ the subclass of continuous functions from G to A and satisfy the conditions: first, there exists a compact subset $K\subset G$ such that*** $\operatorname{supp}(\psi)\subset KN$ ***; second, for all $\psi\in C_C(G,\mathrm{A},\tau)$ the equality $\psi(ng)=\psi(g)\tau(n)^*$ holds for all $n\in N$ and $g\in G$.***

Let μ_N and $\mu_{G/N}$ be Haar measure on N and G/N, respectively, then there the equality

$$\int_G \psi(g)\,d\mu(g)=\int_{G/N}\int_N \psi(hn)\,d\mu_N(n)\,d\mu_{G/N}(h) \quad (15)$$

holds for all $\psi\in C_C(G)$.

The convolution product $\psi*\varphi\in C_C(G,\mathrm{A},\tau)$ is given by

$$\psi*\varphi(h)=\int_{G/N}\psi(k)\omega(k,\varphi(k^{-1}h))\,d\mu_{G/N}(k)\ , \quad (16)$$

and function

$$\psi^*(h)=\Delta(G/N,\ \ h^{-1})\,\omega\left(h,\left(\psi(h^{-1})^*\right)\right),$$
$$\psi^*\in C_C(G,\mathrm{A},\tau). \quad (17)$$

So, we obtain that ***the set $C_C(G,\mathrm{A},\tau)$ is a $*$-algebra with the convolution operations given by (16) and (17).***

Lemma 2. ***For every function $\psi\in C_C(G,\mathrm{A})$, we define a function $\Upsilon(\psi)\in C_C(G,\mathrm{A},\tau)$ by***

$$\Upsilon(\psi)(g)=\int_N \psi(gn)\tau(gng^{-1})\,d\mu_N(n)\ , \quad (18)$$

thus, there exists a structure-preserving $*$-surjective mapping $\Upsilon : C_C(G,\mathrm{A}) \to C_C(G,\mathrm{A},\tau)$.

Proof. Let N be a normal subgroup of a locally compact group G. A continuous function $\mathrm{B} : G \to R_+$ is called the Bruhat approximate cross-section of G by N if the equality

$$\int_N \mathrm{B}(gn)\, d\mu_N(n) = 1$$

holds for all $g \in G$, and condition that $\operatorname{supp}(\mathrm{B}) \cap satur(K)$ is compact for any compact $K \subset G$. Let $\mathrm{B} : G \to R_+$ be Bruhat approximate cross-section of G by N so we have that $\psi(g) = \mathrm{B}(g)\varphi(g) \in C_C(G,\mathrm{A})$ for any fixed function $\varphi \in C_C(G,\mathrm{A},\tau)$ and the equality

$$\Upsilon(\psi)(g) = \int_N \psi(gn)\varphi(gn)\tau\left(gng^{-1}\right) d\mu_N(n) = \varphi(g)$$

holds, and so Υ is a surjection.

Now, we show that $\Upsilon(\psi^*) = \Upsilon(\psi)^*$ so we write

$$\Upsilon(\psi)^*(g) = \Delta\left(G/N,\ h^{-1}\right)\omega\left(h, \left(\Upsilon(\psi)\left(g^{-1}\right)^*\right)\right) =$$
$$= \Delta\left(G/N,\ h^{-1}\right) \times \omega\left(h, \int_N \psi\left(g^{-1}n\right)\tau\left(g^{-1}ng\right) d\mu_N(n)\right)^* =$$
$$= \Delta\left(G/N,\ h^{-1}\right) \times \int_N \omega\left(h, \tau\left(g^{-1}n^{-1}g\right)\psi\left(g^{-1}n\right)^*\right) d\mu_N(n) =$$
$$= \Delta\left(G/N,\ h^{-1}\right) \times \int_N \omega\left(n^{-1}h, \psi\left(g^{-1}n\right)^* \tau(n)^*\right) d\mu_N(n) =$$
$$= \Delta\left(G/N,\ h^{-1}\right)\Delta\left(GN,\ \left(h^{-1}\right)\right) \times \int_N \omega\left(hn^{-1}, \psi\left(ng^{-1}\right)^* \tau\left(gn^{-1}g^{-1}\right)^*\right) d\mu_N(n) =$$
$$= \Delta\left(G,\ h^{-1}\right)\int_N \psi^*(gn)\tau\left(gng^{-1}\right) d\mu_N(n)$$
$$= \Upsilon(\psi)^*(g).$$

Similarly, since

$$\Upsilon(\psi * \varphi)(g) = \int_N \int_G \psi(k)\omega\left(k, \varphi\left(k^{-1}gn\right)\tau\left(gng^{-1}\right)\right) d\mu(k) d\mu_N(n)$$
$$= \int_{G/N} \Upsilon(\psi)(k)\omega\left(k, \Upsilon(\varphi)\left(k^{-1}g\right)\right) d\mu_{G/N}(k) =$$
$$= \Upsilon(\psi) * \Upsilon(\varphi)(g),$$

we have $\Upsilon(\psi * \varphi) = \Upsilon(\psi) * \Upsilon(\varphi)$.

Assume covariant representation (π, ρ) preserves τ, so that $\pi(e, nI_H) = nI_H$ for all $n \in N$, we define a norm

$$\|\varphi\|_\tau = \sup\left\{\left\|\pi \times^\tau \rho(\varphi)\right\| \ : (\pi, \rho) \text{ preserves } \tau\right\}$$

for all $\varphi \in C_C(G,\mathrm{A},\tau)$.

The norm of the quotient product $\mathrm{A} \times_\omega^\tau G$ is given as $\|\Upsilon(\psi)\|_\tau$. Thus, we obtain the following theorem.

Theorem 1. ***Let*** $(\mathrm{A}, G, \omega, \tau)$ ***be a twisted dynamical system. Then, the completion of the set*** $C_C(G,\mathrm{A},\tau)$ ***with respect to the norm defined as***

$$\|\varphi\|_\tau = \sup\left\{\left\|\pi \times^\tau \rho(\varphi)\right\| \ : (\pi, \rho) \text{ preserves } \tau\right\}$$

coincides with the crossed-product $\mathrm{A} \times_\omega^\tau G \overset{def}{=} \mathrm{A} \times_\omega G / \left(I \ \cap \mathrm{A} \times_\omega G\right)$, ***and*** $*$***-surjective mapping*** $\Upsilon : C_C(G,\mathrm{A}) \to C_C(G,\mathrm{A},\tau)$ ***can be***

extended to quotient mapping $\tilde{\Upsilon}\ :\ \mathsf{A}\times_{\omega}G\to\mathsf{A}\times_{\omega}^{\tau}G$.

4. An example of the Schrodinger representation

Let G be a locally compact group and let set $\mathsf{A}(G)$ be a C^{*}-algebra of all bounded left translation invariant and left uniformly continuous functions such that $\omega(h,\psi)(g)=\psi(h^{-1}g)$.

Definition. ***Let*** $\tau:G\to G$ ***be a measurable mapping, then we define a binal operation given by***

$$\left(\psi_{1}\square^{\tau}\psi_{2}\right)(p,x)=$$
$$=\int_{G}\psi_{1}\left(\tau(z)^{-1}\tau(x)p,z\right)\psi_{2}\left(\tau\left(z^{-1}x\right)^{-1}z^{-1}\tau(x)p,z^{-1}x\right)d\mu(z)$$

for all functions $\psi_{1},\psi_{2}\in L^{1}(G,\mathsf{A})$***; and the involution define by***

$$\psi_{1}^{\square^{\tau}}(p,x)=\psi_{2}\left(\tau\left(x^{-1}\right)^{-1}x^{-1}\tau(x)p,x^{-1}\right).$$

We define a crossed product $\mathsf{A}\times_{\omega}^{\tau}G$ ***as the enveloping*** C^{*}***-algebra of the Banach*** $*$***-algebra*** $L^{1}(G,\mathsf{A})$ ***with the convolution product*** $\square^{\tau}$ ***and completion in the universal norm***

$$\|\psi\|=\sup_{P}\left\{\|P(\psi)\|:P:L^{1}(G,\mathsf{A})\to LB(H,H)\right\}.$$

The space $C_{C}(G,\mathsf{A})$ of all continuous functions $G\mapsto\mathsf{A}$ with compact support is a dense subalgebra of $\mathsf{A}\times_{\omega}^{\tau}G$, where $\tau:G\to G$ is a measurable function.

The Schrodinger representation $\left(\pi,\Lambda,L^{2}(G)\right)$ is given by

$$\left(\Lambda(z)\phi\right)(x)=\phi\left(z^{-1}x\right)$$

and

$$\pi(\psi)\phi=\psi\phi,$$

for any bounded function ϕ, so that $\pi(\psi)$ is a multiplication operator.

The covariant representation is given by the integral

$$\left(\pi\times^{\tau}\Lambda\right)(\psi)=$$
$$\int_{G}\pi\left(\omega\left(\tau(z),\psi(z)\right)\right)\Lambda(z)d\mu(z)=$$
$$=\int_{G}\psi\left(\tau\left(xz^{-1}\right)^{-1}x,xz^{-1}\right)\phi(z)d\mu(z)\overset{def}{=}$$
$$=Sch_{\Lambda}^{\tau}(\psi)(x)$$

defined for $\psi\in L^{1}(G,\mathsf{A})$ and $\phi\in L^{2}(G)$.

Let $\mathrm{B}^{\tau}(\mathsf{A})$ be an enveloping C^{*}-algebra of $\left(id(\mathsf{A})\otimes_{pr}F\right)\left(\mathsf{A}\otimes_{pr}L^{1}(G)\right)$ where operation $\otimes_{pr}$ means the projective tensor product and F is a Fourier transform. Then, there exists an extension $\mathsf{F}^{\tau}(\mathsf{A})$ of $id(\mathsf{A})\otimes_{pr}F$ such that $\mathsf{F}^{\tau}(\mathsf{A}):\mathsf{A}\times_{\omega}^{\tau}G\to\mathrm{B}^{\tau}(\mathsf{A})$.

There are well-known statements that: for the locally compact group G to be amenable it is necessary and sufficient that the left-regular representation was faithful representation in $C^{*}(G)$; assuming G is a locally compact amenable group then the reduced crossed product coincides with the universal crossed product.

Theorem 2. ***The*** C^{*}***-algebra*** $Sch_{\Lambda}^{\tau}\left(\mathsf{A}\times_{\omega}^{\tau}G\right)$ ***is isomorphic the reduced crossed product*** $\left\lfloor\mathsf{A}\times_{\omega}^{\tau}G\right\rfloor_{red}$. ***Assume that the group*** G ***is amenable then the representation*** $\mathrm{B}^{\tau}(\mathsf{A})\mapsto Sch_{\Lambda}^{\tau}\left(\mathsf{A}\times_{\omega}^{\tau}G\right)\subset LB\left(L^{2}(G)\right)$ ***define by***

$$Sch_{\Lambda}^{\tau}(\psi)(x)=$$
$$\int_{G}\psi\left(\tau\left(xz^{-1}\right)^{-1}x,xz^{-1}\right)\phi(z)d\mu(z)$$

is faithful.

Proof. The Fourier transform is an isomorphic mapping. The reduced crossed product $\lfloor \mathsf{A} \times_{\omega}^{\tau} G \rfloor_{red}$ can be defined as the range of the left regular $*$-representation in $LB\left(L^2\left(G\times G\right)\right)$. The faithfulness of representation $\mathsf{B}^{\tau}\left(\mathsf{A}\right) \mapsto Sch_{\Lambda}^{\tau}\left(\mathsf{A} \times_{\omega}^{\tau} G\right) \subset LB\left(L^2\left(G\right)\right)$ follows from the equality of universal and reduced crossed products for the amenable locally compact groups.

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