Aspects of Solvability Theory for Quasilinear Parabolic Systems in Specific Form with the Singular Coefficients

MYKOLA YAREMENKO

National Technical University of Ukraine, Pryrichna 17G, 117, 04123, Kiev, UKRAINE

Abstract: In this paper, we study a quasilinear parabolic system in the form $\partial_t \vec{u} = \nabla_i (\alpha_{ij}(x, t, \vec{u}) \nabla_j \vec{u}) + \vec{b}(x, t, \vec{u}, \nabla\vec{u})$, where $\vec{u}(x, t)$ is an unknown N-dimensional vector over a domain $D_T = \Omega \times [0, T]$, we assume the weak general conditions on the structural coefficients, demanding that the singular term satisfy the formboundary conditions.

Keywords: Quasilinear Partial Differential Equation, Holder solution, regularity theory, form-bounded, Parabolic equation, Weak Solution, a Priori Estimation.

Received: September 8, 2022. Revised: May 12, 2023. Accepted: Jane 7, 2023. Published: July 7, 2023.

1. Introduction

The subject of this paper is the solvability of the quasilinear parabolic system

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i,j=1,\dots,l} \nabla_i \left(a_{ij}(x,\ t,\ \vec{u})\,\nabla_j \vec{u}\right) + \vec{b}(x,\ t,\ \vec{u},\ \nabla\vec{u})$$

defined on $R^l \times [0,\ T]$, $l > 2$ with elliptic matrix $a(x,\ t,\ \vec{u})$ and N-dimension singular vector $\vec{b}$.

The existence of solutions to boundary problems for quasilinear parabolic equations has been intensively studied during the last decades, see list of references [1 - 37]. These studies produce several universal methods such as methods of the fixed point introduced in the works of Leray and Schauder [6], the perturbation method, and the method of a priory estimation and their combinations. In the linear case, fundamental results were obtained in the works of Nash, Degiorgi [8, 9], Moser, and Aronson with further development produced by Zhang, Qian, Xi [34 - 37], and many others. The quasilinear case is less explored and presents great interest due to the plethora of applications in signal processing and quantum physics. Some foundational questions were explored in [6] by Ladyzenskaja and Solonnikov.

The main progress in linear parabolic theory is the extension to general linear parabolic equation containing lower order term Nash-Degiorgi results, so the conditions on the coefficient, which guarantee certain regularity of solutions were formulated in terms of form-boundary functions and Kato functional classes. In colloquial terms, a function $f : R^l \to R$, $f \in L^2{}_{loc}$ is said to be form-bounded if there are positive constants β, $c(\beta)$ such that inequality

$$\|f\varphi\|_2^2 \le \beta\,\|\nabla\varphi\|_2^2 + c(\beta)\,\|\varphi\|_2^2$$

holds for all $\varphi \in C_0^\infty$; the Kato class K_ν^l, $\nu > 0$ consists of all functions f such that $\left\|(\lambda - \Delta)^{-1}|f|\right\|_\infty \le \nu$. We consider a simple linear equation

$$\left(\frac{\partial}{\partial t} - \nabla a \nabla + f\cdot\nabla\right) u(x,\ t) = 0, \quad (x,\ t) \in R^l \times [0,\ \infty)$$

then the heat kernel of this simple linear equation satisfies two Gaussian bounds if the coefficient f is form-bounded and diverges of f belongs to some Kato class.

In the present article, we consider a more complex case of the quasilinear parabolic system with singular coefficients given in the specific form (1), we establish sufficient conditions for the initial-boundary value problem $\vec{u}|_\Gamma = \phi|_\Gamma$, $\phi \in C^{2,1}(\{(x,\ t) : x \in \partial\Omega, \quad t \in [0,\ T]\})$ for quasilinear parabolic system (1) has a unique solution $\vec{u} \in H^{\alpha,\ \frac{\alpha}{2}}(clos(D_T))$. The Heinz example

$$\begin{aligned}\partial_t u^1 - \partial_{xx} u^1 &= u^1\left(\left(\partial_x u^1\right)^2 + \left(\partial_x u^2\right)^2\right)\\ \partial_t u^2 - \partial_{xx} u^2 &= u^2\left(\left(\partial_x u^1\right)^2 + \left(\partial_x u^2\right)^2\right),\end{aligned}$$

with the solution $u^1 = \cos(mx)$ and $u^2 = \sin(mx)$ that does not satisfy the condition $\max\limits_{[0,\ 2\pi]} |\nabla\vec{u}|$, this shows the necessity of some additional growth conditions for a nonlinear system with an unknown vector in contrast to the case of a single equation, in this work such conditions are

$$\left|\frac{\partial a_{ij}}{\partial u^k}\nabla_i u^k + \frac{\partial a_{ij}}{\partial x_i}\right| \le \mu(|\vec{u}|)\,(1 + |\nabla\vec{u}|)\,,$$

$$\left|\vec{b}\right| \le \left(\varepsilon(|\vec{u}|) + \theta(|\nabla\vec{u}|\,,\ |\vec{u}|)\right)(1 + |\nabla\vec{u}|)^2$$

with $\lim\limits_{|\nabla\vec{u}|\to\infty} \theta(|\nabla\vec{u}|\,,\ |\vec{u}|) = 0$ and $\varepsilon(M_1)$ is small enough.

2. Problem formalization

Consider a quasilinear parabolic system in the specific divergent form

$$\frac{\partial}{\partial t}\vec{u}=\sum_{i,j=1,...,l}\nabla_i\left(a_{ij}\left(x,\ t,\ \vec{u}\right)\nabla_j\vec{u}\right)+\vec{b}\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right),\tag{1}$$

in the domain $(x,\ t)\in D_T=\Omega\times[0,\ T]$, $\vec{u}\left(x,\ t\right)=\left(u^1\left(x,\ t\right),\ ...,\ u^N\left(x,\ t\right)\right)$ is an unknown N-dimensional vector in $clos\left(D_T\right)$, $l\geq 3$; $\vec{b}\ :\ \Omega\times[0,\ T]\times R^N\times R^l\times R^N\to R^N$ is a known vector-function. Functions a_{ij} comprise a symmetric $l\times l$-matrix uniformly elliptic, namely,

$$\nu\left(\vec{u}\right)\xi^2\leq a_{ij}\xi_i\xi_j\leq\mu\left(\vec{u}\right)\xi^2\tag{2}$$

for all $(x,\ t)\in R^l\times[0,\ T]$ and all $\xi\in R^l$.

We formulate the restrictions on the measurable structural coefficients of the system (1) as

$$a_{ij}\left(x,\ t,\ \vec{u}\right)\vec{k}_i\vec{k}_j\geq\nu\left(|\vec{u}|\right)\left|\vec{k}\right|^2-\gamma_0\left(x,\ t\right)\tag{3}$$

$$\left|a_{ij}\left(x,\ t,\ \vec{u}\right)\vec{k}_j\right|\leq\mu\left(|\vec{u}|\right)\left|\vec{k}\right|+\gamma_1\left(x,\ t\right)\tag{4}$$

$$\left|\vec{b}\left(x,\ t,\ \vec{u},\ \vec{k}\right)\right|\leq\tilde{\mu}\left(|\vec{u}|\right)\left|\vec{k}\right|^2+\gamma_2\left(x,\ t\right)\tag{5}$$

where $\nu\left(\tau\right)$, $\mu\left(\tau\right)$ and $\tilde{\mu}\left(\tau\right)$ are given positive continuous functions.

Definition 1. ***A function $\vec{f}\ :\ D_T\to R^N$ is said to be form-boundary or belongs to the class $PK\left(\beta\right)$ if there exist some positive constants β and $c\left(\beta\right)$ such that the inequality***

$$\begin{aligned}&\int_{[0,\ T]}\int_\Omega\left|\vec{f}\vec{\varphi}\right|^2dxdt\leq\\&\leq\beta\int_{[0,\ T]}\int_\Omega\left|\nabla\vec{\varphi}\right|^2dxdt+c\left(\beta\right)\int_{[0,\ T]}\int_\Omega\left|\vec{\varphi}\right|^2dxdt\end{aligned}\tag{6}$$

holds all functions $\vec{\varphi}\ :\ R^l\times[0,\ T]\to R^N$, $\vec{\varphi}\in C_0^\infty$.

Definition 2. ***We call real-valued vector-function $\vec{u}\left(x,\ t\right)$ a weak bounded solution to system (1) if $\vec{u}\in V_{1,0}^2\left(D_T\right)$, $\underset{(x,\ t)\in D_T}{ess\max}\left|\vec{u}\left(x,\ t\right)\right|<\infty$ and***

$$\begin{aligned}&\int_\Omega\vec{u}\left(x,\ t\right)\vec{\varphi}\left(x,\ t\right)dx\Big|_0^T=\\&=\int_{[0,\ T]}\int_\Omega\vec{u}\partial_t\vec{\varphi}dxdt-\\&-\int_{[0,\ T]}\int_\Omega a_{ij}\left(x,\ t,\ \vec{u}\right)\nabla_j\vec{u}\nabla_i\vec{\varphi}dxdt+\\&+\int_{[0,\ T]}\int_\Omega\vec{b}\vec{\varphi}dxdt\end{aligned}\tag{7}$$

for all $\vec{\varphi}\in C_0^\infty$.

We assume the function $\vec{u}$ is a weak bounded solution to system (1) so that from (7), we obtain an integral inequality

$$\begin{aligned}&\int_\Omega\vec{u}\left(x,\ t\right)\vec{\varphi}\left(x,\ t\right)dx\Big|_0^T+\\&+\int_{[0,\ T]}\int_\Omega a_{ij}\left(x,\ t,\ \vec{u}\right)\nabla_j\vec{u}\nabla_i\vec{\varphi}dxdt\leq\\&\leq\int_{[0,\ T]}\int_\Omega\vec{u}\partial_t\vec{\varphi}dxdt\\&+\int_{[0,\ T]}\int_\Omega\left(\tilde{\mu}\left|\nabla\vec{u}\right|^2+\gamma_2\left(x,\ t\right)\right)\left|\vec{\varphi}\right|dxdt,\end{aligned}\tag{8}$$

where $\vec{\varphi}\in C_0^\infty\left(D_T\right)\cap W_{1,2}^2\left(D_T\right)$.

We construct N_1 functions $\varpi^m\left(x,\ t\right)=\phi^m\left(u^1\left(x,\ t\right),\ ...,\ u^N\left(x,\ t\right)\right)$, $m=1,...,\ N_1$, where $\phi^1\left(u^1,\ ...,\ u^N\right),\ ....,\ \phi^{N_1}\left(u^1,\ ...,\ u^N\right)$ are continuously differentiable over their domains, and such that functions $\varpi^m\left(x,\ t\right)=\phi^m\left(u^1\left(x,\ t\right),\ ...,\ u^N\left(x,\ t\right)\right)$, $m=1,...,\ N_1$ satisfy the following conditions:

$$1)\underset{(x,\ t)\in D_T}{ess\max}\left|\varpi^m\left(x,\ t\right)\right|<M_1,\varpi^m\in V_{1,0}^2\left(D_T\right);$$

2) for an arbitrary cylinder $D_{2\rho}=B\left(2\rho\right)\times\left[\tilde{t},\ \tilde{t}+\tau\right]\subset D_T$ and a point $t_1\in\left(\tilde{t},\ \tilde{t}+\tau\right)$ there is a number $\tilde{m}$ such that

$$\begin{aligned}&osc\left\{\varpi^m\left(x,\ t\right),\quad D_{2\rho}\right\}\geq\\&\geq\delta_1\max_{k=1,...,\ N}osc\left\{u^k\left(x,\ t\right),\quad D_{2\rho}\right\}\end{aligned}\tag{9}$$

and

$$\begin{aligned}&\mu\left\{\begin{array}{l}x\in B\left(\rho\right)\ :\ \varpi^m\left(x,\ t_1\right)\leq\\ \leq\underset{D_{2\rho}}{ess\max}\left\{\varpi^m\left(x,\ t\right)\right\}-\\ -\delta_2 osc\left\{\varpi^m\left(x,\ t\right),\quad D_{2\rho}\right\}\end{array}\right\}\geq\\&\geq\left(1-\delta_3\right)c\left(l\right)\rho^l,\end{aligned}\tag{10}$$

where $B\left(\rho\right)$ is a ball, of radius ρ, concentric with $B\left(2\rho\right)$; $\delta_1,\ \delta_2,\ \delta_3$ are positive constants;

3) we denote the Lebesgue measure by μ; for each function ϖ^m, $m=1,...,\ N_1$, we have

$$\begin{aligned}&\max_{t\in\left[\tilde{t},\ \tilde{t}+\tau\right]}\left\|\varpi^m{}_n\left(\cdot,\ t\right)\right\|_{2,B(\rho-\vartheta_1\rho)}^2\leq\\&\leq\max_{t\in\left[\tilde{t},\ \tilde{t}+\tau\right]}\left\|\varpi^m{}_n\left(\cdot,\ \tilde{t}\right)\right\|_{2,B(\rho)}^2+\\&+\breve{c}\left(\frac{1}{\left(\vartheta_1\rho\right)^2}\left\|\varpi^m{}_n\left(\cdot,\ \tilde{t}\right)\right\|_{2,B(\rho)\times\left[\tilde{t},\ \tilde{t}+\tau\right]}^2+\hat{c}\left(\mu\left(\Lambda_{n,\rho}\right)\right)\right),\end{aligned}\tag{11}$$

and

$$\begin{aligned}&\left\|\varpi^m{}_n\left(\cdot,\ t\right)\right\|_{H,\ 2,\ B(\rho-\vartheta_1\rho)\times\left[\tilde{t},\ \tilde{t}+\vartheta_2\tau\right]}^2\leq\\&\leq\breve{c}\left(\frac{1}{\left(\vartheta_1\rho\right)^2}+\frac{1}{\left(\vartheta_2\tau\right)^2}\right)\left\|\omega\varpi^m{}_n\left(\cdot,\ \tilde{t}\right)\right\|_{2,B(\rho)\times\left[\tilde{t},\ \tilde{t}+\tau\right]}^2+\\&+\hat{c}\left(\mu\left(\Lambda_{n,\rho}\right)\right),\end{aligned}\tag{12}$$

$\left\|\cdot\right\|_H$ is the Holder norm, we denote $\varpi^m{}_n\left(x,\ t\right)=\max\left\{\varpi^m\left(x,\ t\right)-n,\quad 0\right\}$ and $\Lambda_{n,\rho}$ is a set of all $x\in B\left(\rho\right)$ such that $\min_{m=1,...,N_1}\varpi^m\left(x\right)>n$ for natural number n.

Since $\gamma_0,\ \gamma_1^2,\ \gamma_2\in PK\left(\beta\right)$ we obtain the estimation

$$\begin{aligned}&\left\|\varpi_n\left(x,\ \tilde{t}+\tau\right)\xi\left(x,\ \tilde{t}+\tau\right)\right\|_{2,B(\rho)}^2+\\&+\nu\left\|\xi\nabla\varpi_n\right\|_{2,\ D_\rho}^2\leq\\&\leq\left\|\varpi_n\left(x,\ \tilde{t}\right)\xi\left(x,\ \tilde{t}\right)\right\|_{2,B(\rho)}^2+\\&+\tilde{c}\left(\int_{D_\rho}\left|\varpi_n\right|^2\left(\left|\nabla\xi\right|^2+\xi\left|\partial_t\xi\right|\right)dxdt\right.\\&\left.+\hat{c}\left(\mu\left(\Lambda_{n,\rho}\right)\right)\right),\end{aligned}\tag{13}$$

where we used

$$\int_{[0,\ T]}\int_\Omega\left|\sqrt{\gamma_2}\xi\right|^2dxdt\leq$$

$$\leq\beta\int_{[0,\ T]}\int_\Omega\left|\nabla\xi\right|^2dxdt+c\left(\beta\right)\int_{[0,\ T]}\int_\Omega\left|\xi\right|^2dxdt,$$

where $\xi\in C_0^\infty\left(D_T\right)$.

3. A priori estimations

We denote

$$\vec{u}_{\overline{h}}(x,\ t)=\frac{1}{h}\int_{[t-h,\ t]}\vec{u}(x,\ \tau)\,d\tau \qquad (14)$$

and

$$\vec{u}_{h}(x,\ t)=\frac{1}{h}\int_{[t,\ t+h]}\vec{u}(x,\ \tau)\,d\tau. \qquad (15)$$

By taking $\vec{\varphi}=\left(\vec{u}\exp\left(\lambda\left|\vec{u}_h\right|^2\right)\xi^2(x)\right)_{\overline{h}}$ in (8), then taking the limit $h\to 0$, we obtain

$$\begin{aligned}&\tfrac{1}{2\lambda}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2(x)\,dx\Big|_{t_1}^{t_2}+\\&+\tfrac{\lambda\nu}{2}\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2\left(\nabla\left(|\vec{u}|^2\right)\right)^2dxdt+\\&+\nu\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2|\nabla\vec{u}|^2dxdt\le\\&\le\tfrac{\lambda}{2}\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2\gamma_0dxdt+\\&+\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2\gamma_0dxdt+\\&+2\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\\&(\mu|\nabla\vec{u}|+\gamma_1)|\vec{u}|\xi|\nabla\xi|\,dxdt+\\&+\int_{[t_1,\ t_2]}\int_{B(2\rho)}\left(\tilde{\mu}|\nabla\vec{u}|^2+\gamma_2\right)|\vec{u}|\exp\left(\lambda|\vec{u}|^2\right)\xi^2dxdt\end{aligned} \qquad (16)$$

and further we have

$$\begin{aligned}&\tfrac{1}{2\lambda}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2(x)\,dx\Big|_{t_1}^{t_2}+\\&+\tfrac{\lambda\nu}{2}\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2\left(\nabla\left(|\vec{u}|^2\right)\right)^2dxdt+\\&+\nu\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda|\vec{u}|^2\right)\xi^2|\nabla\vec{u}|^2dxdt\le\\&\le\int_{[t_1,\ t_2]}\int_{B(2\rho)}\exp\left(\lambda{M_1}^2\right)\\&\left(\left(\tfrac{\lambda}{2}+1\right)\gamma_0+\gamma_1^2+M_1\gamma_2\right)\xi^2dxdt+\\&+2\mu M_1\exp\left(\lambda{M_1}^2\right)\int_{[t_1,\ t_2]}\int_{B(2\rho)}|\nabla\vec{u}|\xi|\nabla\xi|\,dxdt+\\&+\tilde{\mu}M_1\exp\left(\lambda{M_1}^2\right)\int_{[t_1,\ t_2]}\int_{B(2\rho)}|\nabla\vec{u}|^2\xi^2dxdt\le\\&\le\mu M_1\exp\left(\lambda{M_1}^2\right)\int_{[t_1,\ t_2]}\int_{B(2\rho)}|\nabla\xi|^2dxdt+\\&+(\mu+\tilde{\mu})M_1\exp\left(\lambda{M_1}^2\right)\int_{[t_1,\ t_2]}\int_{B(2\rho)}|\nabla\vec{u}|^2\xi^2dxdt+\\&+c_1\exp\left(\lambda{M_1}^2\right)\left(\beta\int_{[0,\ T]}\int_\Omega|\nabla\xi|^2dxdt+\right.\\&\left.+c(\beta)\int_{[0,\ T]}\int_\Omega|\xi|^2dxdt\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\int_{[t_1,\ t_2]}\int_{B(2\rho)}\xi^2|\nabla\vec{u}|^2dxdt\le c_1\rho^l+\\&+c_2(t_2-t_1)\rho^l\left(\max|\nabla\xi|^2+\max\xi^2\right).\end{aligned}$$

If $\eta(x,\ t)\le 1$ and equals zero on the boundary then

$$\begin{aligned}&\max_{t\in\left[\tilde{t}-\tau,\ \tilde{t}\right]}\int_{B(\rho)}\left|\varpi^m{}_n\eta\right|^2dx+\\&+\int_{\left[\tilde{t}-\tau,\ \tilde{t}\right]}\int_{B(2\rho)}\left|\nabla\varpi^m{}_n\right|^2\eta^2dxdt\le\\&\le c_1\int_{\left[\tilde{t}-\tau,\ \tilde{t}\right]}\int_{B(2\rho)}\left|\varpi^m{}_n\right|^2\left(|\nabla\eta|^2+\eta\left|\partial_t\eta\right|\right)dxdt+\\&+c_1\left(\int_{[0,\ T]}\int_\Omega|\nabla\xi|^2dxdt+\int_{[0,\ T]}\int_\Omega|\xi|^2dxdt\right).\end{aligned} \qquad (17)$$

The Holder estimation of $\vec{u}$ follows from

$$\begin{aligned}&osc\{\varpi^m,\quad D_\rho\}\le\\&\le(1-\vartheta)\,osc\{\varpi^m,\quad D_{2\rho}\}+\vartheta_2\rho^{\vartheta_1},\end{aligned} \qquad (18)$$

where constants $\vartheta,\ \vartheta_1,\ \vartheta_2$ are depending on the structural coefficients.

Proposition 1. ***Let function*** $\vec{u}\in C^{2,1}(D_T)$ ***be a solution to the system*** (1) ***and let function*** $\vec{u}$ ***equals zero on the boundary. Let structural coefficients of the system*** (1) ***satisfy conditions*** (2)-(5) ***and***

$$\left|\frac{\partial a_{ij}}{\partial u^k}\nabla_iu^k\nabla_j\vec{u}+\frac{\partial a_{ij}}{\partial x_i}\nabla_j\vec{u}+\vec{b}\right|\le\mu(|\vec{u}|)\left(1+|\nabla\vec{u}|\right)^2, \qquad (19)$$

$$\left|\frac{\partial a_{ij}}{\partial u^k}\nabla_iu^k+\frac{\partial a_{ij}}{\partial x_i}\right|\le\mu(|\vec{u}|)\left(1+|\nabla\vec{u}|\right), \qquad (20)$$

$$\left|\vec{b}\right|\le\left(\varepsilon(|\vec{u}|)+\theta\left(|\nabla\vec{u}|,\ |\vec{u}|\right)\right)\left(1+|\nabla\vec{u}|\right)^2, \qquad (21)$$

where $\lim\limits_{|\nabla\vec{u}|\to\infty}\theta\left(|\nabla\vec{u}|,\ |\vec{u}|\right)=0$ ***and*** $\varepsilon(M_1)$ ***is a small number. Then, the value*** $\max\limits_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}|\nabla\vec{u}(x,\ t)|$ ***estimates by*** $\max\limits_{D_T}|\vec{u}(x,\ t)|=M_1$ ***and functions of structural coefficients,*** $\max\limits_{\Omega}|\nabla\vec{u}(x,\ 0)|$ ***and boundary.***

Proof. We denote $v^k(x,\ t)=u^k(x,\ t)+|\vec{u}(x,\ t)|^2$, then we equality

$$\begin{aligned}&\frac{\partial}{\partial t}v^k=a_{ij}\nabla_i\nabla_jv^k-\\&-\sum_{i,j=1,..l}\ \sum_{m=1,...N}2a_{ij}\nabla_iu^m\nabla_ju^m-B_j\nabla_jv^k-C^k,\end{aligned}$$

where we denote $B_j=\frac{\partial a_{ij}}{\partial u^k}\nabla_iu^k+\frac{\partial a_{ij}}{\partial x_i}$ and $C^k=\sum_m 2b^mu^m+b^k$.

We change the function $\vec{v}$ on the function $v^k=\psi\left(w^k\right)$ where the function ψ given by

$$\psi(z)=const\ \nu(M_1)\ln(z+1).$$

So, applying the standard arguments we obtain

$$\begin{aligned}&\tfrac{\partial}{\partial t}w^k-a_{ij}\nabla_i\nabla_jw^k-\\&-\tfrac{\psi''}{\psi'}\sum_{i,j=1,..l}\sum_{m=1,...N}a_{ij}\nabla_iw^m\nabla_jw^m\le\\&\le c\left(\tfrac{1}{\psi'}+\psi'\left|\nabla w^k\right|^2\right)\end{aligned}$$

where the constant c is strictly positive. Therefore, there are some positive constants $\widehat{c}$ such that

$$\frac{\partial}{\partial t}w^k-a_{ij}\nabla_i\nabla_jw^k\le\widehat{c},$$

both functions $\vec{w}$ and $\vec{v}$ equal zero on the boundary $\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}$ and

$$\begin{aligned}&\tfrac{\partial w^k}{\partial\vec{n}}\Big|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}=\\&=\tfrac{1}{\psi'(w^k)}\tfrac{\partial v^k}{\partial\vec{n}}\Big|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}=\\&=\tfrac{c}{\nu}\tfrac{\partial u^k}{\partial\vec{n}}\Big|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}\end{aligned}$$

reaches its maximum on $\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}$ in the same point $\left(\hat{x},\ \hat{t}\right)$ that $\frac{\partial u^k}{\partial\vec{n}}$.

Next, we take the function $\tilde{\psi}$ such that

$$-a_{ij}\nabla_i\nabla_j\tilde{\psi} < -\widehat{c}$$

for all $x \in \Omega$ and

$$\max_{x\in\Omega}\left\{w^k(x,\ 0) + \tilde{\psi}(x)\right\} = w^k(\hat{x},\ 0) + \tilde{\psi}(\hat{x}),$$

$$\max_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}\left\{\tilde{\psi}(x)\right\} = \tilde{\psi}(\hat{x}).$$

Then, we have

$$\frac{\partial}{\partial t}\left(w^k + \tilde{\psi}\right) < a_{ij}\nabla_i\nabla_j\left(w^k + \tilde{\psi}\right)$$

so that

$$\max_{x\in\Omega}\left\{w^k(x,\ 0) + \tilde{\psi}(x)\right\} = w^k(\hat{x},\ 0) + \tilde{\psi}(\hat{x}) = \tilde{\psi}(\hat{x}),$$

thus

$$\left.\frac{\partial w^k(x,\ t) + \tilde{\psi}(x)}{\partial\vec{n}}\right|_{x=\hat{x}} \geq 0$$

therefore, there is an estimation

$$\left.\frac{\partial\tilde{\psi}(x)}{\partial\vec{n}}\right|_{x=\hat{x}} \geq -\left.\frac{\partial w^k(x,\ t)}{\partial\vec{n}}\right|_{(x,\ t)=(\hat{x},\ \hat{t})},$$

finally, we obtain the value $\max_{k=1,\ldots,N}\max_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}}\left|\frac{\partial u^k}{\partial\vec{n}}\right|$ is bounded.

Theorem 1. ***Let functions*** a_{ij} ***and*** $\vec{b}$ ***satisfy conditions*** (2)-(5) ***and***

$$\left|\frac{\partial a_{ij}}{\partial u^k}\nabla_i u^k\nabla_j\vec{u} + \frac{\partial a_{ij}}{\partial x_i}\nabla_j\vec{u} + \vec{b}\right| \leq \mu(|\vec{u}|)(1 + |\nabla\vec{u}|)^2, \tag{22}$$

$$\left|\frac{\partial a_{ij}}{\partial u^k}\nabla_i u^k + \frac{\partial a_{ij}}{\partial x_i}\right| \leq \mu(|\vec{u}|)(1 + |\nabla\vec{u}|), \tag{23}$$

$$\left|\vec{b}\right| \leq (\varepsilon(|\vec{u}|) + \theta(|\nabla\vec{u}|,\ |\vec{u}|))(1 + |\nabla\vec{u}|)^2, \tag{24}$$

where $\lim_{|\nabla\vec{u}|\to\infty}\theta(|\nabla\vec{u}|,\ |\vec{u}|) = 0$ ***and*** $\varepsilon(M_1)$ ***is a small number. Let the boundary be smooth enough. Then, the value*** $\max_{D_T}|\nabla\vec{u}(x,\ t)| = M_1$ ***can be estimated by functions of structural coefficients and*** $\varepsilon(M_1)$***,*** θ***.***

Proof. In inequality (8), we take $\vec{\varphi} = \left(\vec{u}\exp\left(\lambda|\vec{u}_h|^2\right)\xi^2(x)\right)_{\overline{h}}$ and proceed as (16) we obtain an estimation $\int_{[,\ T]}\int_\Omega\xi^2|\nabla\vec{u}|^2dxdt \leq const$; next, we take $\vec{\varphi} = \nabla_m\left(\xi\ \nabla_m\vec{u}^2\right)$, we have

$$\begin{aligned}
&\tfrac{1}{2}\int_{[0,\ T]}\int_\Omega\xi\partial_t\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)dxdt = \\
&= -\int_{[0,\ T]}\int_\Omega\xi a_{ij}\nabla_m\nabla_i u^k\nabla_m\nabla_j u^k dxdt- \\
&-\tfrac{1}{2}\int_{[0,\ T]}\int_\Omega a_{ij} \\
&\nabla_j\xi\nabla_i\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)dxdt- \\
&-\int_{[0,\ T]}\int_\Omega\tfrac{da_{ij}}{dx_m}\xi\nabla_i u^k\nabla_m\nabla_j u^k dxdt- \\
&-\int_{[0,\ T]}\int_\Omega\tfrac{da_{ij}}{dx_m}\nabla_j\xi\nabla_i u^k\nabla_m u^k dxdt+ \\
&+\int_{[0,\ T]}\int_\Omega b^k\left(\Delta u^k\xi - \nabla_m u^k\nabla_m\xi\right)dxdt,
\end{aligned} \tag{25}$$

where we denote $\nabla_m u^k\nabla_m\xi = \sum_{m=1,\ldots,l}(\nabla_m\xi)\left(\nabla_m u^k\right)$. We take

$$\xi = 2\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^s\eta^2(x)$$

where η is the cutoff for the ball $B(\rho) \subset \Omega,\quad s \geq 0$. Applying conditions, for small ρ, we have

$$\begin{aligned}
&\tfrac{1}{2+s}\int_{B(\rho)}\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^{s+1} \\
&\eta^2 dx\Big|_0^T + \\
&+\nu\hat{c}_1\int_{[0,\ T]}\int_{B(\rho)}|\nabla\nabla\vec{u}|^2 \\
&\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^s\eta^2 dxdt+ \\
&+\nu\hat{c}_1\int_{[0,\ T]}\int_{B(\rho)}\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^{s+2} \\
&\eta^2 dxdt \leq \\
&\leq \check{c}\int_{[0,\ T]}\int_{B(\rho)} \\
&\left(1 + \left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^{s+1}\right)|\nabla\eta|^2 dxdt,
\end{aligned}$$

therefore, we obtain

$$\max_{t\in[0,\ T]}\int_{B(\rho)}\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^{s+1}dx \leq \\ \leq c.$$

Finally, if $\vec{\varphi}(x,\ t) = \nabla_m\xi(x,\ t),\quad \xi|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}} = 0$ then we obtain

$$\begin{aligned}
&\int_{[0,\ T]}\int_\Omega\xi\nabla_m\partial_t u^k dxdt = \\
&= -\int_{[0,\ T]}\int_\Omega a_{ij}\nabla_i\xi\nabla_m\nabla_j u^k dxdt+ \\
&+\int_{[0,\ T]}\int_\Omega\Theta_{mi}^k\nabla_i\xi dxdt,
\end{aligned}$$

where $\Theta_{mi}^k = \frac{\partial a_{ij}}{\partial x_m}\nabla_j u^k + \frac{\partial a_{ij}}{\partial u^d}\nabla_m u^d\nabla_j u^k + b^k\delta_{im}$. We denote $w = \nabla_m u^k,\quad m = 1,\ldots,l;\quad k = 1,\ldots,N$ a solution to the system

$$\partial_t w = \nabla_j\left(a_{ij}\nabla_i w + \Theta_{mi}^k\right),$$

applying linear theory, we are proving the theorem.

Remark. If the cylinder intersects the boundary then we assume that $\max_{\partial D_T}|\nabla\vec{u}(x,\ t)|$ has already been estimated. For a domain that intersects we take in (25)

$$\xi = \begin{cases} 2\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^s\eta^2(x) \\ for\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right) > M_1^2; \\ 0,\quad\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right) \leq M_1^2, \end{cases}$$

so that we are going to obtain the following estimation

$$\max_{t\in[0,\ T]}\int_{B(\rho)\cap\Omega}\left(\sum_{k=1,\ldots,N}\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)^2\right)^{s+1}dx \leq \\ \leq c.$$

Existence of the solution

We consider a boundary problem for the system (1), namely, the function $\vec{u}$ that satisfies

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i,j=1,\ldots,l} \nabla_i \left(a_{ij}\left(x,\ t,\ \vec{u}\right) \nabla_j \vec{u}\right) + \vec{b}\left(x,\ t,\ \vec{u},\ \nabla\vec{u}\right)$$

over the closure of D_T and on the boundary coincides with the given function $\phi \in H^{\tilde{\alpha},\ \frac{\tilde{\alpha}}{2}}\left(clos\left(D_T\right)\right)$.

Theorem 2. ***Let functions*** a_{ij} ***and*** $\vec{b}$ ***satisfy conditions*** (2)-(5) ***with*** $\gamma_0,\ \gamma_1^2,\ \gamma_2 \in PK\left(\beta\right)$ ***and*** (22)-(24) ***with*** $\lim\limits_{|\nabla\vec{u}|\to\infty} \theta\left(|\nabla\vec{u}|,\ |\vec{u}|\right) = 0$ ***and*** $\varepsilon\left(M_1\right)$ ***is a small number. Let*** $\phi \in C^{2,1}\left(\left\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\right\}\right)$***,*** $\max\limits_{x\in\Omega}\left|\nabla\phi\left(x,\ 0\right)\right| < \infty$***,*** $\phi \in H^{\tilde{\alpha},\ \frac{\tilde{\alpha}}{2}}\left(clos\left(D_T\right)\right)$***; and let***

$$\frac{\partial}{\partial t}\phi = \sum_{i,j=1,\ldots,l} \nabla_i \left(a_{ij}\left(x,\ t,\ \phi\right) \nabla_j \phi\right) + \vec{b}\left(x,\ t,\ \phi,\ \nabla\phi\right).$$

Let function a_{ij} ***satisfy the Lipschitz condition at*** $\vec{u}$ ***on any compact.***

Then, there exists a unique solution $\vec{u} \in H^{\alpha,\ \frac{\alpha}{2}}\left(clos\left(D_T\right)\right)$ ***to the problem***

$$\vec{u}|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}\cap\{(x,\ t)\ :\ x\in\Omega,\quad t=0\}} =$$

$$= \phi|_{\{(x,\ t)\ :\ x\in\partial\Omega,\quad t\in[0,\ T]\}\cap\{(x,\ t)\ :\ x\in\Omega,\quad t=0\}}$$

for the system (1)***.***

This theorem can be proven by the Leray–Schauder method with the application estimations obtained in previous chapters. A linearized system is given as

$$\begin{gathered}\tfrac{\partial}{\partial t}\vec{w} = \left(\tau a_{ij}\left(x,\ t,\ \vec{v}\right) + (1-\tau)\,\delta_{ij}\right)\nabla_i\nabla_j\vec{w} + \\ -\tau B\left(x,\ t,\ \vec{v},\ \nabla\vec{v}\right) + (1-\tau)\left(\tfrac{\partial}{\partial t}\phi - \Delta\phi\right),\quad \tau\in[0,\ 1],\end{gathered}$$

where we denote

$$B\left(x,\ t,\ \vec{v},\ \nabla\vec{v}\right) =$$

$$= -\vec{b}\left(x,\ t,\ \vec{v},\ \nabla\vec{v}\right) - \frac{\partial a_{ij}\left(x,\ t,\ \vec{v}\right)}{\partial v^k}\nabla_j\vec{v}\nabla_i v^k -$$

$$-\frac{\partial a_{ij}\left(x,\ t,\ \vec{v}\right)}{\partial x_i}\nabla_j\vec{v},$$

and we consider function $\vec{w}$ to be unknown and $\vec{v}$ to be given.

The linearized system defines the nonlinear operator $\Phi\left(\tau\right)\ :\ \vec{v}\mapsto\vec{w}$ given by $\vec{w} = \Phi\left(\vec{v},\ \tau\right)$, where the function $\vec{w}$ is a solution to the linearized system for each given parameter $\tau\in[0,\ 1]$. The fixed point of the operator Φ at the point $\tau = 1$ is a solution to the boundary problem for system (1). The existence of such a fixed point is guaranteed by the Leray–Schauder theorem, uniqueness follows from the Lipschitz condition straightforwardly by the contradiction method.

References

[1] M. Agueh, Gagliardo-Nirenberg inequalities involving the gradient L2 -norm, C. R. Acad. Sci. Paris, Ser., 346 (2008), 757–762.

[2] H. Amann, Invariant sets and existence theorems for semilinear parabolic and elliptic systems, J. Math. Anal. Appl., 65 (1978), pp. 432–467.

[3] J. Jackson and G, Zitkovic, Existence, and uniqueness for non-Markovian triangular quadratic BSDEs, (2021).

[4] C. Chen, R.M. Strain, H. Yau, T. Tsai, Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations II, Comm. Partial Differential Equations 34 (2009) no. 1-3, 203–232.

[5] M. Kassmann, M. Weidner, The parabolic Harnack inequality for nonlocal equations. arXiv:2303.05975 (2023).

[6] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, Amer. Math. Soc., Vol. 23, (1968).

[7] J. M. Lee, T. Hillena and M. A. Lewis, Pattern formation in prey-taxis systems, J. Biol. Dynamics, 3 (2009), 551–573.

[8] E. DeGiorgi, Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari, Mem. Accad Sc. Torino, C. Sc. Fis. Mat. Natur. 3 (1957), 25-43.

[9] E. DeGiorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellitico, Bull. UMI 4 (1968), 135-137.

[10] J. Dieudonne, History of Functional Analysis, North-Holland, Amsterdam, (1981).

[11] H. Dong, S. Kim, and S. Lee, Estimates for fundamental solutions of parabolic equations in non-divergence form. J. Differential Equations, 340:557–591, (2022).

[12] D., Hongjie, L. Escauriaza, S. Kim, On C1/2,1, C1,2, and C0,0 estimates for linear parabolic operators. J. Evol. Equ. 21 (2021), no. 4, 4641–4702.

[13] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkha user, Basel, (1984).

[14] C. Gordon, L. Webb, and S. Wolpert, One cannot hear the shape of a drum, Bull. Amer. Math. Soc. 27 (1992), 134-138.

[15] R. Landes and V. Mustonen, On the pseudomonotone operators and nonlinear noncoercive problems on unbounded domains, Math. Ann. 248 (1980), 241-246.

[16] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, Berlin, (1989).

[17] L. Nirenberg, Partial differential equations in the first half of the century, in "Development of Mathematics 1900-1950" (J. P. Pier, Ed.), Birkha user, Basel, (1994).

[18] S. Kim, S. Lee, Estimates for Green's functions of elliptic equations in non-divergence form with continuous coefficients. Ann. Appl. Math. 37 (2021), no.

2, 111–130.

[19] M. Li and W. Mao, Finite-time bounded control for coupled parabolic PDE-ODE systems subject to boundary disturbances, Mathematical Problems in Engineering, vol. 2020, Article ID 8882382, 13 p., (2020).

[20] S. Fan, A new general decay rate of the wave equation with memory-type boundary control, Mathematical Problems in Engineering, vol. 2021, Article ID 5571072, 11 p., (2021).

[21] M. Ghattassi and T. M. Laleg, Boundary stabilization of a reaction-diffusion system weakly coupled at the boundary, IFAC-PapersOnLine, vol. 53, no. 2, Article ID 16537, (2020).

[22] C. Bao, B. Cui, X. Lou, W. Wu, and B. Zhuang, Fixed-time stabilization of boundary controlled linear parabolic distributed parameter systems with space-dependent reactivity, IET Control Theory, and Applications, vol. 15, no. 11, pp. 652–667, (2020).

[23] W. L. J. Wang and W. Guo, A backstepping approach to adaptive error feedback regulator design for one-dimensional linear parabolic PIDEs, Journal of Mathematical Analysis and Applications, vol. 503, no. 2, Article ID 125310, (2021)

[24] Z. Qian, G. Xi, Parabolic equations with singular divergence-free drift vector fields, J. London Math. Soc. 100 (1) (2019), 17-40.

[25] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant, J. Diff. Eqns., 257 (2014), 784–815.

[26] T. L. Stepien, E. M. Rutter, and Y. Kuang, A data-motivated density-dependent diffusion model of in vitro glioblastoma growth, Mathematical Biosciences and Engineering, 12 (2015), 1157–1172.

[27] Y. Tao, Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis, Nonl. Aanl.: RWA, 11 (2010), 2056–2064.

[28] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant, J. Diff. Eqns., 257 (2014), 784–815.

[29] F. B. Weissler, Single-point blow-up for a semilinear initial value problem, J. Diff. Eq. 55 (1984), 204-224.

[30] S. Wang, Y. Teng, and P. Perdikaris, Understanding and mitigating gradient flow pathologies in physics-informed neural networks. SIAM J. Sci. Comput. 43, 5 (2021), A3055–A3081.

[31] M.I. Yaremenko, The existence of a solution of evolution and elliptic equations with singular coefficients, Asian Journal of Mathematics and Computer Research, (2017), Vol.: 15, Issue.: 3. pp. 172- 204.

[32] M.I. Yaremenko, Quasi-linear evolution, and elliptic equations, Journal of Progressive Research in Mathematics, Vol.11., N 3., (2017), pp. 1645-1669.

[33] M.I. Yaremenko, The sequence of semigroups of nonlinear operators and their applications to study the Cauchy problem for parabolic equations, Scientific Journal of the Ternopil national technical university N 4 (**??**). (2016), pp. 149-160.

[34] D. Zhang, D., Guo, L., and Karniadakis, G. E. Learning in modal space: Solving time-dependent stochastic PDEs using physics-informed neural networks. SIAM J. Sci. Comput. 42, 2 (2020), A639–A665.

[35] J. A. Goldstein and Qi. S. Zhang, Linear parabolic equations with strongly singular potentials, Trans. Amer. Math. Society, 355 (2002), no. 1, 197-211.

[36] Qi S. Zhang, A strong regularity result for parabolic equations, Comm. Math. Phys. 244 (2004), no. 2, 245–260. MR 2031029 (2005b:35116).

[37] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang, Stability of heat kernel estimates for symmetric non-local Dirichlet forms. Mem. Amer. Math. Soc., 271(1330):v+89, (2021).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.