The Problem of Non-Stationary Thermal Conductivity: Formulation in Derivatives and Solution by the Boundary Element Method

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Abstract: - The formulation of the transient heat conduction problem with respect to the heat flux vector and its solution using the boundary element method is described. The primary advantage is the direct determination of the heat flux without the need for numerical differentiation of the temperature field. A boundary integral equation is derived for the problem formulated in terms of the heat flux vector. The procedure for solving the problem using the boundary element method is detailed, and a system of governing equations is established. A solution to a test problem concerning the transient temperature distribution in a bounded domain is presented. Comparison of the computational results with an analytical solution illustrates the reliability and acceptable engineering accuracy of the obtained solutions.

Key-Words: - Boundary element method, non-stationary thermal conductivity, integral equation, formulation of problem in derivatives, thermal analysis, time-stepping schemes.

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1 Introduction

Non-stationary thermal conductivity, also known as transient heat conduction, is a fundamental phenomenon in heat transfer engineering and material science, [1]. Accurate modeling of transient heat conduction is crucial for the design of structures where temperature and heat flux changes over time significantly affect the stress-strain state, as well as when solving thermoelasticity problems.

Traditional numerical methods for solving transient heat conduction problems often rely on discretization techniques like the finite element method (FEM) [2], [3] or finite difference method (FDM) or meshless method [4]. While these methods are well-established, they typically require the computation of temperature gradients to determine heat fluxes, which involves numerical differentiation.

The boundary element method (BEM) offers an alternative approach by formulating the problem in terms of boundary integrals, reducing the dimensionality of the problem by one. This reduction can lead to decreased computational effort and improved accuracy in certain cases. However, traditional BEM formulations for most problems, including non-stationary heat conduction, are formulated in terms of an unknown function. For example, in the heat conduction problem, the temperature distribution field is sought, in elasticity the displacement field, or in filtration problems the head function. At the same time, when solving these problems, the primary interest is often not in the field itself but in its gradient (the heat flux vector, the head gradient, or mechanical stresses), which is determined indirectly through numerical differentiation of the field, [5], [6], [7], [8] [9]. Numerical differentiation can introduce significant errors, especially in regions with sharp changes in the function or where the grid is coarse, reducing the accuracy of the calculation of the gradient of the original function.

This study presents a formulation of the transient heat conduction problem directly in terms of the heat flux vector. By deriving a boundary integral equation specific to the heat flux, the need for numerical differentiation of temperature fields to obtain flux values is eliminated. This direct approach has the potential to enhance the accuracy of heat flux computations and provide more reliable results for engineering applications where heat flux is of primary interest.

A boundary element method tailored to this formulation is developed, detailing the procedure for solving the transient heat conduction problem with respect to the heat flux vector. A system of governing equations is established, and time-stepping schemes are implemented to handle the transient nature of the problem. To confirm the efficiency of the proposed method, a test problem with an analytical solution was solved, describing the non-stationary temperature distribution in a bounded domain. The calculation results were compared with the analytical solution. The comparison shows that the developed method achieves acceptable engineering accuracy and confirms its reliability for practical application.

The proposed approach offers a significant advantage for engineers and researchers by providing a more direct and potentially more accurate means of calculating heat fluxes in transient heat conduction problems. It opens avenues for improved thermal analysis in various fields.

2 Problem Formulation

This section presents the formulation of the non-stationary heat conduction problem and derives the boundary integral equations for the temperature field gradient.

2.1 Formulation of the Transient Heat Conduction Problem in Terms of Derivatives of Temperature

Let us consider a flat region *S*, bounded by a contour Γ. Write the classical formulation of the problem for determining the non-stationary temperature distribution *T=T(x,y,t)* in the region *S* as follows:

$$S: k\Delta T + d = \frac{\partial T}{\partial t}\rho c, \tag{1}$$

$$\Gamma_1: T = T_1(t), \tag{2}$$

$$\Gamma_2: k\partial_n T = p(t), \tag{3}$$

$$t = t_0: T = T_0(x, y), \tag{4}$$

where $\alpha = k/\rho c$ – thermal diffusivity coefficient, m^2/sec;

k – thermal conductivity, W/(m·K);
ρ – density, kg/m^3;
c – specific heat capacity, J/(kg·K);
Δ – the Laplace operator on the plane;
d – the function of heat sources;
p – heat flux at the boundary, W/m^2.

A reformulated problem is considered, where the unknowns are the first derivatives of temperature with respect to coordinate, as well as the derivative of temperature with respect to time:

$$q_x = k\partial_x T, \quad q_y = k\partial_y T, \quad \dot{T} = \frac{\partial T}{\partial t}. \tag{5}$$

Then write the equation (1) as follows

$$\partial_x q_x + \partial_y q_y + d = \dot{T}\rho c. \tag{6}$$

The functions q_x, q_y satisfy the equation

$$\partial_x q_y - \partial_y q_x = 0, \tag{7}$$

which is naturally called the equation of compatibility.

If we differentiate equation (6) with respect to *X* and *Y*, and differentiate equation (5) with respect to time *t*, we obtain the following system of equations

$$\begin{aligned} \Delta q_x + \partial_x d &= \partial_x \dot{T}\rho c, \\ \Delta q_y + \partial_y d &= \partial_y \dot{T}\rho c, \\ \partial_t q_x &= k\partial_x \dot{T}, \\ \partial_t q_y &= k\partial_y \dot{T}. \end{aligned} \tag{8}$$

The formulation of the initial boundary value problem for the system of equations (8) requires setting boundary conditions relative to derivatives at each point of the boundary contour Γ. The boundary condition (3) by definition has the following form

$$\begin{aligned} k\partial_n T &= k\left(n_x \partial_x T + n_y \partial_y T\right) = \\ &= k\left(n_x q_x + n_y q_y\right) = p, \end{aligned} \tag{9}$$

where n_x, n_y – projections of the vector of the unit external normal on the coordinate axis.

To formulate another boundary condition, we differentiate condition (2) along the arc of the contour:

$$\begin{aligned} k\partial_\Gamma T &= k\left(t_x \partial_x T + t_y \partial_y T\right) = \\ &= k\left(t_x q_x + t_y q_y\right) = h, \end{aligned} \tag{10}$$

where t_x, t_y – projections of the unit tangent vector to the Γ contour.

To ensure the necessary number of boundary conditions, require condition (6) to be fulfilled at the points of the boundary Γ_1, and condition (7) to be fulfilled at the points of the boundary Γ_2.

To obtain the initial conditions for the heat flux vector, take the gradient from the initial condition $T_0(x, y)$

$$\begin{aligned} k\partial_x T_0 &= q_{0x}(x, y), \\ k\partial_y T_0 &= q_{0y}(x, y). \end{aligned} \tag{11}$$

Finally, for unknown functions q_x, q_y and $\dot{T}$ we have the following equations, boundary and initial conditions:

$$S:\begin{cases}\Delta q_x+\partial_x d=\partial_x\dot{T}\rho c\\ \Delta q_y+\partial_y d=\partial_y\dot{T}\rho c\\ \partial_t q_x=k\partial_x\dot{T}\\ \partial_t q_y=k\partial_y\dot{T}\end{cases}\tag{12}$$

$$\Gamma_1:\begin{cases}k\left(t_x q_x+t_y q_y\right)=h\\ \partial_x q_x+\partial_y q_y+d=\dot{T}\rho c\end{cases}\tag{13}$$

$$\Gamma_2:\begin{cases}k\left(n_x q_x+n_y q_y\right)=p\\ \partial_x q_y-\partial_y q_x=0\end{cases}\tag{14}$$

$$t=t_0:\begin{cases}k\partial_x T_0=q_{0x}(x,y)\\ k\partial_y T_0=q_{0y}(x,y)\end{cases}\tag{15}$$

The vector formulation has the following form:

$$S:\begin{cases}\Delta\boldsymbol{q}+\boldsymbol{\nabla}d=\boldsymbol{\nabla}\dot{T}\rho c\\ \dot{\boldsymbol{q}}=k\boldsymbol{\nabla}\dot{T}\end{cases}\tag{16}$$

$$\Gamma_1:\begin{cases}\boldsymbol{t}\cdot\boldsymbol{q}=h\\ \boldsymbol{\nabla}\cdot\boldsymbol{q}=\dot{T}\rho c-d\end{cases}\tag{17}$$

$$\Gamma_2:\begin{cases}\boldsymbol{n}\cdot\boldsymbol{q}=p\\ \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}}=0\end{cases}\tag{18}$$

$$t=t_0: k\boldsymbol{\nabla}T_0=\boldsymbol{q_0}\tag{19}$$

where $\boldsymbol{\nabla}$ – the gradient operator on the plane;
$\widetilde{\boldsymbol{q}}$ – the vector $\boldsymbol{q}$, rotated by π/2 clockwise.

2.2 Boundary Integral Equations with Respect to the Heat Flux Vector

To obtain the boundary integral equations with respect to the heat flux vector, it is necessary to have an integral representation in the plane for the vector $\boldsymbol{q}$ and the function $\dot{T}$.

Let m and z be two arbitrary points on the plane, $\boldsymbol{R_m},\boldsymbol{R_z}$ are their radius vectors (Figure), t_0 is the initial moment of time, t_F is the final moment of time. Introduce the following notation:

$$\boldsymbol{R}=\boldsymbol{R_z}-\boldsymbol{R_m},\;\; R=|\boldsymbol{R}|=\sqrt{\boldsymbol{R}\cdot\boldsymbol{R}},\quad \tau=t_F-t.\tag{20}$$

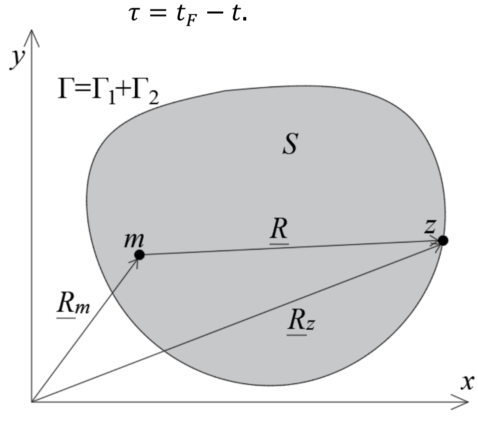

Fig. 1: The scheme for determining the vector $\boldsymbol{R}$

The fundamental solution of the non-stationary problem of heat conduction in the plane has the following form [10]:

$$T^*=\frac{1}{4\pi\alpha\tau}\exp\left(\frac{-R^2}{4\alpha\tau}\right).\tag{21}$$

Equation (21) describes the distribution of temperature over the plane and time from the action of a concentrated pulsed heat source. By definition of a fundamental solution

$$\alpha\Delta_z T^*+\frac{\partial T^*}{\partial t}=-\delta(m,z)\delta(t_F,t),\tag{22}$$

where Δ_z – the Laplace operator with differentiation by the coordinates of the point z;
$\delta(m,z),\delta(t_F,t)$ – Dirac's delta functions in coordinate and time, respectively, having the following property:

$$\int_t\int_S \psi(z,t)\,\delta(m,z)\delta(t_F,t)dS(z)dt= \\ =v(m)\psi(m,t_F),\tag{23}$$

here $\psi(z,t)$ – arbitrary continuous function, the designation *dS(z)* emphasizes that integration is carried out according to the coordinates of the z point:

$$v(m)=\begin{cases}1, & m\in S\\ 0.5, & m\in\Gamma\\ 0 & m\notin S+\Gamma\end{cases}\tag{24}$$

Let us obtain the normal and time derivatives for the function T^*

$$\frac{\partial T^*}{\partial n}=\boldsymbol{\nabla}T^*\cdot\boldsymbol{n}= \\ =\frac{-\boldsymbol{R}\cdot\boldsymbol{n_z}}{8\pi\alpha^2\tau^2}e^{\left[\frac{-R^2}{4\alpha\tau}\right]}=\frac{-\boldsymbol{R}\cdot\boldsymbol{n_z}}{2\alpha\tau}T^*,\tag{25}$$

$$\frac{\partial T^*}{\partial t}=\frac{1}{4\pi\alpha\tau}e^{\left[\frac{-R^2}{4\alpha\tau}\right]}\left(\frac{R^2}{4\alpha\tau^2}-\frac{1}{\tau}\right)= \\ =T^*\left(\frac{R^2}{4\alpha\tau^2}-\frac{1}{\tau}\right).\tag{26}$$

Introduce a vector $\boldsymbol{q}^*=\boldsymbol{b}\cdot T^*$, where $\boldsymbol{b}$ – an arbitrary constant vector. For the vector $\boldsymbol{q}^*$, we have:

$$\alpha\Delta_z\boldsymbol{q}^*+\frac{\partial\boldsymbol{q}^*}{\partial t}=-\boldsymbol{b}\cdot\delta(m,z)\delta(t_F,t),\tag{27}$$

$$\boldsymbol{\nabla}_z\cdot\boldsymbol{q}^*=-\boldsymbol{b}\cdot\frac{\boldsymbol{R}}{8\pi\alpha^2\tau^2}e^{\left[\frac{-R^2}{4\alpha\tau}\right]}==-\frac{\boldsymbol{b}\cdot\boldsymbol{R}}{2\alpha\tau}T^*,\tag{28}$$

$$\boldsymbol{\nabla}_z\cdot\widetilde{\boldsymbol{q}}^*=\boldsymbol{b}\cdot\frac{\widetilde{\boldsymbol{R}}}{8\pi\alpha^2\tau^2}e^{\left[\frac{-R^2}{4\alpha\tau}\right]}=\frac{\boldsymbol{b}\cdot\widetilde{\boldsymbol{R}}}{2\alpha\tau}T^*,\tag{29}$$

$$q_n{}^*=\boldsymbol{n_z}\cdot\boldsymbol{b}\cdot T^*=\boldsymbol{n_z}\cdot\boldsymbol{b}\cdot\frac{1}{4\pi\alpha\tau}e^{\left[\frac{-R^2}{4\alpha\tau}\right]}.\tag{30}$$

Let us use the equation (16)

$$\alpha\Delta\boldsymbol{q} + \alpha\boldsymbol{\nabla}d = \dot{\boldsymbol{q}}. \quad (31)$$

Multiply (27) scalar by $\boldsymbol{q}$, (31) by $\boldsymbol{q}^*$

$$-\boldsymbol{q}\cdot\frac{\partial\boldsymbol{q}^*}{\partial t} = \alpha\Delta\boldsymbol{q}^*\cdot\boldsymbol{q} + \\ +\boldsymbol{q}\cdot\boldsymbol{b}\delta(m,z)\delta(t_F,t), \quad (32)$$

$$\boldsymbol{q}^*\cdot\frac{\partial\boldsymbol{q}}{\partial t} = \alpha\Delta\boldsymbol{q}\cdot\boldsymbol{q}^* + \alpha\boldsymbol{q}^*\cdot\boldsymbol{\nabla}d. \quad (33)$$

Subtract (32) from (33) and integrate over the region S and over time t:

$$\int_{t_0}^{t_F}\int_S\left[\boldsymbol{q}^*\cdot\frac{\partial\boldsymbol{q}}{\partial t} + \boldsymbol{q}\cdot\frac{\partial\boldsymbol{q}^*}{\partial t}\right]dS(z)dt = \\ = \alpha\int_{t_0}^{t_F}\int_S[\Delta\boldsymbol{q}\cdot\boldsymbol{q}^* - \Delta\boldsymbol{q}^*\cdot\boldsymbol{q}]\,dS(z)dt + \\ + \int_{t_0}^{t_F}\int_S[\alpha\boldsymbol{q}^*\cdot\boldsymbol{\nabla}d - \boldsymbol{q}\cdot\boldsymbol{b} \\ \cdot\delta(m,z)\delta(t_F,t)]\,dS(z)dt. \quad (34)$$

Using the delta function property (23), we get

$$\int_{t_0}^{t_F}\int_S\left[\boldsymbol{q}^*\cdot\frac{\partial\boldsymbol{q}}{\partial t} + \boldsymbol{q}\cdot\frac{\partial\boldsymbol{q}^*}{\partial t}\right]dS(z)dt = \\ = \alpha\int_{t_0}^{t_F}\int_S[\Delta\boldsymbol{q}\cdot\boldsymbol{q}^* - \Delta\boldsymbol{q}^*\cdot\boldsymbol{q}]\,dS(z)dt + \\ + \int_{t_0}^{t_F}\int_S[\alpha\boldsymbol{q}^*\cdot\boldsymbol{\nabla}d]\,dSdt - v(m)\boldsymbol{b}\cdot\boldsymbol{q}(m,t_F). \quad (35)$$

Using integration by parts formula for the left side of equation (35), we obtain:

$$\int_{t_0}^{t_F}\left[\boldsymbol{q}^*\cdot\frac{\partial\boldsymbol{q}}{\partial t} + \boldsymbol{q}\cdot\frac{\partial\boldsymbol{q}^*}{\partial t}\right]dt = \boldsymbol{q}^*\cdot\boldsymbol{q}|_{t=t_0}^{t=t_F} = \\ = 0 - \boldsymbol{q}^*\cdot\boldsymbol{q}_0(x,y,t=t_0). \quad (36)$$

For integral

$$\int_{t_0}^{t_F}\int_S[\Delta\boldsymbol{q}\cdot\boldsymbol{q}^* - \Delta\boldsymbol{q}^*\cdot\boldsymbol{q}]\,dS(z)dt \quad (37)$$

get the following representation, [11]

$$\int_{t_0}^{t_F}\int_S(\Delta\boldsymbol{q}\cdot\boldsymbol{q}^* - \Delta\boldsymbol{q}^*\cdot\boldsymbol{q})\,dS(z)dt = \\ = \int_{t_0}^{t_F}\int_\Gamma({q_n}^*\,\boldsymbol{\nabla}\cdot\boldsymbol{q} + {q_t}^*\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}})d\Gamma(z)dt - \\ - \int_{t_0}^{t_F}\int_S(\boldsymbol{\nabla}\cdot\boldsymbol{q}^*\,\boldsymbol{\nabla}\cdot\boldsymbol{q} + \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}^*}\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}})dS\,(z)\,dt - \\ - \left[\begin{array}{l}\int_{t_0}^{t_F}\int_\Gamma(q_n\,\boldsymbol{\nabla}\cdot\boldsymbol{q}^* + q_t\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}}^*)d\Gamma(z)dt - \\ - \int_{t_0}^{t_F}\int_S(\boldsymbol{\nabla}\cdot\boldsymbol{q}\,\boldsymbol{\nabla}\cdot\boldsymbol{q}^* + \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}}\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}^*})dS\,(z)\,dt\end{array}\right].$$

Equation (35) will take the following form

$$v(m)\boldsymbol{b}\cdot\boldsymbol{q}(m,t_F) = \int_S\boldsymbol{q}^*\cdot\boldsymbol{q}_0\,dS + \\ +\alpha\int_{t_0}^{t_F}\int_\Gamma({q_n}^*\,\boldsymbol{\nabla}\cdot\boldsymbol{q} + {q_t}^*\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}})d\Gamma(z)\,dt - \\ -\alpha\int_{t_0}^{t_F}\int_\Gamma(q_n\,\boldsymbol{\nabla}\cdot\boldsymbol{q}^* + q_t\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}}^*)d\Gamma(z)\,dt + \\ + \int_{t_0}^{t_F}\int_S[\alpha\boldsymbol{q}^*\cdot\boldsymbol{\nabla}d]\,dSdt. \quad (38)$$

Let the source function $d = 0$, then substituting (17)-(18) and (28)-(29) in (38), get (discarding an arbitrary vector $\boldsymbol{b}$):

$$v(m)\boldsymbol{q}(m,t_F) = \int_S T^*\cdot\boldsymbol{q}_0\,dS(z) + \\ +k\int_{t_0}^{t_F}\int_\Gamma\boldsymbol{n_z}\cdot T^*\,\dot{T}d\Gamma(z)\,dt - \\ -\alpha\int_{t_0}^{t_F}\int_\Gamma\left[\frac{T^*}{2\alpha\tau}\left(q_t\widetilde{\boldsymbol{R}} - q_n\boldsymbol{R}\right)\right]d\Gamma(z)\,dt. \quad (39)$$

The equation (39) is the desired representation of the vector $\boldsymbol{q}$ on the plane.

Obtain a boundary integral for the function $\dot{T}$. To do this, use equation (1)

$$k\Delta T + d = \dot{T}\rho c. \quad (40)$$

Multiply (22) by $\dot{T}$, (40) by $\dot{T}^*$

$$-\dot{T}\dot{T}^* = \alpha\dot{T}\Delta T^* + \dot{T}\delta(m,z)\delta(t_F,t), \quad (41)$$

$$\dot{T}\dot{T}^* = \alpha \Delta T \dot{T}^* + \frac{\dot{T}^* d}{\rho c}. \tag{42}$$

Add (41) and (42), integrate over the region S and over time t

$$\begin{gathered}\int_{t_0}^{t_F} \int_S \left[\frac{\dot{T}^* d}{\rho c} + \dot{T}\delta(m,z)\delta(t_F,t)\right] dSdt + \\ +\alpha \int_{t_0}^{t_F} \int_S \left[\Delta T \cdot \dot{T}^* + \Delta T^* \cdot \dot{T}\right] dSdt = 0.\end{gathered} \tag{43}$$

Using the delta function property (23) and Green's first formula, we reduce equation (43) to the form:

$$\begin{gathered}v(m)\dot{T}(m,t_F) + \int_{t_0}^{t_F} \int_S \frac{\dot{T}^* d}{\rho c} dS(z)dt + \\ +\alpha \int_{t_0}^{t_F} \int_\Gamma \left[\dot{T}\frac{\partial T^*}{\partial n} + \dot{T}^* \frac{\partial T}{\partial n}\right] d\Gamma(z)dt - \\ -\alpha \int_{t_0}^{t_F} \int_S \left[\dot{\boldsymbol{q}} \cdot \boldsymbol{\nabla} T^* + \boldsymbol{\nabla}\dot{T}^* \cdot \boldsymbol{q}\right] dS(z)dt = 0.\end{gathered} \tag{44}$$

Using integration by parts formula and put the source function $d=0$. We will get:

$$\begin{gathered}v(m)\dot{T}(m,t_F) + \alpha \int_{t_0}^{t_F} \int_\Gamma \dot{T}\frac{\partial T^*}{\partial n} d\Gamma(z)dt = \\ = \alpha \int_S \boldsymbol{q_0} \cdot \boldsymbol{\nabla} T^* dS(z) - \\ -\frac{1}{\rho c}\int_{t_0}^{t_F} \int_\Gamma \dot{T}^* q_n \, d\Gamma(z)dt.\end{gathered} \tag{45}$$

Substituting formulas (25)-(26) into (45), obtain:

$$\begin{gathered}v(m)\dot{T}(m,t_F) - \\ -\alpha \int_{t_0}^{t_F} \int_\Gamma \dot{T}\frac{\boldsymbol{R} \cdot \boldsymbol{n_z}}{2\alpha\tau} T^* d\Gamma(z)dt = \\ = -\alpha \int_S \boldsymbol{q_0} \cdot \frac{\boldsymbol{R}}{2\alpha\tau} T^* dS(z) - \\ -\frac{1}{\rho c}\int_{t_0}^{t_F} \int_\Gamma T^* \left(\frac{R^2}{4\alpha\tau^2} - \frac{1}{\tau}\right) q_n \, d\Gamma(z)dt.\end{gathered} \tag{46}$$

Equations (39) and (46) form the desired system of equations for three unknowns: the heat flux vector $\boldsymbol{q}$ and the rate of temperature change $\dot{T}$.

To compute heat flux vector $\boldsymbol{q}$ at any point $m \in S$, it is necessary to know the values of the normal and tangential components q_n and q_t, as well as the rate of temperature change $\dot{T}$, at every point on the boundary Γ.

Let us consider the main initial-boundary value problems and the functions known on the boundary

$\Gamma = \Gamma_1$: $\begin{matrix} q_t = h \\ \dot{T} = g \end{matrix}$ – I initial-boundary value problem, where $g = \dot{T}_1$ – known function,

$\Gamma = \Gamma_2$: $q_n = p$ – II initial-boundary value problem.

First Boundary Value Problem. Let $m \in \Gamma$ and scalar-multiply equation (39) by the unit normal vector $\boldsymbol{n_m}$ at point m. We obtain:

$$\begin{gathered}0{,}5 q_n(m,t_F) - \\ -\alpha \int_{t_0}^{t_F} \int_\Gamma \left(q_n \frac{\boldsymbol{n_m} \cdot \boldsymbol{R}}{2\alpha\tau} T^*\right) d\Gamma(z)\, dt = \\ = \int_S T^* \boldsymbol{n_m} \cdot \boldsymbol{q}_0 \, dS + \\ +k \int_{t_0}^{t_F} \int_\Gamma \boldsymbol{n_m} \cdot \boldsymbol{n_z} T^* g d\Gamma(z)\, dt - \\ -\alpha \int_{t_0}^{t_F} \int_\Gamma \left(h \frac{\boldsymbol{n_m} \cdot \widetilde{\boldsymbol{R}}}{2\alpha\tau} T^*\right) d\Gamma(z)\, dt.\end{gathered} \tag{47}$$

For the first boundary value problem, the unknown function is the normal derivative of the heat flux vector q_n on the boundary. By determining the unknown function q_n from equation (47), the value of the vector $\boldsymbol{q}$ at any point $m \in S$ can be calculated using formula (39).

Second Boundary Value Problem. For the second boundary value problem, the unknown functions are q_t and $\dot{T}$. We scalar-multiply equation (39) by the unit tangential vector $\boldsymbol{t}$ at point $m \in \Gamma$ (taking into account that $\boldsymbol{t} \cdot \widetilde{\boldsymbol{R}} = -\boldsymbol{n} \cdot \boldsymbol{R}$). Additionally, we will use equation (46).

$$
\begin{gathered}
0{,}5 q_t(m, t_F) - \\
-\alpha \int_{t_0}^{t_F} \int_{\Gamma} q_t \, \frac{\boldsymbol{n_m} \cdot \boldsymbol{R}}{2\alpha\tau} T^* d\Gamma(z)\, dt - \\
-k \int_{t_0}^{t_F} \int_{\Gamma} \boldsymbol{t_m} \cdot \boldsymbol{n_z} \cdot T^* \, \dot{T} d\Gamma(z)\, dt = \\
= \int_S T^* \cdot \boldsymbol{t_m} \cdot \boldsymbol{q}_0 \, dS(z) - \\
-\alpha \int_{t_0}^{t_F} \int_{\Gamma} \left(p \, \frac{\boldsymbol{t_m} \cdot \boldsymbol{R}}{2\alpha\tau} T^* \right) d\Gamma(z)\, dt,
\end{gathered} \tag{48}
$$

$$
\begin{gathered}
0{,}5 \dot{T}(m, t_F) - \\
-\alpha \int_{t_0}^{t_F} \int_{\Gamma} \dot{T} \frac{\boldsymbol{R} \cdot \boldsymbol{n_z}}{2\alpha\tau} T^* \, d\Gamma(\mathrm{z}) dt = \\
= -\alpha \int_S \boldsymbol{q_0} \cdot \frac{\boldsymbol{R}}{2\alpha\tau} T^* \, dS(z) - \\
-\frac{1}{\rho c} \int_{t_0}^{t_F} \int_{\Gamma} T^* \left(\frac{R^2}{4\alpha\tau^2} - \frac{1}{\tau} \right) p \, d\Gamma(z) dt \,.
\end{gathered} \tag{49}
$$

Equations (48)-(49) form the system necessary to solve the second boundary value problem. In the first step, the rate of temperature change $\dot{T}$ should be determined using equation (49). Then, using equation (48), the function q_t can be calculated.

The numerical solution of equations (47)-(49) can be carried out using standard BEM schemes typically employed for solving ordinary boundary integral equations with respect to the function *T*. In this case, equations (47)-(49), along with the integral representations (39), (46) provide a way to determine the derivatives of the function *T*, both on the boundary and within the domain, without requiring numerical differentiation. In [12], the convergence and stability of BEM in solving boundary integral equations (47)-(49) with similar kernels are mathematically proved.

3 Numerical Solution of the Test Problem using the Boundary Element Method

To verify the described methodology, we will perform the calculation of the first initial boundary value problem that has an analytical solution. The task conditions according to Figure are as follows:

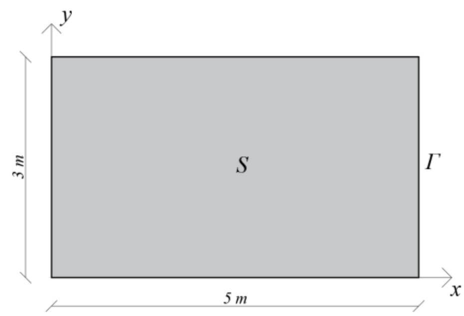

Fig. 2: Computational scheme of the test problem
initial condition $T(x, y, 0) = 0\ °K$
boundary condition $T|_{\Gamma} = 20 sin\left(\frac{4\pi}{3}\, t\right)\ °K$
thermal conductivity k = 1.25 W/(m·K)
density ρ = 1 kg/m^3
specific heat capacity c = 1 J/(kg·K)

Let's write down the discrete form of equation (47), using "constant" boundary elements both in time and coordinate. We divide the boundary Γ into *N* boundary elements, the time interval into *F* steps, the area *S* into *L* cells and present equation (47) in the following form (Figure).

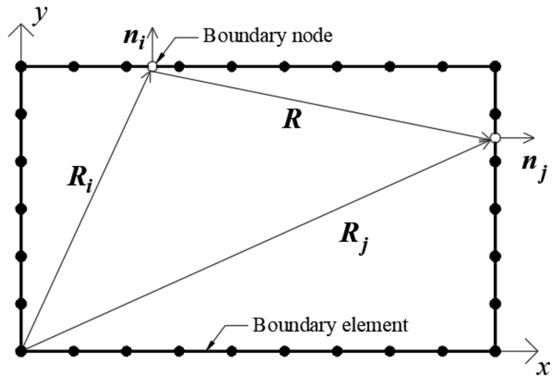

Fig. 3: Domain sampling

$$
\begin{gathered}
0{,}5 q_n(i, t_F) - \\
-\alpha \sum_{j=1}^{N} \sum_{f=1}^{F} \int_{\Gamma_j} \int_{t_{f-1}}^{t_f} q_{nj} \, \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\alpha\tau} T^* d\Gamma(j)\, dt = \\
= \sum_{l=1}^{L} \int_{S_l} T^* \boldsymbol{n_i} \cdot \boldsymbol{q}_0 \, dS + \\
+k \sum_{j=1}^{N} \sum_{f=1}^{F} \int_{\Gamma_j} \int_{t_{f-1}}^{t_f} \boldsymbol{n_i} \cdot \boldsymbol{n_j} T^* \, \dot{T} d\Gamma(j)\, dt - \\
-\alpha \sum_{j=1}^{N} \sum_{f=1}^{F} \int_{\Gamma_j} \int_{t_{f-1}}^{t_f} h \, \frac{\boldsymbol{n_i} \cdot \widetilde{\boldsymbol{R}}}{2\alpha\tau} T^* d\Gamma(j)\, dt
\end{gathered} \tag{50}
$$

or in matrix form

$$\sum_{f=1}^{F} \boldsymbol{K_{fF}Q_f} = \boldsymbol{B_0Q_0} + \sum_{f=1}^{F} \boldsymbol{G_{fF}\dot{T}_f} - \\ - \sum_{f=1}^{F} \boldsymbol{P_{fF}H_f}. \tag{51}$$

Since in the problem under consideration the initial temperature distribution in the region S is zero and the boundary condition $T(t)$ does not depend on the coordinate (the function $h = 0$ at any given time), the matrices $\boldsymbol{Q_0}$ and $\boldsymbol{H_f}$ are zero. Equation (51) will take the following form

$$\sum_{f=1}^{F} \boldsymbol{K_{fF}Q_f} = \sum_{f=1}^{F} \boldsymbol{G_{fF}\dot{T}_f}. \tag{52}$$

When calculating the unknown boundary values at the time $t = t_F$, it is necessary to determine the matrices $\boldsymbol{K_{fF}}$ and $\boldsymbol{G_{fF}}$ for $f = 1,\ 2, \dots, F$. The matrices from $\boldsymbol{K_{1F}}$ to $\boldsymbol{K_{(F-1)F}}$ and from $\boldsymbol{G_{1F}}$ to $\boldsymbol{G_{(F-1)F}}$ will sequentially multiply the specified and previously computed values of the functions q_n and $\dot{T}$ (calculated at the previous time step), represented as vectors with unknown components. Thus, with a constant time step, only two new matrices need to be determined at each new step, while all other matrices are defined in previous steps and stored in the computer memory.

The coefficients of the matrices $\boldsymbol{K}$ and $\boldsymbol{G}$ consist of expressions:

$$k_{fFij} = \\ = \frac{\delta_{fF}\delta_{ij}}{2} - \alpha \int_{\Gamma_j} \int_{t_{f-1}}^{t_f} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\alpha\tau} T^* d\Gamma(j)\, dt, \tag{53}$$

$$g_{fFij} = k \int_{\Gamma_j} \int_{t_{f-1}}^{t_f} \boldsymbol{n_i} \cdot \boldsymbol{n_j} T^*\, d\Gamma(j)\, dt, \tag{54}$$

where δ_{fF}, δ_{ij} – Kronecker deltas.

Time integration in expressions (53) and (54) can be performed analytically [9]. The integral in the expression (53) is equal to:

$$\int_{t_{f-1}}^{t_f} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\alpha\tau} T^* dt = - \int_{t_{f-1}}^{t_f} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{\boldsymbol{n_j} \cdot \boldsymbol{R}} \frac{\partial T^*}{\partial n} dt = \\ = \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\pi\alpha R^2} \int_{t_{f-1}}^{t_f} \frac{R^2}{4\alpha\tau^2} \exp\left(\frac{-R^2}{4\alpha\tau}\right) dt = \\ = \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\pi\alpha R^2} \left[\exp(-a_{f-1}) - \exp(-a_f)\right], \tag{55}$$

where $a_f = R^2/\left[4\alpha(t_F - t_f)\right]$.

The integral in the equation (54) is equal to [9]

$$\int_{t_{f-1}}^{t_f} \boldsymbol{n_i} \cdot \boldsymbol{n_j} T^*\, dt = \\ = \frac{\boldsymbol{n_i} \cdot \boldsymbol{n_j}}{4\pi\alpha} \left[E_1(a_{f-1}) - E_1(a_f)\right], \tag{56}$$

where E_1 – integral exponential function [9].

The diagonal coefficients g_{FFii} in equation (55) contain integrals with singularities. The analytical solution is as follows [9]

$$g_{FFii} = \\ = \frac{l}{4\pi} \left[\begin{array}{c} 2 - C - \ln\gamma + \\ + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\gamma^n}{n(2n+1)n!} \end{array} \right], \tag{57}$$

where l – the length of the boundary element;
C – Euler's constant;
$\gamma = l^2/16\alpha\Delta t_F$.

The diagonal coefficients k_{FFii} are identically equal to 1/2, since on one boundary element the vectors $\boldsymbol{n}$ and $\boldsymbol{R}$ are orthogonal to each other.

Thus, the expressions for determining the coefficients in the influence matrices can be represented as follows:

$$\left[\begin{array}{l} k_{fFij} = -l \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\pi R^2} [e^{-a_{f-1}} - e^{-a_f}] \\ g_{fFij} = l\rho c \frac{\boldsymbol{n_i} \cdot \boldsymbol{n_j}}{4\pi} \left[E_1(a_{f-1}) - E_1(a_f)\right]' \\ \quad \text{if } i \neq j, \quad t_f \neq t_F \end{array} \right. \\ \left[\begin{array}{l} k_{FFij} = -l \frac{\boldsymbol{n_i} \cdot \boldsymbol{R}}{2\pi R^2} e^{-a_{f-1}} \\ g_{FFij} = l\rho c \frac{\boldsymbol{n_i} \cdot \boldsymbol{n_j}}{4\pi} E_1(a_{f-1})' \\ \quad \text{if } i \neq j, \quad t_f = t_F \end{array} \right. \\ \left[\begin{array}{l} k_{FFii} = 0.5 \\ g_{FFii} = equation\ (57), \\ \text{if } i = j, \quad t_f = t_F \end{array} \right. \\ \left[\begin{array}{l} k_{fFii} = 0 \\ g_{fFii} = \frac{l}{4\pi} \left(\begin{array}{c} ln\frac{\gamma_2}{\gamma_1} + \\ + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\gamma_{f-1}^n - \gamma_f^n)}{n(2n+1)n!} \end{array} \right). \\ \quad \text{if } i = j, \quad t_f \neq t_F \end{array} \right. \tag{58}$$

Since the test problem has two axes of symmetry, the calculation results are presented for one quarter of the computational domain, as shown in Figure .

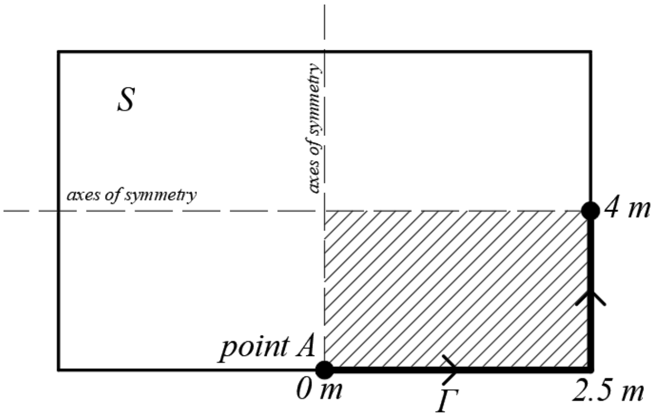

Fig. 4: Part of the computational domain for displaying calculation results

The analytical solution of the test problem is described by the following formula:

$$T(x,y,t) = 20\,sin(\omega t) -$$
$$-\sum_{n=1,3,5,\ldots}^{\infty}\ \sum_{m=1,3,5,\ldots}^{\infty} \frac{1280}{3mn\pi}\cdot$$
$$\cdot\frac{\alpha\lambda_{mn}\left[cos(\omega t) - e^{-\alpha\lambda_{mn}t}\right] + \omega\, sin(\omega t)}{\alpha^2\lambda_{mn}^2 + \omega^2} \qquad (59)$$
$$\cdot\, sin\left(\frac{m\pi x}{L_x}\right)\cdot sin\left(\frac{n\pi y}{L_y}\right),$$

where $\lambda_{mn} = (m\pi/L_x)^2 + (n\pi/L_y)^2$;
$\omega = 4\pi/3$;
L_x, L_y – dimensions of the calculation area;
α – thermal diffusivity coefficient.

Let us consider the distribution of heat flux along the boundary Γ and over time *t*. The calculation results are compared with a known analytical solution. The calculations were performed with successive reductions in the time discretization step (0.1, 0.05, 0.01 sec) and the size of the boundary element (0.5, 0.25, 0.1, 0.05 m). Figure 5 and Figure 6 show the dependence of heat flux on time at point *A* for different sizes of the boundary element *l* and time steps *Δt*.

Figure 7 and Figure 8 show the distribution of the heat flux along the boundary at time t = 1.5 second for different spatial and temporal discretizations.

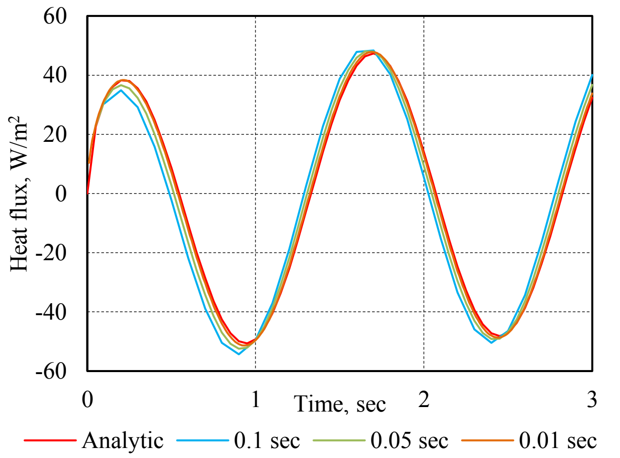

Fig. 5: The dependence of the heat flux on time at point *A* at different *Δt*

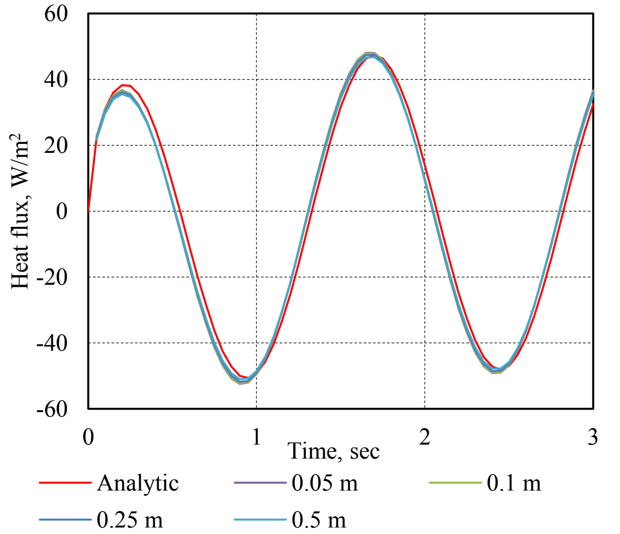

Fig. 6: The dependence of the heat flux on time at point *A* at different *l*

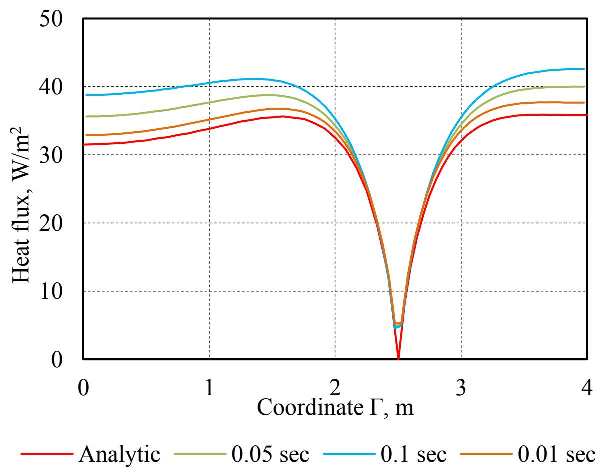

Fig. 7: The distribution of the heat flux along the contour at time *t = 1.5 sec* with different time step and the size of the boundary element *l = 0.05 m*

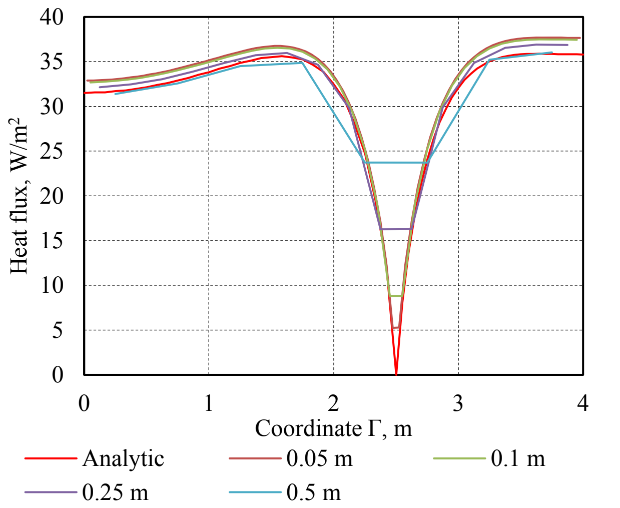

Fig. 8: The distribution of the heat flux along the contour at time *t = 1.5 sec* with a different size of the boundary element and a time step *Δt = 0.01 sec*

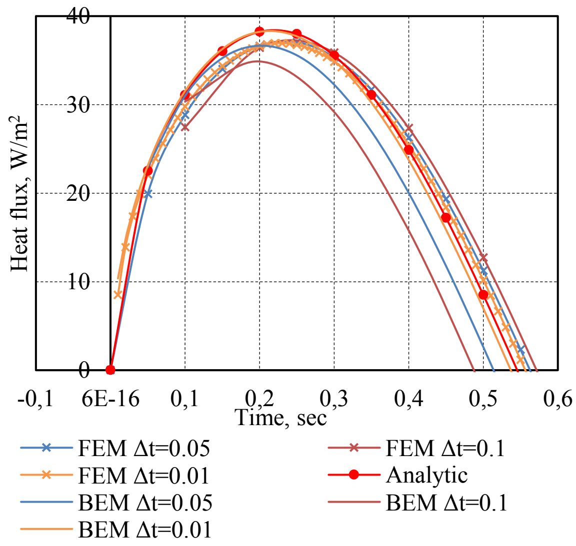

Fig. 9: The dependence of the heat flux on time at point *A* at different *Δt.* FEM/BEM comparison

The error of the numerical method is estimated by the following expression:

$$\rho(\Gamma,t) = \\ = \frac{\left|q_{n,BEM}(\Gamma,t) - q_{n,analytic}(\Gamma,t)\right|}{\left|q_{n,analytic}(\Gamma,t)\right|} 100\%. \qquad (60)$$

Table 1 shows the results of estimating the error of the numerical method depending on the time step with the size of the boundary element *l=0.25 m*.

Table 1. The error of the numerical method for different sizes of boundary elements and time steps

ρ_{max}, %			
Δt, sec / *l, m*	*0.01*	*0.05*	*0.1*
0.25	***3.00***	***10.29***	***15.56***

Additionally, the calculation results were compared with the finite element method. A similar time discretization is considered. The discretization of the computational domain is performed in such a way that the number of unknowns in the FEM corresponds to the number of unknowns in the BEM. Figure 9 shows a comparison of the heat flux at point A for the first 0.6 seconds of the calculation.

4 Conclusion

In this study, we have developed a novel approach to solving transient heat conduction problems by formulating them directly in terms of the heat flux vector. By deriving a boundary integral equation specific to the heat flux, we eliminated the need for numerical differentiation of temperature fields, which is a common source of errors in traditional methods. The tailored boundary element method (BEM) we proposed effectively handles the transient nature of the problem through the implementation of time-stepping schemes.

Our method was validated by solving a test problem involving transient temperature distribution in a bounded domain and comparing the results with analytical solutions and finite element calculation. The comparison demonstrated that our approach achieves acceptable engineering accuracy, confirming its reliability for practical applications.

The direct calculation of heat fluxes offers significant advantages for engineering and research applications where heat flux is of primary interest. It enhances the accuracy of thermal analyses and reduces computational effort by avoiding indirect calculations and numerical differentiation. This method has the potential to improve thermal management in various fields, including civil and mechanical engineering, and also to increase the accuracy of solving thermoelasticity problems.

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The authors equally contributed to the present research at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare.