A Layered-Shell Model of Anisotropic Composites: Extension of the Milgrom and Shtrikman Model

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Abstract: - In this work, the extension of the Milgrom and Shtrikman model to anisotropic composite materials containing n-layered hollow ellipsoidal inclusions, is presented. The effective properties of such materials are determined using the Green function techniques and interfacial operators. Here, the basic unit of the microstructure is a hollow system of contacting concentric ellipsoidal shells, each of which is made of one of the components. Space is packed with such units of different sizes, but the same proportions; the cavity within each such shell system is then packed with similar systems and this continues in an infinite nesting sequence. In the final configuration, the effective properties are inside and outside the basic unit of layered shells *(n+1)*. For $n=2$ and in the case of isotropic material, it is shown that the effective compressibility covers all ranges of the Hashin-Strikman bounds.

Key-Words: - Layered Shells, composite materials, anisotropy, ellipsoidal inclusions, Micromechanics, Milgrom and Shtrikman Model, Four-Phase Model, Hashin-Shtrikman bounds.

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1 Introduction

The multilayered model is widely studied in the literature. [1] extend the composite sphere model, [2] to n-layered spherical inclusion and give an exact solution of the effective elastic properties. [3], studied the thermo-elastic behavior of such materials. [4], obtained a general form for the piezoelectric properties of n-layered ellipsoidal inclusion. [5] and [6], give another form of the problem of multicoated inclusion.

Historically, [7] was the first to study theoretically the effect of an interphase on local stress and strain fields. More recently, [8] developed a new analytical method based on Green's function technics [9] and interfacial operators [10] for the determination of effective elastic moduli of the so-called four-phase model.

Here, an extension of [11] is given for a composite consisting of hollow multilayered ellipsoidal inclusions. Space is packed with such units of different sizes, but the same proportions; the cavity within each such shell system is then packed with similar systems and this continues in an infinite nesting sequence. All phases are assumed homogeneous and anisotropic and perfect bonding is supposed at the interfaces. It shows that the obtained effective properties depend on the evaluation of the localization tensors.

2 Micromechanical Modelling

Following [11] and using the generalized self-consistent scheme (GSCS) [12] the effective properties are inside and outside the composite inclusion (Fig.1). The elementary problem described in Fig. 1 is constituted by n ellipsoidal shells surrounded by a matrix layer and the whole composite inclusion is embedded in the effective medium with unknown elastic properties $\boldsymbol{C}^{eff}$. In this composite inclusion, $\boldsymbol{c}^1 = \boldsymbol{C}^{eff}$ denotes the elastic properties of ellipsoidal inclusion and $\boldsymbol{c}^2$, $\boldsymbol{c}^3$,...., $\boldsymbol{c}^k$,….$\boldsymbol{c}^{n+1}$ denote the elastic properties of outer shells.

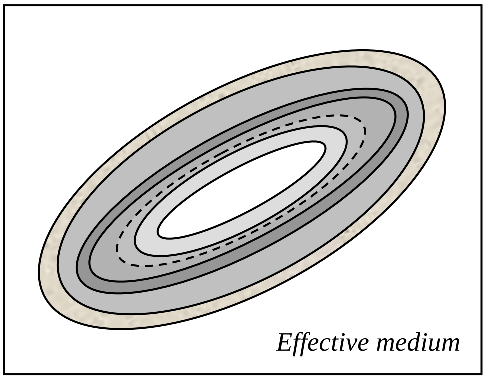

Fig. 1: The *(n+1)* phase model

The present model takes a step beyond the multi-layered ellipsoid model in that the equivalent medium is both inside and outside the basic unit of layered shells (Figure 1).

2.1 Integral Equation

To solve this problem, the kinematical integral equation [9] linking the elastic local strain $\boldsymbol{\varepsilon}(x)$ with the global uniform strain $\boldsymbol{E}$, is given by:

$$\boldsymbol{\varepsilon}(x) = \boldsymbol{E} - \int_V \boldsymbol{\Gamma}(x - x'): \delta\boldsymbol{c}(x'): \boldsymbol{\varepsilon}(x')dV' \quad (1)$$

Where:

$$\delta\boldsymbol{c}(x) = \sum_{k=1}^{n+1}\left(\boldsymbol{c}^k - \boldsymbol{c}^{eff}\right) \quad (2)$$

The average strain $\boldsymbol{\varepsilon}^J$ of composite inclusion in Fig.1 is given by:

$$\boldsymbol{\varepsilon}^J = \sum_{k=1}^{n+1} f_k \boldsymbol{\varepsilon}^k \quad (3)$$

$\boldsymbol{\varepsilon}^k$ and f_k denote the average strain and volume fraction of phase k, respectively.
The composite inclusion of volume V_J consists of all phases, such that:

$$V_J = \sum_{k=1}^{n+1} V_k \quad (4)$$

On the other hand, $\boldsymbol{\varepsilon}^J$ is calculated by using equation (1):

$$\boldsymbol{\varepsilon}^J = \boldsymbol{E} - \boldsymbol{T}^J\left(\boldsymbol{c}^{eff}\right): \sum_{k=1}^{n+1} f_k \Delta\boldsymbol{c}^{(k/eff)}: \boldsymbol{\varepsilon}^k \quad (5)$$

The tensor $\boldsymbol{T}^J\left(\boldsymbol{c}^{eff}\right)$ is deduced from [13]:

$$\boldsymbol{T}^J\left(\boldsymbol{c}^{eff}\right) = \frac{1}{V_J}\int_{V_J}\int_{V_J} \boldsymbol{\Gamma}(x - x')dV'dV \quad (6)$$

Combining equations (3) and (5), one can give:

$$\boldsymbol{E} = \sum_{k=1}^{n+1} f_k\left[\boldsymbol{I} + \boldsymbol{T}^J\left(\boldsymbol{c}^{eff}\right): \Delta\boldsymbol{c}^{(k/eff)}\right]: \boldsymbol{\varepsilon}^k \quad (7)$$

The concentrations tensor A^J for the composite inclusion and the concentrations tensors a^k for each phase k can be introduced [14] so that:

$$\boldsymbol{\varepsilon}^J = A^J: \boldsymbol{E} \quad (8)$$

$$\boldsymbol{\varepsilon}^k = a^k: \boldsymbol{\varepsilon}^J \quad (9)$$

with from equation (3):

$$\sum_{k=1}^{n+1} f_k a^k = \boldsymbol{I} \quad (10)$$

where $\boldsymbol{I}$ is the unit fourth order tensor. From equations (8) and (9), one can also write:

$$\boldsymbol{\varepsilon}^k = A^k: \boldsymbol{E} \quad (11)$$

where:

$$A^k = a^k: A^J \quad (12)$$

Using equation (7), it comes:

$$A^J = \left(\boldsymbol{I} + \boldsymbol{T}^J(\boldsymbol{c}^{eff}): \left(\sum_{k=1}^{n+1} f_k \Delta\boldsymbol{c}^{(k/eff)}: a^k\right)\right)^{-1} \quad (13)$$

Equation (13) expresses a relation between tensors A^J and a^k. To solve the problem, another equation should be derived.

2.2 Interfacial Operators

Perfect bonds between all phases are assumed and then the displacement and traction vectors are continuous through the interfaces. Using the elastic properties of two phases (k) and $(k+1)$, the strain jump through their common interface is written as follows [10]:

$$\boldsymbol{\varepsilon}^{k+1}(x) - \boldsymbol{\varepsilon}^k(x) = \boldsymbol{P}(\boldsymbol{c}^{k+1}): (\boldsymbol{c}^k - \boldsymbol{c}^{k+1}): \boldsymbol{\varepsilon}^k(x) \quad (14)$$

$\boldsymbol{P}\left(\boldsymbol{c}^{k+1}\right)$ is the interfacial operator.

For each level (k), $\Omega_k = \mathrm{V}_1 \cup \ldots \cup \mathrm{V}_k$ denotes the volume of the composite formed by the phases 1 to k. Then, in order to solve the problem $\boldsymbol{\varepsilon}^k(x)$ is substituted by the averaged value $\boldsymbol{\varepsilon}^k$. Thus, by performing the average strain over the phase $(k + 1)$ of volume V_{k+1} denoted $\boldsymbol{\varepsilon}^{k+1}$, the following recurrence relation at each level (k) from eq. (14) is giving by:

$$\boldsymbol{\varepsilon}^{k+1} = \left(I + \boldsymbol{T}^{k+1}\left(\boldsymbol{c}^{k+1}\right): \left(\boldsymbol{c}^{\Omega_k} - \boldsymbol{c}^{k+1}\right)\right): \boldsymbol{\varepsilon}^{\Omega_k} \quad (15)$$

where:

$$\boldsymbol{T}^{k+1}\left(\boldsymbol{c}^{k+1}\right) = \frac{1}{V_{k+1}}\int_{V_{k+1}} \boldsymbol{P}^{k+1}\left(\boldsymbol{c}^{k+1}\right)dV \quad (16)$$

and:

$$\boldsymbol{\varepsilon}^{\Omega_k} = \sum_{i=1}^{k} \frac{V_k}{\Omega_k}\boldsymbol{\varepsilon}^i = \frac{\sum_{i=1}^{k} f_i \boldsymbol{\varepsilon}^i}{\sum_{i=1}^{k} f_i} \quad (17)$$

for $n = 2$, $\boldsymbol{\varepsilon}^2$ is given by:

$$\boldsymbol{\varepsilon}^2 = \left(I + \boldsymbol{T}^2(\boldsymbol{c}^2): (\boldsymbol{c}^1 - \boldsymbol{c}^2)\right): \boldsymbol{\varepsilon}^1 \quad (18)$$

Tensor $\boldsymbol{T}^{k+1}\left(\boldsymbol{c}^{k+1}\right)$ is given by [15]:

$$\boldsymbol{T}^{k+1}(\boldsymbol{c}^{k+1}) = \boldsymbol{T}^{\Omega_k}(\boldsymbol{c}^{k+1}) - \frac{\sum_{i=1}^{k} f_i}{f_{i+1}}[\boldsymbol{T}^{\Omega_{k+1}}(\boldsymbol{c}^{k+1}) - \boldsymbol{T}^{\Omega_k}(\boldsymbol{c}^{k+1})] \quad (19)$$

where:

$$\boldsymbol{T}^{\Omega_k}\left(\boldsymbol{c}^{k+1}\right) = \frac{1}{\Omega_k}\int_{\Omega_k} \boldsymbol{\Gamma}\left(\boldsymbol{c}^{k+1}\right)dV \quad (20)$$

$$\boldsymbol{T}^{\Omega_{k+1}}\left(\boldsymbol{c}^{k+1}\right) = \frac{1}{\Omega_{k+1}}\int_{\Omega_{k+1}} \boldsymbol{\Gamma}\left(\boldsymbol{c}^{k+1}\right)dV \quad (21)$$

From equation (15), the following form is obtained:

$$\boldsymbol{\varepsilon}^{k+1} = \boldsymbol{\varepsilon}^{\Omega_k} - \boldsymbol{T}^{k+1}\left(\boldsymbol{c}^{k+1}\right): \Delta\boldsymbol{c}^{(k+1/\Omega_k)}: \boldsymbol{\varepsilon}^{\Omega_k} \quad (22)$$

Where;

$$\Delta \boldsymbol{c}^{(k+1/\Omega_k)} = \boldsymbol{c}^{k+1} - \boldsymbol{c}^{\Omega_k} \tag{23}$$

The Hooke's law implies that:

$$\Delta \boldsymbol{c}^{(k+1/\Omega_k)} : \boldsymbol{\varepsilon}^{\Omega_k} = \frac{\sum_{i=1}^{k} f_i \Delta \boldsymbol{c}^{(k+1/i)} : \boldsymbol{\varepsilon}^i}{\sum_{i=1}^{k} f_i} \tag{24}$$

By using equations (17), (22) and (24), the expression of a^{k+1} is obtained:

$$a^{k+1} = \frac{\sum_{i=1}^{k} f_i\, w^{(k+1/i)} : a^i}{\sum_{i=1}^{k} f_i} \tag{25}$$

where:

$$w^{(k+1/i)} = I - \boldsymbol{T}^{k+1}\left(\boldsymbol{c}^{k+1}\right) : \Delta \boldsymbol{c}^{(k+1/i)} \tag{26}$$

Then, by recurrence, equation (25) is transformed into the following equation:

$$a^{k+1} = X^{k+1} : a^1 \tag{27}$$

with the recurrence relations for X^{k+1}:

$$X^{k+1} = \frac{\sum_{i=1}^{k} f_i\, w^{(k+1/i)} : X^i}{\sum_{i=1}^{k} f_i} \tag{28}$$

Thus, it is sufficient to derive a^1 to completely solve the problem. This is done by applying equation (10) such that:

$$a^1 = \left(\sum_{k=1}^{n+1} f_k X^k\right)^{-1} \tag{29}$$

3 Framing of Any Compressibility by [16]

The composite under consideration is an isotropic material consisting of isotropic matrix containing isotropic spherical and coated inclusions**.** The resulting four phase model is shown in Fig.2 where the inclusion 1, the interphase 2 and the matrix layer 3 are characterized by the radii r_1, r_2, and r_3, respectively. The interphases 2 and 3 are characterized by the elastic moduli c^2, and c^3 respectively. The inclusion and the equivalent homogeneous medium are characterized by the effective elastic moduli $\boldsymbol{C}^{eff}$.

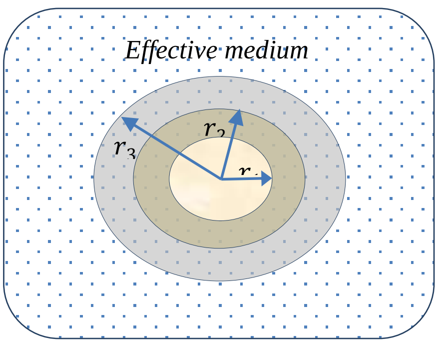

Fig. 2: Four-phase model

The volume fraction of the sphere occupied by the components [11] is given by:

$$p = \frac{r_3^3 - r_1^3}{r_3^3} \tag{30}$$

The special case $p = 1$ is the well-known coated sphere model. Another special case is $p \ll 1$ (very thin shells).

The volume fractions of phases 1 and 2 are, respectively:

$$f_1 = \frac{r_1^3}{r_3^3} \tag{31}$$

$$f_2 = \frac{r_2^3 - r_1^3}{r_3^3} \tag{32}$$

The effective elastic moduli [8] of the composite of Fig.2 are given by:

$$\boldsymbol{C}^{eff} = \boldsymbol{c}^3 + f_1(\boldsymbol{C}^{eff} - \boldsymbol{c}^3) : \boldsymbol{A}^{(1)} + f_2(\boldsymbol{c}^2 - \boldsymbol{c}^3) : \boldsymbol{A}^{(2)} \tag{33}$$

For isotropic elastic bodies, the tensors $\boldsymbol{c}^2$, $\boldsymbol{c}^3$, $\boldsymbol{C}^{eff}$, $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are written as sum of volumetric and deviatoric parts $\boldsymbol{J}$ and $\boldsymbol{K}$ and are given by the following relations:

$$\boldsymbol{c}^2 = 3k_2 \boldsymbol{J} + 2\mu_2 \boldsymbol{K} \tag{34}$$

$$\boldsymbol{c}^3 = 3k_3 \boldsymbol{J} + 2\mu_3 \boldsymbol{K} \tag{35}$$

$$\boldsymbol{C}^{eff} = 3k^{eff} \boldsymbol{J} + 2\mu^{eff} \boldsymbol{K} \tag{36}$$

$$\boldsymbol{A}^{(1)} = M^{(1)} \boldsymbol{J} + D^{(1)} \boldsymbol{K} \tag{37}$$

$$\boldsymbol{A}^{(2)} = M^{(2)} \boldsymbol{J} + D^{(2)} \boldsymbol{K} \tag{38}$$

where k and μ are bulk and shear moduli and the tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ result from the decomposition of the unit tensor $\boldsymbol{I}$ such that:

$$\boldsymbol{I} = \boldsymbol{J} + \boldsymbol{K} \tag{39}$$

Basing on the forgoing equations, one can express the effective compressibility as follow:

$$k^{eff} = k_3 + f_1\left(k^{eff} - k_3\right) : M^{(1)} + f_2(k_2 - k_3) : M^{(2)} \tag{40}$$

$M^{(1)}$ and $M^{(2)}$ denote the volumetric parts of tensors $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, respectively, and are given by:

$$M^{(1)} = m^1 M \tag{41}$$

$$M^{(2)} = m^2 M \tag{42}$$

Where:

$$= \begin{pmatrix} 1 + \frac{3f_1(k^{eff} - k^{eff})}{3k^{eff}+4\mu^{eff}} m^{(1)} \\ + \frac{3f_2(k_2 - k^{eff})}{3k^{eff}+4\mu^{eff}} m^{(2)} + \frac{3f_3(k_3 - k^{eff})}{3k^{eff}+4\mu^{eff}} m^{(3)} \end{pmatrix}^{-1} \tag{43}$$

$$m^{(1)} = \begin{pmatrix} f_1 + f_2 \frac{4\mu^{eff}+3k^{eff}}{3k_2+4\mu_2} \\ + \frac{f_3}{f_1+f_2}\left(f_1 \frac{4\mu_3+3k^{eff}}{3k_3+4\mu_3} + f_2 \frac{4\mu_3+3k_2}{3k_3+4\mu_3}\frac{4\mu_2+3k^{eff}}{3k_2+4\mu_2}\right) \end{pmatrix}^{-1} \tag{44}$$

$$m^{(2)} = \begin{pmatrix} f_2 + f_1 \frac{3k_2+4\mu_2}{4\mu_2+3k^{eff}} \\ + \frac{f_3}{f_1+f_2}\left(f_1 \frac{4\mu_3+3k^{eff}}{3k_2+4\mu_3}\frac{3k_2+4\mu_2}{4\mu_2+3k^{eff}} + f_2 \frac{4\mu_3+3k_2}{3k_2+4\mu_3}\right) \end{pmatrix}^{-1} \tag{45}$$

$$m^{(3)} = \begin{Bmatrix} f_1\left(\frac{f_1}{f_1+f_2}\frac{4\mu_3+3k^{eff}}{3k_3+4\mu_3} + \frac{f_2}{f_1+f_2}\frac{4\mu_3+3k_2}{3k_3+4\mu_3}\frac{4\mu_2+3k^{eff}}{3k_2+4\mu_2}\right)^{-1} \\ +f_2\left(\frac{f_1}{f_1+f_2}\frac{4\mu_3+3k^{eff}}{3k_3+4\mu_3}\frac{4\mu_2+3k_2}{3k^{eff}+4\mu_2} + \frac{f_2}{f_1+f_2}\frac{4\mu_3+3k_2}{3k_3+4\mu_3}\right)^{-1} \\ +f_3 \end{Bmatrix}^{-1} \tag{46}$$

For a material having $k_2 = 8GPa$, $\mu_2 = 4GPa$, $k_3 = 1GPa$, $\mu_3 = 0.5GPa$ and varying p between 0 and 1, an exact model that cover the whole range of allowed values of the effective compressibility is given in Fig.3.

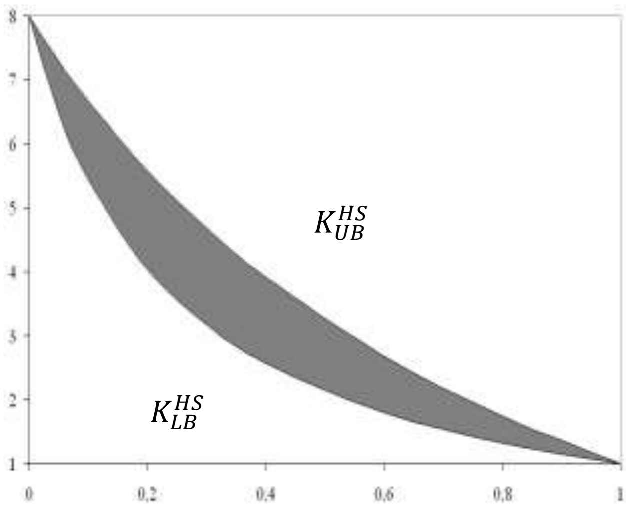

Fig. 3: Framing of any compressibility by [16]

4 Conclusion

The expression for the effective properties of a new model for anisotropic composite materials containing n -layered hollow ellipsoidal inclusions, is derived. The obtained results present the extension of the Milgrom and Shtrikman model who gave an expression for the effective response matrix of isotropic composites.

Analytical formulations obtained using the integral equation, interfacial operators and the obtained effective properties require the estimation of the strain localization tensors in each phase of the multi-layered inclusion. It is shown that any compressibility of isotropic material lies within the Hashin-Shtrikman bounds.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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Conflict of Interest

The author has no conflict of interest to declare.

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