Saint-Venant Torsion of non-homogeneous Orthotropic Right Angle Triangle Cross Section

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Abstract: This paper deals with the analytical solution of the non-homogeneous, orthotropic right-angle triangle cross-section. The shear moduli of the elastic materials are linear function of the elastic materials are linear functions of the cross-sectional coordinates. Explicit formulas are given for Prandtl's stress function, torsion function, shearing stresses, and torsional rigidity. The formulation of the solution is based on Saint-Venant's theory of uniform torsion and the application of Prandtl's stress function.

Key-Words: torsion, non-homogeneous, orthotropic, stress function, right angle triangle, Saint-Venant's theory, numerical example.

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1 Introduction

The comprehensive understanding of uniform torsion often denoted as Saint-Venant's torsion or uniform torsion theory has been thoroughly elucidated in the existing literature, as evidenced by references, [1], [2], [3], and, [4]. Building upon this theoretical foundation, a significant contribution to the field is found in the thesis work documented in, [5], where numerical analyses of torsion for both homogeneous and non-homogeneous rectangular cross-sections are conducted with the aid of the finite-volume method. This work extends the applicability of torsional analysis to diverse cross-sectional geometries.

Furthermore, the exploration of torsional behavior extends to arbitrarily shaped orthotropic beams in a journal paper documented in, [6]. Here, a hybrid finite element approach is employed to simulate the torsion of orthotropic beams with irregular shapes, adding a layer of complexity and realism to the analysis. The utilization of such advanced numerical methods showcases a commitment to capturing the nuances of real-world structural behavior and highlights the versatility of these techniques in accommodating various geometrical and material complexities. Overall, these references collectively contribute to the expanding body of knowledge in the realm of torsional analysis and simulation techniques.

This paper constitutes a significant contribution to the understanding of Saint-Venant's torsion, as it introduces a novel analytical solution tailored for the torsional analysis of non-homogeneous orthotropic cross sections specifically characterized by a right-angled triangle shape. In this distinctive study,

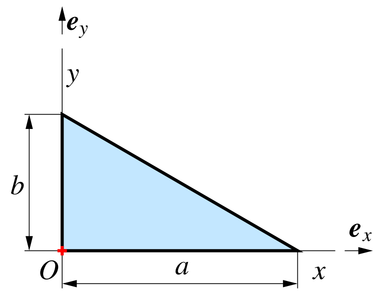

Fig. 1: Orthotropic non-homogeneous crossection

the principal directions of orthotropy are skillfully aligned parallel to the hypotenuse of the right-angled triangle, introducing a unique and specialized geometric configuration that reflects the intricacies of real-world structural elements.

To provide a visual context for the discussed cross-section, Figure 1 is included, offering a detailed illustration of the considered geometry. This depiction serves as a valuable reference for readers, aiding in the comprehension of the geometric intricacies under examination. By focusing on the torsion of a right-angled triangle with non-homogeneous orthotropic properties, this paper delves into a specific and nuanced aspect of structural mechanics, enhanc-ing our understanding of torsional behavior in materi-als with varying properties across different directions.

The analytical solution proposed in this study not only contributes to the theoretical framework of Saint-Venant's torsion but also opens avenues for further exploration and comparison with numerical methods. The specialized nature of the considered cross-section adds a layer of complexity to the analysis, making the findings particularly relevant for applications in which right-angled triangles with non-uniform orthotropic properties play a crucial role. Overall, this paper enriches the existing literature by offering a detailed and tailored approach to the analysis of torsional behavior in non-homogeneous orthotropic structures.

Lekhitskii's works have extensively explored solutions for Saint-Venant's torsion, particularly emphasizing orthotropic and non-homogeneous cross sections in various beam configurations, as documented in references, [1], [2]. In alignment with this theme, the present paper takes a focused approach, addressing Saint-Venant's torsion specifically within the context of non-homogeneous orthotropic right-angled triangles. An essential aspect of the formulation involves representing the shear moduli of the cross-section, denoted as

$$G_x(x) = g_x\, x, \qquad G_y(y) = g_y\, y, \tag{1}$$

where g_x and g_y capture the spatial variation along the x-axis and y-axis, respectively. This representation underscores the non-uniform and orthotropic nature of the cross-sectional material properties.

Uniquely, this paper contributes to the field by providing an analytical solution to Saint-Venant's torsion problem within the specific constraints of the considered non-homogeneous orthotropic right-angled triangle cross-section. By doing so, it addresses a nuanced aspect of structural mechanics, offering insights that are particularly relevant to scenarios involving materials with varying properties along different axes. The analytical approach presented in this work serves as a valuable addition to the existing body of knowledge, enriching our understanding of torsional behavior in non-homogeneous orthotropic structures with specialized geometric configurations.

2 Governing equations

The torsion problem can be effectively expressed in the realm of stresses by employing the requisite equilibrium equation. This equation finds exact satisfaction through the stress function $U = U(x, y)$. For a more comprehensive exploration of this approach, readers are encouraged to refer to the in-depth discussion provided in Lekhitskii's book, as outlined in reference, [1].

In the context of solid orthotropic cross sections, the formulation devised by Prandtl gives rise to a specific boundary value problem, denoted by

$$\frac{\partial}{\partial x}\left(\frac{1}{G_y(x,y)}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{G_x(x,y)}\frac{\partial U}{\partial y}\right) = -2 \qquad (x,y) \in A, \tag{2}$$

$$U(x,y) = 0 \qquad (x,y) \in A, \tag{3}$$

where A is the cross-section of the beam, x and y are the cross sectional coordinates. $G_x = G_x(x, y)$ and $G_y = G_y(x, y)$ are the shear moduli of the orthotropic non-homogeneous cross-section, while $U = U(x, y)$ is the Prandtl's stress function of the considered solid cross-section.

This formulation encapsulates the intricacies of the torsional analysis within the framework of solid orthotropic materials. The boundary value problem serves as a critical element in delineating the conditions and constraints that govern the stress function U, shedding light on the behavior of the material under torsional loads. By delving into Prandtl's formulation, researchers and practitioners can gain valuable insights into the nuanced dynamics and responses exhibited by solid orthotropic cross sections, contributing to a deeper understanding of torsional phenomena in this particular structural context.

From basic elasticity equations, the corresponding non-vanishing shear stress components are $\tau_{xz} = \tau_{xz}(x, y)$ and $\tau_{yz} = \tau_{yz}(x, y)$ which can be obtained as, according to e.g., [1], [2], [5],

$$\tau_{xz} = \vartheta\frac{\partial U}{\partial y}, \qquad \tau_{yz} = -\vartheta\frac{\partial U}{\partial x} \qquad (x,y) \in A. \tag{4}$$

In equation (4) ϑ is the rate of twist. The torsional rigidity of the cross-section is denoted by S its expression in terms of stress function $U = U(x, y)$ is as follows

$$S = 2\int_A U \,\mathrm{d}A, \tag{5}$$

$$S = \int_A \left[\frac{1}{G_x}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{1}{G_y}\left(\frac{\partial U}{\partial y}\right)^2\right] \mathrm{d}A. \tag{6}$$

The connection between the torsion function $\omega = \omega(x, y)$ and stress function $U = U(x, y)$ is described with the following equations, for more details see, [1], [2].

$$G_x\left(\frac{\partial \omega}{\partial x} - y\right) = \frac{\partial U}{\partial y}, \quad G_y\left(\frac{\partial \omega}{\partial y} + x\right) = -\frac{\partial U}{\partial x} \qquad (x,y) \in A \cup \partial A. \tag{7}$$

From equation (7) it follows that $\omega = \omega(x, y)$ satisfies the undermentioned boundary value problem

$$\frac{\partial}{\partial x}\left[G_x\left(\frac{\partial \omega}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[G_y\left(\frac{\partial \omega}{\partial y} + x\right)\right] = 0 \quad (x, y) \in A, \quad (8)$$

$$n_x G_x\left(\frac{\partial \omega}{\partial x} - y\right) + n_y G_y\left(\frac{\partial \omega}{\partial y} + x\right) = 0 \quad (x, y) \in \partial A. \quad (9)$$

The expression of the torsional rigidity S in terms of $\omega = \omega(x, y)$ can be obtained from equation (10) or equation (11)

$$S = \int_A \left[x G_y\left(\frac{\partial \omega}{\partial y} + x\right) - y G_y\left(\frac{\partial \omega}{\partial x} - y\right)\right] \,\mathrm{d}A, \quad (10)$$

$$S = \int_A \left[G_x\left(\frac{\partial \omega}{\partial x} - y\right)^2 + G_y\left(\frac{\partial \omega}{\partial y} + x\right)^2\right] \,\mathrm{d}A. \quad (11)$$

3 Analytical Solution

The boundary value problem, articulated by equations (1) and (2), lends itself to an analytical solution, sought in the form of the expression

$$U(x, y) = C\, xy\, (bx + ay - ab) \quad (x, y) \in A \cup \partial A. \quad (12)$$

Substituting equations (1) and (12) into equation (2), we solve the unknown constant C

$$C = -\frac{g_x g_y}{a g_y + b g_x}, \quad (13)$$

Therefore, the complete expression for the stress function $U(x, y)$ is determined as:

$$U(x, y) = -\frac{g_x g_y}{b g_x + a g_y}(bx^2 y + axy^2 - abxy). \quad (14)$$

The application of the formulated stress function leads to the derivation of shearing stresses, as represented by the following equations

$$\tau_{xz}(x, y) = -\vartheta \frac{g_x g_y}{b g_x + a g_y}(bx^2 + 2axy - abx), \quad (15)$$

$$\tau_{yz}(x, y) = \vartheta \frac{g_x g_y}{b g_x + a g_y}(2bxy + ay^2 - aby). \quad (16)$$

Furthermore, the shear stress resultant $\tau_z = \tau_z(x, y)$ is obtained from the following expressions

$$\tau_z(x, y) = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \vartheta \frac{g_x g_y}{b g_x + a g_y} \cdot \sqrt{(bx^2 + 2axy - abx)^2 + (2bxy + ay^2 - aby)^2} \quad (x, y) \in A \cup \partial A. \quad (17)$$

This detailed analysis provides a systematic breakdown of the solution process, offering a clear understanding of the obtained expressions and their significance in characterizing the torsional behavior within the specified non-homogeneous orthotropic cross-section.

Substitution equation (14) into the formula (4) gives

$$S = \frac{g_x g_y a^3 b^3}{60(a g_y + b g_x)}. \quad (18)$$

The expression of the torsional function is obtained as a solution of the system of partial differential equations (7). Solution of system of partial differential equations (7) which satisfies the condition $\omega(0, 0) = 0$ is as follows

$$\omega(x, y) = -\frac{g_y}{a g_y + b g_x}\left(\frac{b}{2}x^2 - abx\right) + \frac{g_x}{a g_y + b g_x}\left(2bxy + \frac{a}{2}y^2 - aby\right) - xy \quad (x, y) \in A \cup \partial A. \quad (19)$$

4 Numerical Example

This section provides a straightforward example to demonstrate that the formulas are functional and effective in obtaining the precise solution for the torsion of a non-homogeneous right-angled triangle. The numerical example below incorporates the data used in our illustration:

$$a = 0.8\,[\mathrm{m}], \quad b = 0.35\,[\mathrm{m}],$$

$$g_x = 5 \times 10^{10}\,[\mathrm{Nm}^{-3}], \quad g_y = 9 \times 10^{10}\,[\mathrm{Nm}^{-3}].$$

Figure 2 displays the contour lines of the stress function denoted as $U = U(x, y)$. Upon applying formula (12), the calculated result yields

$$S = 1.839\,530\,730 \times 10^7\,[\mathrm{Nm}^2]. \quad (20)$$

providing valuable insight into the system's stress distribution based on the given stress function.

In Figure 3, the contours of the shearing stress resultant, denoted as $\tau_z = \tau_z(x, y)$, are illustrated, providing a visual representation of the distribution of shearing stresses across the analyzed cross-section. Simultaneously, Figure 4 displays the contour lines of

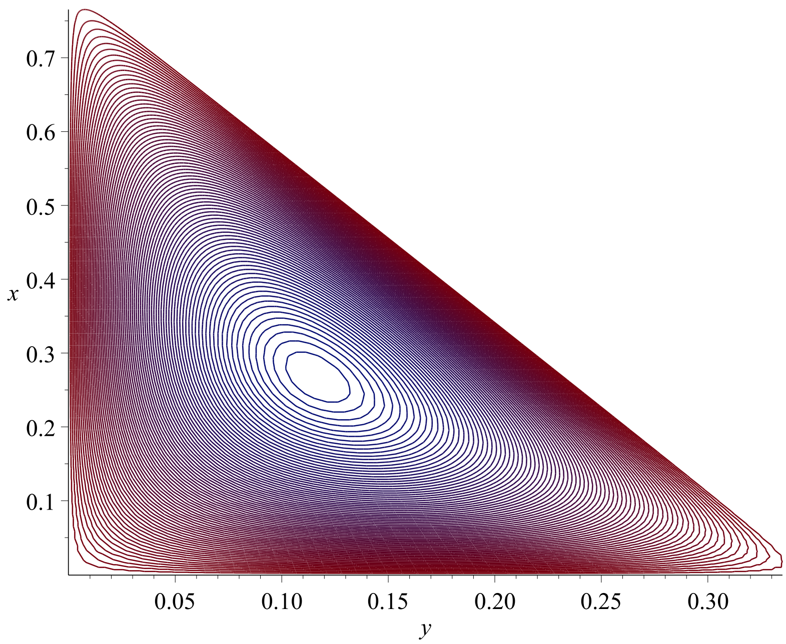

Fig. 2: The contour lines of the stress function.

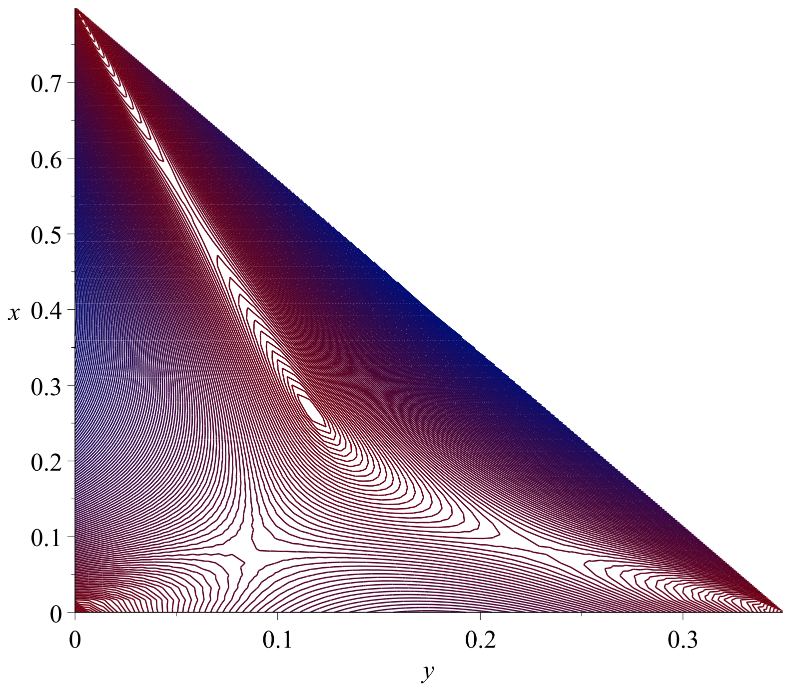

Fig. 3: The plots of contour lines of the shearing stress resultant.

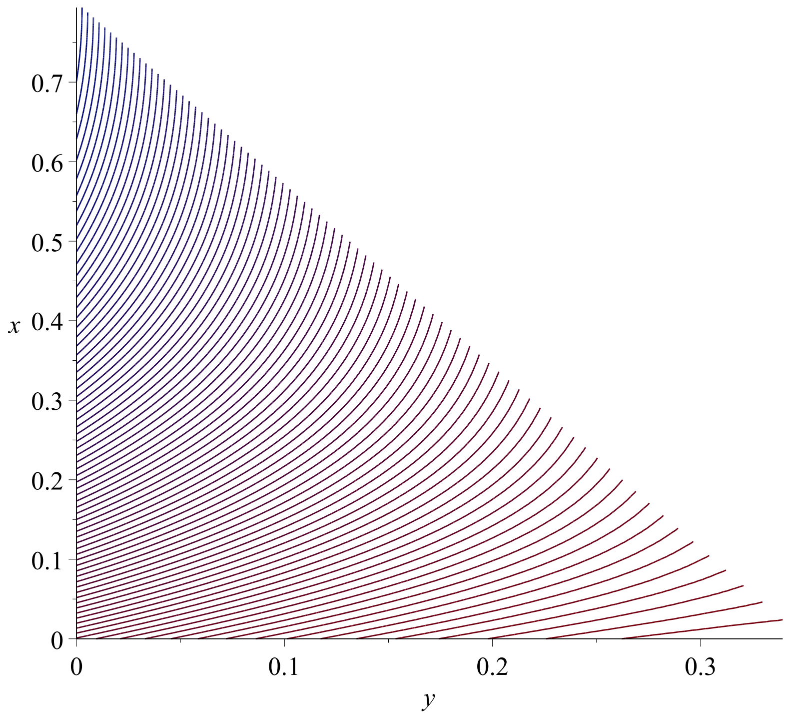

Fig. 4: The plots of the contour lines of the torsion function.

the torsional function $\omega = \omega(x, y)$, offering insights into the torsional characteristics within the examined region.

For a more detailed exploration of the shear stress resultant, specifically along the x-axis within the range $0 \leq x \leq a$, Figure 5 showcases the variation of $\tau(x, 0)$ as a function of x. This depiction allows a focused examination of the shear stress resultant concerning the horizontal position across the specified interval. Collectively, these figures contribute to a comprehensive understanding of the shearing and torsional behaviors within the system under investigation.

5 Conclusion

This paper introduces a comprehensive analytical solution for the uniform torsion analysis applied to a non-homogeneous orthotropic right-angled triangle cross-section. The analytical approach presented in this study serves as a valuable benchmark for validating solutions derived through various numerical methods. Notably, it provides a means to assess the accuracy and reliability of results obtained using techniques such as the finite element method, [6], finite difference method, [5], collocation method, and other numerical approaches commonly employed in the analysis of orthotropic non-homogeneous beams.

By offering a closed-form analytical solution, this work contributes to the validation and verification processes in structural mechanics, allowing researchers and practitioners to corroborate numerical findings against an established analytical framework. Such validation is crucial for ensuring the fidelity of numerical simulations and enhancing confidence in the results obtained through computational methods. The analytical solution proposed herein not only adds to the theoretical foundation of uniform torsion in non-homogeneous orthotropic materials but also facilitates a comparative analysis with numerical outcomes, fostering a deeper understanding of the structural behavior of these specialized cross-sections.

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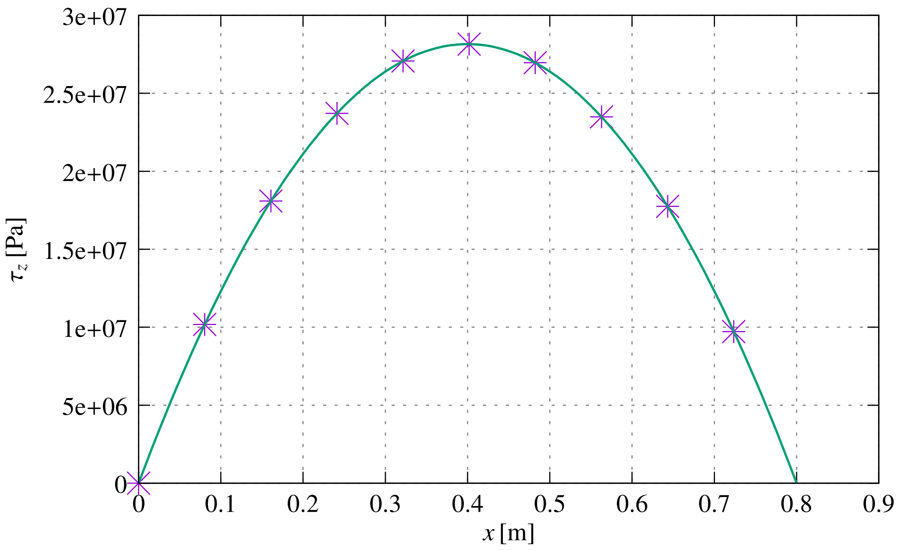

Fig. 5: The plot of shearing stress resultant on the axis of x for $0 \leq x \leq a$.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The examination and formal analysis were conducted by István Ecsedi and Attila Baksa. István Ecsedi took charge of implementing the algorithm for the example, while Attila Baksa assumed responsibility for the validation and visualization of the results. The collaborative effort of all authors is evident in the writing process, encompassing the creation of the original draft, thorough review, and meticulous editing of the paper.

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