Torsion of the Truncated Hollow Orthotropic Elastic Body of Rotation

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Abstract: - This paper deals with the torsion of the body of rotation. The meridian section of the body is bounded by two ellipses and two straight lines which are perpendicular to the axis of rotation of the body. The material of the body is elastic and cylindrical orthotropic. To solve the torsion problem, the theory of the torsion of shafts of varying circular cross-sections is used, which was developed by Mitchell and Töppl. An analytical solution is given for the shearing stresses and circumferential displacement. A numerical example illustrates the application of the presented analytical solution. The results of this paper can be used as a benchmark solution to verify the accuracy of the results computed by finite element simulations and finite different methods.

Key-Words: - Torsion of body of rotation, orthotropic, elastic, variable cross-section

Received: March 12, 2023. Revised: August 12, 2023. Accepted: September 13, 2023. Published: October 13, 2023.

1 Introduction

The torsional deformation of a body of rotation is a very important topic in the mechanics of structures, [1], [2], [3]. The book, [4], and the book, [5], presents many works on the torsion of elastic shafts of varying circular cross section. For a body of rotations whose boundary surfaces are coordinate surfaces of an orthogonal curvilinear coordinate system the closed-form solutions are derived from the torsional boundary value problem, [6]. There are several works on the problem of the torsion deformations of elastic bodies of rotation, [7], [8], [9], [10]. It is not the aim of this paper to provide a detailed list of the literature on this topic. In this paper, the torsion of the hollow truncated body of rotation is considered. The meridian section of the orthotropic elastic body is bounded by two ellipses and two straight lines that are perpendicular to the axis of rotation of the body. The formulation of the torsional boundary value problem is given in the cylindrical coordinate system $Or\varphi z$ (Figure 1). The meridian section of the truncated hollow body of rotation is shown in Figure 2. The body of rotation occupies the space domain B in the three-dimensional space

$$B = \left\{ \begin{matrix} (r,\varphi,z) | -L \leq z \leq L, \\ R_1(z) \leq r \leq R_2(z), 0 \leq \varphi \leq 2\pi \end{matrix} \right\}, \quad (1)$$

where

$$R_1(z) = \sqrt{a_{44}(t_1 - \frac{z^2}{a_{55}})}, \quad (2)$$

$$R_2(z) = \sqrt{a_{44}(t_2 - \frac{z^2}{a_{55}})}, \quad (3)$$

$$t = \frac{r^2}{a_{44}} + \frac{z^2}{a_{55}}, \quad (4)$$

$$t_1 = \frac{R_1^2}{a_{44}} + \frac{z^2}{a_{55}}, \quad t_2 = \frac{R_2^2}{a_{44}} + \frac{z^2}{a_{55}}. \quad (5)$$

In equations (2), (3), (4) and (5) a_{44}, a_{55} are the shear flexibility coefficients of the material of the body of rotation, respectively.

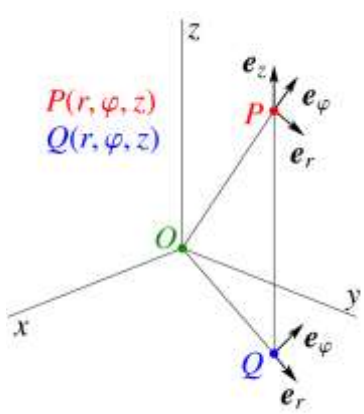

Fig. 1: *Cylindrical coordinates* (r,φ,z).

The boundary curve of the body B is ∂A (see Figure 2)

$$\partial A = \partial A_1 \cup \partial A_2 \cup \partial A_3 \cup \partial A_4, \quad (6)$$

$$\partial A_1 = \{(r,z) | r = R_1(z), -L \leq z \leq L\}, \quad (7)$$

$$\partial A_2 = \{(r,z) | r = R_2(z), -L \leq z \leq L\}, \quad (8)$$

$$\partial A_3 = \{(r,z) | R_1(z) \leq r \leq R_2(z), z = L\}, \quad (9)$$

$$\partial A_4 = \{(r,z) | R_1(z) \leq r \leq R_2(z), z = -L\}, \quad (10)$$

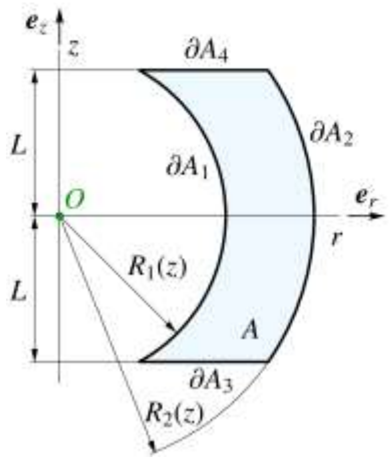

Fig. 2: Meridian section of the truncated hollow body of rotation

2 Formulation of the Problem

Michell-Föppl's theory of the torsion of shafts of varying circular cross-sections is based on the following displacement field, [1], [2], [3], [4], [7].

$$\boldsymbol{v}(r,\varphi,z) = v(r,\varphi,z)\boldsymbol{e}_\varphi \quad (r,\varphi,z) \in B. \quad (11)$$

The unit vectors of the cylindrical coordinate system are denoted by $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_z$ (Figure 1). The non-zero infinitesimal strains are the shearing strains $\gamma_{r\varphi}$ and $\gamma_{\varphi z}$

$$\gamma_{r\varphi} = \frac{\partial v}{\partial r} - \frac{v}{r} = r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = r\frac{\partial \psi}{\partial r}, \quad (12)$$

$$\gamma_{\varphi z} = \frac{\partial v}{\partial z} = r\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = r\frac{\partial \psi}{\partial z}, \quad (13)$$

where

$$\psi(r,z) = \frac{v(r,z)}{r}. \quad (14)$$

For orthotropic homogeneous linearly elastic material according to Hooke's law, we have for the shearing stresses $\tau_{r\varphi}$ and $\tau_{\varphi z}$

$$\tau_{r\varphi} = a_{55}\gamma_{r\varphi}, \quad \tau_{z\varphi} = a_{44}\gamma_{z\varphi}. \quad (15)$$

Since there are no body forces, the condition of the mechanical equilibrium is described by the following stress equilibrium equation (Figure 2)

$$\frac{\partial \tau_{r\varphi}}{\partial r} + 2\frac{\tau_{r\varphi}}{r} + \frac{\partial \tau_{rz}}{\partial z} = 0 \quad (r,z) \in A. \quad (16)$$

Let the general solution of equilibrium equation (9) in terms of first-order stress function $U = U(r,z)$ can be represented as

$$\tau_{r\varphi} = -\frac{1}{r^2}\frac{\partial U}{\partial z}, \quad \tau_{z\varphi} = \frac{1}{r^2}\frac{\partial U}{\partial z}. \quad (17)$$

A combination of equations (12), (13), (15) and (16) gives

$$a_{44}\frac{\partial^2 U}{\partial r^2} - a_{44}\frac{3}{r}\frac{\partial U}{\partial r} + a_{55}\frac{\partial^2 U}{\partial z^2} = 0 \quad (r,z) \in A \quad (18)$$

The solution of this partial differential equation which satisfies the boundary condition

$$U(r,z) = 0 \quad (r,z) \in A_1, \quad (19)$$

$$U(r,z) = \frac{T}{2\pi} \quad (r,z) \in A_2 \quad (20)$$

is as follows

$$U(r,z) = \frac{T}{2\pi}\frac{\left(\frac{r^2}{a_{44}} + \frac{z^2}{a_{55}}\right)^{\frac{3}{2}} - t_1^{\frac{3}{2}}}{t_2^{\frac{3}{2}} - t_1^{\frac{3}{2}}} \quad (r,z) \in A \cup \partial A. \quad (21)$$

Application of formula (17) gives for the shearing stresses

$$\tau_{r\varphi}(r,z) = -\frac{3T}{2\pi}\frac{z\sqrt{\frac{r^2}{a_{44}} + \frac{z^2}{a_{55}}}}{a_{55}r^2\left(t_2^{\frac{3}{2}} - t_1^{\frac{3}{2}}\right)}, \quad (22)$$

$$\tau_{z\varphi}(r,z) = -\frac{3T}{2\pi}\frac{\sqrt{\frac{r^2}{a_{44}} + \frac{z^2}{a_{55}}}}{a_{44}r\left(t_2^{\frac{3}{2}} - t_1^{\frac{3}{2}}\right)}. \quad (23)$$

The resultant of shearing stress

$$\tau_\varphi(r,z) = \sqrt{\tau_{r\varphi}^2 + \tau_{z\varphi}^2} = \frac{3T}{2\pi}\frac{\sqrt{(a_{44}z^2 + a_{55}r^2)(a_{44}^2 z^2 + a_{55}^2 r^2)}}{\sqrt{a_{44}^3 a_{55}^3}\left(t_2^{\frac{3}{2}} - t_1^{\frac{3}{2}}\right)r^2}. \quad (24)$$

Determination of the function $\psi = \psi(r,z)$ is based on the following equation

$$\frac{\partial \psi}{\partial r} = -\frac{1}{a_{55}r^3}\frac{\partial U}{\partial z}, \quad \frac{\partial \psi}{\partial z} = \frac{1}{a_{44}r^3}\frac{\partial U}{\partial r}. \quad (25)$$

The solution of the system of partial differential equations (25) for $\psi = \psi(r,z)$

$$\psi(r,z) = \frac{3T}{4\pi} \frac{z\sqrt{\frac{r^2}{a_{44}} + \frac{z^2}{a_{55}}}}{\left(t_2^{3/2} - t_1^{3/2}\right) z^2}$$

$$\frac{3T}{4\pi}\left[\frac{\ln\left(\frac{z}{\sqrt{a_{55}}}\sqrt{\frac{r^2}{a_{44}}+\frac{z^2}{a_{55}}}\right)}{\pi\sqrt{\frac{a_{44}^2}{a_{55}}}\left(t_2^{\frac{3}{2}}-t_1^{\frac{3}{2}}\right)r^2} - \frac{\ln\sqrt{\frac{r^2}{a_{44}}}}{\pi\sqrt{\frac{a_{44}^2}{a_{55}}}\left(t_2^{\frac{3}{2}}-t_1^{\frac{3}{2}}\right)}\right]. \quad (26)$$

The expression of the circumferential displacement $v = v(r,z)$ is

$$v(r,z) = r\,\psi(r,z). \quad (27)$$

3 Numerical Example

The following data is used in the numerical example

$$a_{44} = 0.02 \times 10^{-8}\,\frac{\mathrm{m}^2}{\mathrm{N}}, \quad a_{55} = 10^{-9}\,\frac{\mathrm{m}^2}{\mathrm{N}},$$

$$t_1 = 0.02 \times 10^{10}\ \mathrm{N}, \quad t_2 = 0.04 \times 10^{10}\ \mathrm{N},$$

$$T = 2000\ \ \mathrm{Nm}, \quad L = 0.5\ \mathrm{m}.$$

The contour lines of the stress function $U = U(r,z)$ and the function of $\psi = \psi(r,z)$ are shown in Figure 3 and Figure 4. The contour lines of the shearing stress are resultant and are presented in Figure 5.

A simple computation gives the following values of stress resultant

$$\tau_\varphi(R_1(0),0) = 65\,283.54652\ \mathrm{Pa} \quad (28)$$

$$\tau_\varphi(R_2(0),0) = 65\,283.54652\ \mathrm{Pa}, \quad (29)$$

$$\tau_\varphi(R_1(L),L) = 1.2309836 \times 10^5\ \mathrm{Pa}, \quad (30)$$

$$\tau_\varphi(R_2(L),L) = 1.2309836 \times 10^5\ \mathrm{Pa}, \quad (31)$$

4 Conclusions

An analytical solution is presented for the problem of torsion of the truncated body of rotation. The curved boundary surfaces of the considered body are ellipsoids of rotation. The material of the body of rotation is linearly elastic homogenous and orthotropic. It is assumed that the deformations are small and the formulation of the linearized theory of elasticity can be used. A numerical example illustrates the presented theory of the torsion of the orthotropic body of rotation.

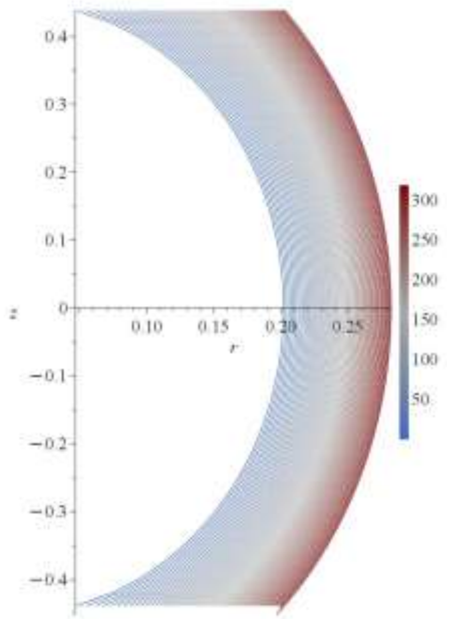

Fig. 3: *The contour lines of stress function $U(r,z)$.*

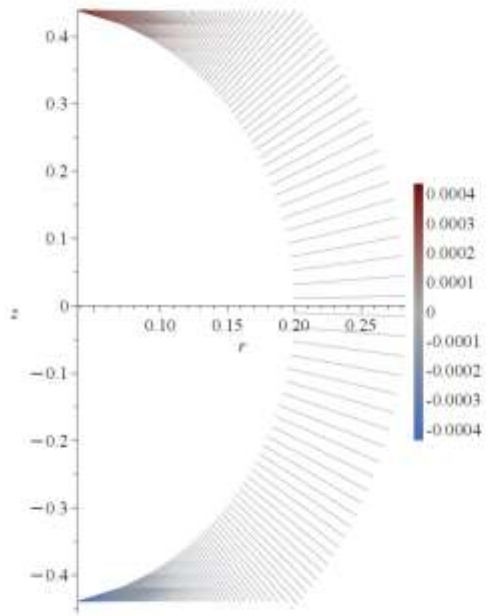

Fig. 4: The contour lines of the stress function $\psi(r,z)$.

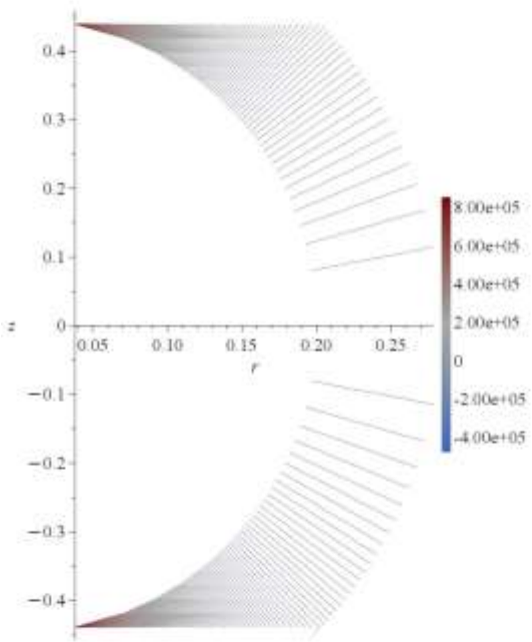

Fig. 5: The contour lines of the shear stress resultant.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- István Ecsedi and Attila Baksa carried out the investigation and the formal analysis. István Ecsedi has implemented the algorithm for all the examples.
- Attila Baksa and Marwen Habbachi were responsible for the validation and visualization of the results.

All authors have been writing the paper with original draft, review, and editing.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The author(s) received no financial support for the research, authorship, and/or publication of this article. Furthermore, on behalf of all authors, the corresponding author states that there is no conflict of interest.

Conflict of Interest

The authors have no conflict of interest to declare.