A Special Note on the Error Functions Defined in Certain Domains of the Complex Plane and Some of Their Implications

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Abstract: - The fundamental object of this specific research is firstly to introduce certain requisite information in relation to the error functions (with complex (or real) parameters), which possess various extensive roles and responsibilities in applications of science, engineering, and technology, next to determine various consequences of those complex functions and also to reveal (or point out) some of those possible consequences. Finally, to present certain recommendations about the possible extent of the scope of this special research note for the relevant researchers.

Key-Words: - Special functions, the complex error functions, z-plane, analytic functions, complex series expansions, complex inequalities, complex powers.

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1 Introduction and Necessary Basic Information

As it is well known from published literature, special functions (together with integral transformations identified by special functions) undertake a wide range of roles associated with various extensive subjects of mathematical analysis, the theory of differential and integral equations, approximation theory, and to many other fields of pure and applied mathematics for nearly all sciences. To see some of those relevant roles and their scope, it will be sufficient to take a look at the fundamental resources given in references in [1], [3], [7], [9], [12], [13], [20], [28], [30], [32], [34], [37].

Naturally, as centuries old, these fundamental subjects are under intense development, for use, especially in pure and applied mathematics, statistics, physics, engineering, and computer science. This also stimulates continuous interest for relevant researchers in those fields. Shortly, the main aim of those functions (and transformations specified special functions) is to foster further growth by providing a means for the scientific publication of important research on all aspects of those subjects. For those special functions, transformations, and also some of their extensive applications, one can also look over a great deal of properties, extensive relations, elementary results, various implications, and also theories pointed out, [3], [7], [11], [13], [14], [17], [18], [21], [23], [25], [28], [30], in the references of this paper.

In particular, the classical error functions are well-known functions as some special types in the families consisting of all special functions with complex (or real) parameters. In the written literature, we generally encounter these special functions with real parameters. But, in this special-scientific research, the error functions with complex parameters will be considered for our possible-special results. Specially, for the concerned researchers, we offer the earlier papers given in [3], [4], [5], [6], [18], [19], [26], [33], [38].

As well as the basic theory in relation to error functions (or analysis), these special functions are also made allowances for probability, statistics, applied mathematics, mathematical physics, and a vast number of other theoretical (or practical) applications. For instance, the Fresnel integrals, which are derived with the help of those functions, are very important functions and they are also used in the theory of optics. For some of those special forms, one may also examine [1], [7], [9], [11].

In particular, we point out here that the main purpose of this scientific note is firstly to consider these error functions, then to put forward some basic theories in relation to those special functions, and also to focus on some of their possible implications. For elementary results associated with those error functions and also various implications (or extra applications) of their different forms, one may also refer to the academic studies presented in [2], [8], [12], [15], [16], [19], [22], [32], [33], [34], [35], [36], [38], [39].

For this scientific research, as extra information, there is a need to introduce various basic notations, definitions, and also lemmas.

Let us now begin to inform our readers about that special information.

First of them, as is known, by the familiar notations given by

$$\mathbb{R} \quad \text{and} \quad \mathbb{C},$$

are the set of the real numbers and the set of the complex numbers, respectively.

In addition, by the best-known notation $\mathbb{U}$, we describe the open unit disk in the complex plane, which is of the form given by

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

Let also

$$\mathbb{C}^* := \mathbb{C} - \{0\}$$

and

$$\mathbb{R}^* := \mathbb{R} - \{0\}.$$

Second of all, the complex error function is represented by the notation $\text{erf}(z)$ and also identified by the integral given by

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds. \tag{1}$$

In the same time, in general, as the second definition of the complex function just above, it is defined by using the MacLaurin series expansion of the function with the variable:

$$f(s) = e^{-s^2},$$

which is given by

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)\ell!} z^{2\ell+1}, \tag{2}$$

where $z \in \mathbb{C}$.

Additionally, the complex-complementary error function is symbolized by the notation $\text{erfc}(z)$ and described by the integral given by

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2} ds. \tag{3}$$

Moreover, as its second definition of that function, it is also defined by considering the MacLaurin series expansion of the mentioned function with the complex parameter z, which is given by

$$\text{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)\ell!} z^{2\ell+1}, \tag{4}$$

where $z \in \mathbb{C}$.

More particularly, by considering the well-known result given by

$$\int_0^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$$

and the relation between the definitions precisely defined as in (1) and (2), the special relationships are given by

$$\begin{aligned} \text{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-s^2} ds - \int_z^{\infty} e^{-s^2} ds \right) \\ &= 1 - \text{erfc}(z) \end{aligned} \tag{5}$$

can be easily received, where $z \in \mathbb{C}$.

We privately note here that, for the complex error functions and also various comprehensive relations between those special functions, one can look over the remarkable studies, which also possess different type results, in the references given by [1], [3], [4], [5], [6], [10], [17], [19], [26], [29], [38], [39].

2 Lemmas and Main Results and Related Consequences

This section is related to our main results and some of their possible consequences. Two lemmas will be important for our research, which are also introduced just below. Those are required for setting and also proving our main results, which will be determined by certain types of complex error functions introduced between (1)-(5). In particular, as theoretical information, one may also check the basic investigations presented in [17], [24], [27], for the auxiliary theorems, which are Lemma 2.1 and Lemma 2.2, respectively. In particular, we offer to center on some of the earlier results given by [18], for our readers.

Lemma 2.1. Let $\omega \in \mathbb{C}^*$ and $w \in \mathbb{C}$. Then, the assertion is given by

$$\omega^w = |\omega|^{\Re e(w)}\, e^{-\operatorname{Arg}(\omega)\,\Im m(w)} \times e^{i[\Re e(w)\operatorname{Arg}(\omega)+\Im m(w)\log|\omega|]} \tag{6}$$

holds.

Lemma 2.2. Let $\omega \in \mathbb{U}$ and also let $\Delta(\omega)$ be an analytic function in the domain $\mathbb{U}$ and satisfy the condition $\Delta(0) = 1$. If there exists a point ω_0 in $\mathbb{U}$ such that

$$\Re e\{\Delta(\omega)\} > 0 \quad \text{for } |\omega| < |\omega_0| < 1 \tag{7}$$

and

$$\Delta(\omega_0) \neq 0 \quad \text{and} \quad \Re e\{\Delta(\omega_0)\} = 0, \tag{8}$$

then

$$\Delta(\omega_0) = iv \quad \text{and} \quad \omega_0\Delta'(\omega_0) = i\Omega\Delta(\omega_0), \tag{9}$$

where

$$u \geq \frac{1}{2},\ v \in \mathbb{R}^*,\ \Omega := u\left(v + \frac{1}{v}\right). \tag{10}$$

Theorem 2.1. If any one of the cases of the assertion given by

$$\left|\left[z\frac{d}{dz}\{\operatorname{erfc}(z)\}\right]^w\right| \begin{cases} < \dfrac{e^{-\pi\,\Im m(w)}}{2^{\Re e(w)}} & \text{if } \Re e(w) \geq 0 \\ \geq \dfrac{e^{-\pi\,\Im m(w)}}{2^{\Re e(w)}} & \text{if } \Re e(w) \leq 0 \end{cases} \tag{11}$$

is satisfied, then the assertion given by

$$\Re e\{\operatorname{erfc}(z)\} > 0 \tag{12}$$

is satisfied, where $z \in \mathbb{U}$ and $w \in \mathbb{U}$ and the main value of the power just above is taken cognizance of its principal ones.

Proof. For the proof of the theorem above, we want to use Lemma 2.2 together with Lemma 2.1 here. Firstly, by considering the definition in (4), let us now define a function $\Delta(z)$ being of the form given by

$$\Delta(z) = \operatorname{erfc}(z), \tag{13}$$

where $z \in \mathbb{U}$.

It is quite obvious that both the function $\Delta(z)$ just above is analytic in the complex domain $\mathbb{U}$ and it satisfies the condition $\Delta(0) = 1$ as it has been stated in Lemma 2.2. Moreover, the differentiating of both sides of the statement given in (13) easily gives us that

$$z\frac{d}{dz}\{\Delta(z)\} = z\frac{d}{dz}\{\operatorname{erfc}(z)\}$$

and

$$\left[z\frac{d}{dz}\{\operatorname{erfc}(z)\}\right]^w = \left[z\frac{d}{dz}\{\Delta(z)\}\right]^w, \tag{14}$$

where $w \in \mathbb{C}$, $z \in \mathbb{U}$ and each one of the main values of the complex power located in Theorem 2.1 is made account of their principal ones.

Now we suppose that there exists a point like z_0 in $\mathbb{U}$ satisfying the conditions indicated as in (8), which consist of the mentioned hypotheses in Lemma 2.2. Then, under the conditions presented by (10), in the light of the mentioned assertion given in (6) and the special information given in (9), from (14), it follows the relationships presented by

$$\begin{aligned} \left[z\frac{d}{dz}\{\operatorname{erfc}(z)\}\right]^w\Big|_{z:=z_0} &= [z\Delta'(z)]^w\big|_{z:=z_0} \\ &= [z_0\Delta'(z_0)]^w \\ &= [i\Omega\Delta(z_0)]^w, \end{aligned} \tag{15}$$

where $w \in \mathbb{C}$ and $z \in \mathbb{U}$.

Indeed, through the instrumentality of the mentioned information given in (9) and (10), from assertion (15), one then arrives at:

$$\begin{aligned} \left|\left[z_0\frac{d}{dz}\{\operatorname{erfc}(z_0)\}\right]^w\right| &= \left|[i\Omega\Delta(z_0)]^w\right| \\ &= \left|[-u(v^2+1)]^w\right| \\ &= \Big||-u(v^2+1)|^{\Re e(w)} \\ &\quad \times e^{-\operatorname{Arg}[-u(v^2+1)]\,\Im m(w)+i\Theta}\Big| \end{aligned}$$

$$= \left| [u(v^2+1)]^{\Re\mathrm{e}(\mathrm{w})} \times e^{-\pi\,\Im m(\mathrm{w})+i\Theta} \right|$$

$$= [u(v^2+1)]^{\Re\mathrm{e}(\mathrm{w})} \times e^{-\pi\,\Im m(\mathrm{w})}$$

$$\begin{cases} \geq \frac{e^{-\pi\,\Im m(\mathrm{w})}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \geq 0 \\ < \frac{e^{-\pi\,\Im m(w)}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \leq 0 \end{cases}, \qquad (16)$$

where

$$u \geq \frac{1}{2} \quad , \quad v \in \mathbb{R}^* \quad , \quad w \in \mathbb{C}$$

and

$$\Theta := \pi\,\Re\mathrm{e}(\mathrm{w}) + \ln[u(v^2+1)]\,\Im m(\mathrm{w}).$$

Nonetheless, as centering on each one of the mentioned cases, which express the hypotheses of Theorem 2.1 constituted in (11), it is clearly seen that each one of those cases determined as in (16) is in contradiction with each one of those inequalities given by (11), respectively.

Hence, this says us that there is no point like z_0 belonging to $\mathbb{U}$, which satisfies the condition of the mentioned theorem given by (8). This means that the inequality given by (7) holds for all points z belonging to $\mathbb{U}$.

Therefore, the assertion constituted as in (13) immediately requires the inequality given by (12), which is the provision of Theorem 2.1. Hereby, this completes the related proof of Theorem 2.1.

As it has been indicated in the part of the introduction of the first section, there are many relationships (or many special functions) in associating with the (complex) error functions. For those functions, as some examples, especially, one may look over some of them given by the earlier investigations given in [1], [3], [7], [9], [18], [20], [21], [29], [31], [39].

In addition, for those many special relationships or functions highlighted there, a number of special conclusions (or suggestions) that may be relevant to this particular study may be of interest to our researchers. From this point of view, only one of them has been presented here, which includes the relation between those complex error functions given by (5). Of course, it is possible to expose others. Now we want to inform relevant researchers about some of them.

As the first consequence of this extensive investigation, all fundamental results (or each one of their possible implications) can be reconstituted as their equal forms by taking into consideration any one of various complex series expansions of those complex error functions which were presented in (2) and (4).

The important roles of the related results containing such series expansions can also be very important, for example, in approximate calculations. For some of them, one may check the different-type results presented in [2], [15], [20], [21], [22], [23], [36], [39].

Moreover, as an elementary-special example, by considering the information presented by Lemma 2.1 and also making use of the basic form of the series expansion given by (4), one of the implications, namely, Theorem 2.1 can be then reconstituted as in the form given by theorem just below.

Theorem 2.2. If any one of the cases of the assertion given by

$$\left(\frac{2}{\sqrt{\pi}}\right)^{\Re e(w)} \left| \left[\sum_{\ell=0}^{\infty} \frac{(-1)^{1+\ell}}{\ell!}\, z^{2\ell+1} \right]^{\mathrm{w}} \right|$$

$$\begin{cases} < \frac{e^{-\pi\,\Im m(\mathrm{w})}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \geq 0 \\ \geq \frac{e^{-\pi\,\Im m(\mathrm{w})}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \leq 0 \end{cases}$$

is occurred, then the assertion is given by

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} \Re e\left\{ z^{2\ell+1} \right\} < \frac{\sqrt{\pi}}{2}$$

is occurred, where $z \in \mathbb{U}$, $\mathrm{w} \in \mathbb{C}$, and the main value of the complex power (just above) are allowed as its principal ones.

As the second consequence of this comprehensive research, in consideration of those relationships between the error functions which are introduced by the forms given in (1)-(5), a large number of related consequences can be also represented (or determined) for relevant researchers.

In specially, for some of them, by considering one of the well-known inequalities which are given by

$$\Re e(z) \leq |\Re e(z)| \leq |z| \qquad (17)$$

and

$$\Im m(z) \leq |\Im m(z)| \leq |z|\,, \qquad (18)$$

or, by using the special relation specified by (5), namely, which is:

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)\,,$$

some of the possible implications of those theorems just above can be next represented for some z belonging to the set $\mathbb{U}$. Since the desired proof of each of them can be easily created, they are omitted here.

Theorem 2.3. If any one of the cases of the statement presented by

$$\left|\left[z\frac{d}{dz}\{1-\mathrm{erf}(z)\}\right]^{\mathrm{w}}\right| \equiv \left|\left[-z\frac{d}{dz}\{\mathrm{erf}(z)\}\right]^{\mathrm{w}}\right| \begin{cases} < \dfrac{e^{-\pi\,\Im m(\mathrm{w})}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \geq 0 \\ \geq \dfrac{e^{-\pi\,\Im m(\mathrm{w})}}{2^{\Re\mathrm{e}(\mathrm{w})}} & \text{if } \Re\mathrm{e}(\mathrm{w}) \leq 0 \end{cases}$$

is true, then the assertion given by

$$\Re e\{\mathrm{erf}(z)\} < 1$$

is true, where $z \in \mathbb{U}$, $\mathrm{w} \in \mathbb{U}$ and the value of the complex power is taken into account as its principal ones.

Finally, as more special consequences of our main results or certain recommendations for the relevant researchers, by selecting the possible values of the parameters used in all our results, it also possesses the admissible potential to uncover (or reconstitute) a vast number of those special results. Here we also want to present only two of those special implications.

In Theorem 2.3, with the help of the information relating to the complex exponential presented by the form in (6) and also by selecting the value of the mentioned parameter w as in $w := \Omega + i0$, where the parameter Ω belongs to the mentioned set $\mathbb{R}^*$, as one of the other implications, the following Proposition 2.1 can be firstly reobtained as one of the special consequences in relation with those complex error functions.

Proposition 2.1. If any of the cases of the assertion given by

$$\left|\left[-z\frac{d}{dz}\{\mathrm{erf}(z)\}\right]^{\Omega}\right| \equiv |z\,\mathrm{erf}'(z)|^{\Omega} \begin{cases} < 2^{-\Omega} & \text{when } \Omega > 0 \\ \geq 2^{-\Omega} & \text{when } \Omega < 0 \end{cases}$$

holds true, then the assertion given by

$$\Re e\{\mathrm{erf}(z)\} < 1,$$

holds true, where $z \in \mathbb{U}$ and the main value of the power just above are made allowances for its principal ones.

As has been indicated before, in the light of the mentioned information in connection with the well-known inequalities constituted as in (17) and (18) and by considering the sensible-real values of the mentioned parameter w, it can be also re-determined various comprehensive-special implications of our essential results above. For an extra exmaple, by selecting the value of the mentioned parameter w as $w := 1$ in Proposition 2.1, the following-two-more-special consequences can be then represented as one of the various possible implications of this special investigation.

Proposition 2.2. If any one of the implications given by

$$\Re e\{z\,\mathrm{erf}'(z)\} \leq |\Re e\{z\,\mathrm{erf}'(z)\}| < \frac{1}{2}$$

and

$$\Im m\{z\,\mathrm{erf}'(z)\} \leq |\Im m\{z\,\mathrm{erf}'(z)\}| < \frac{1}{2}$$

is provided, then the implication is given by

$$\Re e\{\mathrm{erf}(z)\} < 1$$

is provided, where $z \in \mathbb{U}$.

In addition, in view of the mentioned information pointed out in (17) and (18) and by making use of the series expansion of the complex-error function presented in (4) for the function with the complex variable z, which has been constituted as in Proposition 2.2, the following special consequence can be last reconstituted.

Proposition 2.3. If any one of the inequalities given by

$$\Re e\left\{\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\,z^{2\ell+1}\right\} \leq \left|\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\,\Re e\{z^{2\ell+1}\}\right| < \frac{\sqrt{\pi}}{4}$$

and

$$\Im m\left\{\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}z^{2\ell+1}\right\} \leq \left|\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\,\Im m\{z^{2\ell+1}\}\right|$$

$$< \frac{\sqrt{\pi}}{4}$$

holds, then the inequality given by

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} \Re e\{z^{2\ell+1}\} < \frac{\sqrt{\pi}}{2}$$

also holds, where $z \in \mathbb{U}$. Of course, for the complex powers just above, the values of their values are considered as their principal ones.

3 Concluding Remarks

In this special section, some reminders, special information, and extra suggestions will be mentioned. Since this comprehensive scientific research note is directly concerned with the error functions (with complex parameters), certain necessary information in relation to those special functions has been first introduced in the first chapter and some theorems together with two main lemmas have been then presented in the second chapter.

In addition, it has been concentrated on some of the meaningful implications of our fundamental results and various recommendations, which will be related to theoretical and applied applications of those error functions together with various special functions. Shortly, we think that all of our main results and their possible consequences will be interesting for our readers.

More particularly, as one of the extra suggestions for interested researchers, we would also like to remind those about the definition of one type of the other error functions, which is known as the imaginary error function in the literature and its series expansion in the complex form.

The imaginary error function with complex variable is denoted by

$$\mathrm{erf}(z) \quad (z \in \mathbb{C})$$

and it is also defined by

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{s^2} ds \ ,$$

and, by using this complex function just above, its series expansion series can be also determined by the form given by

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)\ell!} \ z^{2\ell+1} \ ,$$

where $z \in \mathbb{U} \subset \mathbb{C}$.

As a final word of this special section, we would also like to emphasize that this special function erfi(z) with complex variable z introduced above can be re-evaluated within the scope of our main results. At the same time, each one of its relevant potential-specific results, which can be obtained (or revealed), will also lead to various specific conclusions, as we did in the second chapter. Of course, for each of them, it will be necessary to realize that various special efforts should be made by the researchers who are interested in those relevant scientific fields.

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