Embedded Spaces of Hermite Splines

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Abstract: - This paper is devoted to the processing of large numerical signals which arise in different technical problems (for example, in positioning systems, satellite maneuvers, in the prediction a lot of phenomenon, and so on). The main tool of the processing is polynomial and nonpolynomial splines of the Hermite type, which are obtained by the approximation relations. These relations allow us to construct splines with approximate properties, which are asymptotically optimal as to N-width of the standard compact sets. The interpolation properties of the mentioned splines are investigated. Such properties give opportunity to obtain the solution of the interpolation Hermite problems without solution of equation systems. The calibration relations on embedded grids are established in the case of deleting the grid knots and in the case of the addition of the last one. A consequence of the obtained results is the embedding of the Hermite spline spaces on the embedded grids. The mentioned embedding allows us to obtain wavelet decomposition of the Hermite spline spaces.

Key-Words: - polynomial splines, non-polynomial splines, Hermite problem

1 Introduction

Many technical studies are associated with large volumes of source information, usually presented as a sequence of discrete samples of a gigantic volume (10^6 -10^{12} numbers). The tasks of extracting useful information from such sequences are very difficult tasks. The extracted information (called the main stream) has a significantly smaller volume and can easily be transferred from source to recipient. The recipient can evaluate the degree of usefulness of this information and decide on the need to obtain the original stream or part thereof. Such a problem arises in measurements in positioning systems [1], in flight control systems of aircraft [2], and in the processing and registration of tactile images [3]. This problem also arises in more mundane things, for example, when processing signals from car suspensions [4]. When processing images use the fast discrete Fourier transform [5], the problem of quickly transmitting basic information also arises. Large amounts of numerical information are used in the design of satellites [7], the solution of their maneuvering using gravity [8], and among planets and clouds of various particles [11] - [12]. A similar problem arises in the prediction of various phenomena (see, for example, [9] - [10]).. As mentioned above, the solution to this problem is to form a much shorter sequence (main stream) in which the main information is concentrated. Taking into account additional characteristics of the initial stream allows improving the information content of the main stream. Such characteristics may be the rate of change of the original stream, expressed by a sequence of different relations or a sequence of values of the derivative of the original analog signal. It is the use of the mentioned values that is proposed in this paper. The approach under consideration leads to a Hermitian approximation of the first height.

Recently, much attention has been paid to the application of wavelets to solving various problems. As it is known, the wavelet-Galerkin method is a useful tool for solving differential equations mainly because the conditional number of the stiffness matrix is independent of the matrix size and thus the number of iterations for solving the discrete problem by the conjugate gradient method is small. The authors of paper [13] have recently proposed a quadratic spline wavelet basis that has a small conditional number and short support. The authors used this basis in the Galerkin method to solve the second-order elliptic problems with Dirichlet boundary conditions in one and two dimensions. They achieve the L_2-error of order $O(h^4)$, where h is the step size, by an appropriate post-processing. The rate of convergence is the same as the rate of convergence for the Galerkin method with cubic spline wavelets. They show theoretically, as well as numerically, that the presented method outperforms the Galerkin method with other quadratic or cubic spline wavelets. Furthermore, they present local post-processing for example of the equation with Dirac measure on the right-hand side.

In paper [14], to solve the problem of the high gradient adaptive analysis of the ship straight structure, a meshless local Petrov-Galerkin method based on a B-spline wavelet is proposed. The approximation function of the structural displacement field quantity is solved by employing the least squares method and the weighted residual method, and the governing equation and stiffness equation were established. Based on the meshless local Petrov-Galerkin method, an m-order B-spline function is used as the wavelet basis function to construct the approximation function of the ship structure displacement field, and a two-scale decomposition technology is used to decompose the high gradient component and the low scale component in the stress field. The high scale component is used to express the high gradient component in the stress field.

Composite materials, with characteristics of light weight and high strength, are useful in manufacturing. Therefore, precise design and analysis is the first key procedure in composite applications. Improper analysis or use of composite materials may cause serious failures. In paper [15], the wavelet finite element method (WFEM) based on B-spline wavelet on the interval (BSWI) is constructed for the precise analysis of laminated plates and shells, which gives a guidance in design and application of composite structures. First, FEM formulations are derived from the generalized potential energy function based on the generalized variational principle and virtual work principle. Then, BSWI scaling functions are used as an interpolation function to discretize the solving displacement field variables. At the same time, a transformation matrix is constructed and used to translate the meaningless wavelet coefficients into physical space. At last, the static analysis results can be obtained by solving the FEM formulations.

The authors of paper [16] present a new biorthogonal wavelet transform using splines performed in a 'lifting' manner. Specifically, polynomial splines of different order were used in the lifting constructions. They study the influence of the order of filters using polynomial splines on image compression in order to choose the best wavelet transforms. In addition, a comparative study of these transforms is done firstly with the biorthogonal B9/7 transform which is frequently used in image compression and secondly with the existing B-spline based transforms. They show through experimental results that their proposed wavelet transforms outperform the existing ones in image compression.

In detailed aerodynamic design optimization, a large number of design variables in geometry parameterization are required to provide sufficient flexibility and obtain the potential optimum shape. However, with the increasing number of design variables, it becomes difficult to maintain the smoothness on the surface which consequently makes the optimization process progressively complex. In paper [17], smoothing methods based on B-spline functions are studied to improve the smoothness and design efficiency. The wavelet smoothing method and the least square smoothing method are developed through coordinate transformation in a linear space constructed by B-spline basis functions. In these two methods, smoothing is achieved by a mapping from the linear space to itself such that the design space remains unchanged. A design example is presented where aerodynamic optimization of a supercritical airfoil is conducted with smoothing methods included in the optimization loop. Affirmative results from the design example confirm that these two smoothing methods can greatly improve quality and efficiency compared with the existing conventional non-smoothing method.

In paper [18], Hermite wavelets are used to develop a numerical procedure for numerical solutions of two-dimensional hyperbolic telegraph equation. In the first stage, the author rewrote the second order hyperbolic telegraph equation as a system of partial differential equations by introducing a new variable and then using finite difference approximation author discretized time-dependent variables. After that, the Hermite wavelets series expansion is used for discretization of space variables. With this approach, finding the solution of a two-dimensional hyperbolic telegraph equation is transformed to finding the solution of two algebraic system of equations. The solution of these systems of algebraic equations gives Hermite wavelet coefficients. Then by inserting these coefficients into the Hermite wavelet series expansion, numerical solutions can be acquired consecutively. The main goal of this paper is to indicate that the Hermite wavelet-based method is suitable and efficient for a two-dimensional hyperbolic telegraph equation as well as other types of hyperbolic partial differential equations such as wave and sinh-Gordon equations. The obtained results corroborate the applicability and efficiency of the proposed method.

To numerically solve the Burgers' equation, in this paper [19] the authors propose a general method for constructing wavelet bases on the interval [0,1] derived from symmetric biorthogonal multiwavelets on the real line. In particular, they obtain wavelet bases with simple structures on the interval [0,1] from the Hermite cubic splines. In comparison with all other known constructed wavelets on the interval [0,1], the authors constructed wavelet bases on the interval [0,1] from the Hermite cubic splines. The result not only has good approximation and symmetry properties with extremely short supports, but also employs a minimum number of boundary wavelets with a very simple structure. These desirable properties make them to be of particular interest in numerical algorithms. They constructed wavelet bases on the interval [0,1] which are then used to solve the nonlinear Burgers' equation. The method is based on the finite difference formula combined with the collocation method. Therefore, the proposed numerical scheme in this paper is abbreviated as MFDCM (Mixed Finite Difference and Collocation Method). Some numerical examples are provided to demonstrate the validity and applicability of the proposed method which can be easily implemented to produce a desired accuracy.

In [20] for cubic splines with nonuniform nodes, which split with respect to the even and odd nodes, these cubic splines are used to obtain a wavelet expansion algorithm in the form of the solution to a three-diagonal system of linear algebraic equations for the coefficients. Computations by hand are used to investigate the application of this algorithm for numerical differentiation. The results are illustrated by solving a prediction problem.

The overview presented here shows the importance of taking into account the smoothness for the discussed functions and their derivatives. This paper, discusses spaces of polynomial and nonpolynomial splines suitable for solving the Hermite interpolation problem (with first-order derivatives) and for constructing a wavelet decomposition. Such splines we call Hermitian type splines of the first level. The basis of these splines is obtained from the approximation relations under the condition connected with the minimum of multiplicity of covering every point of (α, β) (almost everywhere) with the support of the basis splines. Thus these splines belong to the class of minimal splines. This paper is ideologically similar to the papers [21], in which the spaces of the splines of the Lagrangian type are constructed.

Here we consider the processing of flows that include a stream of values of the derivative of an approximated function which is very important for good approximation. Also we construct a splash decomposition of the Hermitian type splines on a non-uniform grid.

The approximation and interpolation formulas for the stream under consideration are constructed. The obtained basis functions have compact support, and the addition of one node leads to an increase in the dimension of the spline space by two units (two basic wavelets are added to the previous basis). Sometimes to discuss the situation connected with a segment [a,b] ⊂ (α,β) is difficult.\Therefore we can solve this problem using our method by restricting all functions on the segment.

2 Splines of the Hermit Type

Let $\varphi(t) = ([\varphi]_0(t), [\varphi]_1(t), [\varphi]_2(t), [\varphi]_3(t))^T$ be a four-component vector function with components $[\varphi]_i(t)$ from space $C^1(\alpha,\beta)$, $i = 0, 1, 2, 3$. Let condition (A) be fulfilled:

$$\text{(A)} \qquad W(x,y;\varphi) \stackrel{\text{def}}{=} \det(\varphi(x), \varphi'(x), \varphi'(y)) \neq 0 \quad \forall x,y \in (\alpha,\beta), x \neq y.$$

Let X be set of nodes such that

$$X: \alpha < \cdots < x_{-1} < x_0 < x_1 < \cdots < \beta; \quad (1)$$

where $\alpha \stackrel{\text{def}}{=} \lim_{j\to-\infty} x_j, \beta \stackrel{\text{def}}{=} \lim_{j\to+\infty} x_j$.

Let us denote

$$G \stackrel{\text{def}}{=} \cup_{j\in\mathbb{Z}}(x_j, x_{j+1}), \varphi_j \stackrel{\text{def}}{=} \varphi(x_j), \varphi'_j \stackrel{\text{def}}{=} \varphi'(x_j).$$

Suppose functions $\omega_j(t)$, $t \in G$, $j \in \mathbb{Z}$, are the solution of the system of equations, which we call the approximation relations

$$\sum(\varphi'_{j+1}\omega_{2j-1}(t) + \varphi_{j+1}\omega_{2j}(t)) = \varphi(t). \quad (2)$$

Here we suppose that

$$\text{supp}\,\omega_{2j-1} \subset [x_j, x_{j+2}], \quad \text{supp}\,\omega_{2j} \subset [x_j, x_{j+2}] \forall j \in \mathbb{Z}. \quad (3)$$

We obtain from (2) - (3) for $t \in (x_k, x_{k+1})$ with fixed $k \in \mathbb{Z}$

$$\varphi'_k \omega_{2k-3}(t) + \varphi_k \omega_{2k-2}(t) + \\ + \varphi'_{k+1} \omega_{2k-1}(t) + \varphi_{k+1} \omega_{2k}(t) = \varphi(t). \quad (4)$$

Due to property (A), the solution of system (4) is unique. Let $t \in (x_k, x_{k+1})$. In this case the solution of the system can be written in the form:

$$\omega_{2k-3}(t) = \frac{\det(\varphi(t), \varphi_k, \varphi'_{k+1}, \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k-2}(t) = \frac{\det(\varphi'_k \varphi(t), \varphi'_{k+1}, \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k-1}(t) = \frac{\det(\varphi'_k, \varphi_k, \varphi(t), \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k}(t) = \frac{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi(t))}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})}.$$

Now if $k = q$, $k = q+1$, we get for any $q \in \mathbb{Z}$:

$$\omega_{2q-1}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi(t), \varphi_{q+1})}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}, \quad (5)$$

for $t \in (x_q, x_{q+1})$,

$$\omega_{2q-1}(t) = \frac{\det\left(\varphi(t), \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2}\right)}{\det\left(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2}\right)}, \quad (6)$$

for $t \in (x_{q+1}, x_{q+2})$,

$$\omega_{2q}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi(t))}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}, \quad (7)$$

for $t \in (x_q, x_{q+1})$,

$$\omega_{2q}(t) = \frac{\det(\varphi'_{q+1}, \varphi(t), \varphi'_{q+2}, \varphi_{q+2})}{\det(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}, \quad (8)$$

for $t \in (x_{q+1}, x_{q+2})$.

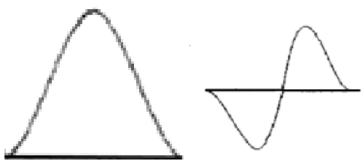

Figure 1: The plot of the basis spline $\omega_{2q-1}(t)$ (left), the plot of the basis spline $\omega_{2q}(t)$ (right)

Plots of the basis spline $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ are shown in Figure 1.

Theorem 1. Let $\varphi \in C^1(\alpha,\beta)$ and let condition (A) be fulfilled; then for any $q \in Z$ functions $\omega_{2q-1}(t)$ и $\omega_{2q}(t)$, which are given by (3) and (5) – (8), can be continued by continuity for the entire interval (α,β) to class functions $C^1(\alpha,\beta)$. In addition, the following relations are valid:

$$\omega_{2q-1}(x_q) = 0, \omega_{2q-1}(x_{q+1}) = 0, \omega_{2q-1}(x_{q+2}) = 0, \quad (9)$$

$$\omega'_{2q-1}(x_q) = 0, \omega'_{2q-1}(x_{q+1}) = 1, \ \omega'_{2q-1}(x_{q+2}) = 0, \quad (10)$$

$$\omega_{2q}(x_q) = 0, \omega_{2q}(x_{q+1}) = 1, \omega_{2q}(x_{q+2}) = 0, \quad (11)$$

$$\omega'_{2q}(x_q) = 0, \omega'_{2q}(x_{q+1}) = 1, \omega'_{2q}(x_{q+2}) = 0, \quad (12)$$

where previous designations are used for continued functions.

Proof. Calculating the corresponding one-sided limits from the functions $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ and their derivatives at nodes x_q, x_{q+1} and x_{q+2} with the help of representations (3) and (5) - (8), we conclude that all the statements in the theorem are valid (see also [21]).

Remark 1. If the components $[\varphi(t)]_i$ of the vector $\varphi(t)$ are given by the equations $[\varphi(t)]_i = t^i$, then the functions $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ represent the known interpolation basis of the cubic Hermitian spline space.

The space

$$S^1_\varphi(X) \stackrel{\text{def}}{=} \{u | u = \textstyle\sum c_j \omega_j \ \ \forall c_j \in \mathbb{R}^1, j \in \mathbb{Z}\}$$

is called the spline space of Hermitian type (of the first level). In view of the property (A), the functions $\omega_j, j \in \mathbb{Z}$, are linearly independent. The set $\omega_j, j \in \mathbb{Z}$, is called the main basis of the space $S^1_\varphi(X)$.

Remark 2. Relations (9) - (12) can be written in the form

$$\omega_{2s-1}(x_j) = 0, \omega'_{2s-1}(x_j) = \delta_{s+1,j}, \quad (13)$$

$$\omega_{2s}(x_j) = \delta_{s+1,j}, \ \omega'_{2s}(x_j) = 0 \forall s, j \in \mathbb{Z}. \quad (14)$$

3 Calibration relations for the Hermitian type splines

In the set X, consider the subset Y:

$$Y: \ldots < y_{-2} < y_{-1} < y_0 < y_1 < y_2 < \ldots$$

$$\lim_{j\to-\infty} y_j = \alpha, \quad \lim_{j\to+\infty} y_j = \beta .$$

Let $\chi(s)$ denote a monotonically increasing integer function such that

$$y_j = x_{\chi(j)}. \qquad (15)$$

Let $\mathbb{Z}^{\chi}=\chi(\mathbb{Z})$. The introduced function is reversible on $\mathbb{Z}^{\chi}$ and generates the map $Y\mapsto X$ which is the embedding of Y in X. Repeating constructions (2) – (8) using the newly introduced grid Y, the functions w_j, for which

$$\operatorname{supp} w_{2j-1} \subset [y_j, y_{j+2}], \quad \operatorname{supp} w_{2j} \subset [y_j, y_{j+2}] \ \forall j \in \mathbb{Z}. \qquad (16)$$

For a fixed $i \in Z$ for $t \in (y_i, y_{i+1})$, by analogy with (4) we have

$$v'_i w_{2i-3}(t) + v_i w_{2i-2}(t) + v'_{i+1} w_{2i-1}(t) + v_{i+1} w_{2i}(t) = \varphi(t), \qquad (17)$$

where $v_j = \varphi(y_j)$, $v'_j = \varphi'(y_j)$, $\forall j \in \mathbb{Z}$.

We find from relations (16) – (17) for $p \in \mathbb{Z}$

$$w_{2p-1}(t) = \frac{\det(v'_p, v_p, \varphi(t), v_{p+1})}{\det(v'_p, v_p, v'_{p+1}, v_{p+1})}, \qquad (18)$$

for $t \in (y_p, y_{p+1})$,

$$w_{2p-1}(t) = \frac{\det(\varphi(t), v_{p+1}, v'_{p+2}, v_{p+2})}{\det(v'_{p+1}, v_{p+1}, v'_{p+2}, v_{p+2})}, \qquad (19)$$

for $t \in (y_{p+1}, y_{p+2})$,

$$w_{2p}(t) = \frac{\det(v'_p, v_p, v'_{p+1}, \varphi(t))}{\det(v'_p, v_p, v'_{p+1}, v_{p+1})} \qquad (20)$$

for $t \in (y_p, y_{p+1})$,

$$w_{2p}(t) = \frac{\det(v'_{p+1}, \varphi(t), v'_{p+2}, v_{p+2})}{\det(v'_{p+1}, v_{p+1}, v'_{p+2}, v_{p+2})} \qquad (21)$$

for $t \in (y_{p+1}, y_{p+2})$.

It can be shown that for functions (18) – (21) the following equations are valid, they are similar to (13) – (14)

$$w_{2s-1}(y_j) = 0, w'_{2s-1}(y_j) = \delta_{s+1,j}, \qquad (22)$$

$$w_{2s}(y_j) = \delta_{s+1,j}, \ w'_{2s}(y_j) = 0 \ \forall s, \ j \in \mathbb{Z} \qquad (23)$$

Let $q = \chi(i)$, $q + k = \chi(i + 1)$, so that between nodes y_i and y_{i+1} there are nodes x_j, $j = q + 1, q + 2, \ldots, q + k - 1$:

$$y_i = x_q < x_{q+1} < x_{q+2} < \ldots < x_{q+k-1} < x_{q+k} = y_{i+1}. \qquad (24)$$

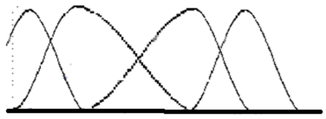

Figure 2: The basis splines $\omega_{2q-1}(t)$ after removing one node.

The basis splines $\omega_{2q-1}(t)$ after removing one node are shown in Figure 2: It was shown in [21] that if one node is removed from the original grid, the coordinate functions w_j associated with the new grid are linear combinations of the original (the mentioned linear combinations are called gauge relations). It follows that when a group of nodes is deleted, the corresponding coordinate functions will also have this property. To determine the coefficients of the gauge relations, we use the biorthogonal system of functionals represented by formulas (22) – (23). Thus, taking into account the arrangement of the supports of the functions w_j, $j \in \{2i-3, 2i-2, 2i-1, 2i\}$ and the functions ω_{2s-3}, ω_{2s-2}, ω_{2s-1}, ω_{2s} , (see formulas (3)), for $t \in (y_i, y_{i+1})$ and we have the representations

$$w_j(t) = \sum_{(y_i,y_{i+1})\cap(x_s,x_{s+2})\neq\emptyset} (c^{(j)}_{2s-1}\omega_{2s-1}(t) + c^{(j)}_{2s}\omega_{2s}(t)), \qquad (25)$$

where $j \in \{2i-3, 2i-2, 2i-1, 2i\}$.

The following statement is true.

Theorem 2. *Let i be a fixed integer and $k = \chi(i+1) - \chi(i) + 1$. Under these conditions, when $t \in (y_i, y_{i+1})$, the following relations are true*

$$w_j(t) = \sum_{s=q-1}^{q+k-1} (w'_j(x_{s+1})\omega_{2s-1}(t) + w_j(x_{s+1})\omega_{2s}(t)), \qquad (26)$$

where $j \in \{2i-3, 2i-2, 2i-1, 2i\}$, $q = \chi(i)$.

Proof. Relation (25) can be rewritten as

$$w_j(t) = \sum_{s=q-1}^{q+k-1}(c_{2s-1}^{(j)}\omega_{2s-1}(t)+c_{2s}^{(j)}\omega_{2s}(t)), \tag{27}$$

where $j \in \{2i-3, 2i-2, 2i-1, 2i\}$.
Substituting $t = x_r$, $r \in \{q, q+1, \ldots, q+k\}$ into formula (27), we have

$$w_j(x_r) = \sum_{s=q-1}^{q+k-1}(c_{2s-1}^{(j)}\omega_{2s-1}(x_r)+c_{2s}^{(j)}\omega_{2s}(x_r)). \tag{28}$$

Using the relations

$$\omega_{2s-1}(x_r) = 0,\ \omega_{2s}(x_r) = \delta_{s+1,r},$$

we find only one nonzero term on the right-hand side of (28); thus, we find a term with the index $s = r - 1$:

$$w_j(x_r) = c_{2r-2}^{(j)}\omega_{2r-2}(x_r) = c_{2r-2}^{(j)}.$$

So,

$$c_{2s}^{(j)} = w_j(x_{s+1})\ \ \forall s \in \{q-1, q, \ldots, q+k-1\}. \tag{29}$$

Differentiating relation (29) and substituting $t = x_r$ into the obtained identity, we find

$$w'_j(x_r) = \sum_{s=q-1}^{q+k-1}(c_{2s-1}^{(j)}\omega'_{2s-1}(x_r) + c_{2s}^{(j)}\omega'_{2s}(x_r)). \tag{30}$$

Considering the relations $\omega'_{2s-1}(x_r) = \delta_{s+1,r}$, $\omega_{2s}(x_r) = 0$, we see that on the right side of the equation (30) there is perhaps only one nonzero term (in this case, it is the first one), thus, we find a term with the subscript $s = r - 1$. Thus, we have $w'_j(x_r) = c_{2r-3}^{(j)}$ and

$$c_{2s-1}^{(j)} = w'_j(x_{s+1})\ \forall s \in \{q-1, q, \ldots, q+k-1\}. \tag{31}$$

Substituting (29) and (31) into (27), we find relations (26).

Theorem 3. *Under the conditions of Theorem 2, relations (26) can be represented in the form*

$$w_{2i-3}(t) = \omega_{2q-3}(t) + \sum_{s'=q+1}^{q+k-1} w'_{2i-3}(x_{s'})\omega_{2s'-3}(t) + \sum_{s'=q+1}^{q+k-1} w_{2i-3}(x_{s'})\omega_{2s'-2}(t). \tag{32}$$

$$w_{2i-2}(t) = \omega_{2q-2}(t) + \sum_{s'=q+1}^{q+k-1} w'_{2i-2}(x_{s'})\omega_{2s'-3}(t) + \sum_{s'=q+1}^{q+k-1} w_{2i-2}(x_{s'})\omega_{2s'-2}(t). \tag{33}$$

$$w_{2i-1}(t) = \sum_{s'=q+1}^{q+k-1} w'_{2i-1}(x_{s'})\omega_{2s'-3}(t) + \sum_{s'=q+1}^{q+k-1} w_{2i-1}(x_{s'})\omega_{2s'-1}(t) + \omega_{2q+2k-3}(t). \tag{34}$$

$$w_{2i}(t) = \sum_{s'=q+1}^{q+k-1} w'_{2i}(x_{s'})\omega_{2s'-3}(t) + \sum_{s'=q+1}^{q+k-1} w_{2i}(x_{s'})\omega_{2s'-2}(t) + \omega_{2q+2k-2}(t). \tag{35}$$

Proof. Formula (26) can be represented as

$$w_j(t) = w'_j(x_q)\omega_{2q-3}(t) + w_j(x_q)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2}(w'_j(x_{s+1})\omega_{2s-1}(t) + w_j(x_{s+1})\omega_{2s}(t)) + w'_j(x_{q+k})\omega_{2q+2k-3}(t) + w_j(x_{q+k})\omega_{2q+2k-2}(t);$$

considering that $x_q = y_i$ and $x_{q+k} = y_{i+1}$ we get

$$w_j(t) = w'_j(y_i)\omega_{2q-3}(t) + w_j(y_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2}(w'_j(x_{s+1})\omega_{2s-1}(t) + w_j(x_{s+1})\omega_{2s}(t)) + w'_j(y_{i+1})\omega_{2q+2k-3}(t) + w_j(y_{i+1})\omega_{2q+2k-2}(t). \tag{36}$$

From formula (36) with $j = 2i - 3$, we have

$$w_{2i-3}(t) = w'_{2i-3}(y_i)\omega_{2q-3}(t) + w_{2i-3}(y_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} w'_{2i-3}(x_{s+1})\omega_{2s-1}(t) + \sum_{s=q}^{q+k-2} w_{2i-3}(x_{s+1})\omega_{2s}(t) + w'_{2i-3}(y_{i+1})\omega_{2q+2k-3}(t) + w_{2i-3}(y_{i+1})\omega_{2q+2k-2}(t). \tag{37}$$

By formulas (22) – (23) we get:

$$w'_{2i-3}(y_i) = 1,$$

$$w_{2i-3}(y_i) = w'_{2i-3}(y_{i+1}) = w_{2i-3}(y_{i+1}) = 0,$$

and therefore identity (37) can be given the form (32). When $j = 2i - 2$, the identity (36) takes the form

$$w_{2i-2}(t) = w'_{2i-2}(y_i)\omega_{2q-3}(t) + w_{2i-2}(y_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} w'_{2i-2}(x_{s+1})\omega_{2s-1}(t) + \sum_{s=q}^{q+k-2} w_{2i-2}(x_{s+1})\omega_{2s}(t) + w'_{2i-2}(y_{i+1})\omega_{2q+2k-3}(t) + w_{2i-2}(y_{i+1})\omega_{q+2k-2}(t). \tag{38}$$

using equalities (22) – (23), we find

$$w'_{2i-2}(y_i) = 0, w_{2i-2}(y_i) = 1,$$

$$w'_{2i-2}(y_{i+1}) = w_{2i-2}(y_{i+1}) = 0,$$

so from (38) we derive the formula (33). Consider (36) with $j = 2i - 1$:

$$\begin{gathered} w_{2i-1}(t) = w'_{2i-1}(y_i)\omega_{2q-3}(t)+ \\ +w_{2i-1}(y_i)\omega_{2q-2}(t)+ \\ +\sum_{s=q}^{q+k-2} w'_{2i-1}(x_{s+1})\omega_{2s-1}(t) + \\ +\sum_{s=q}^{q+k-2} w_{2i-1}(x_{s+1})\omega_{2s}(t) + \\ +w'_{2i-1}(y_{i+1})\omega_{2q+2k-3}(t)+ \\ +w_{2i-1}(y_{i+1})\omega_{2q+2k-2}(t). \end{gathered} \quad (39)$$

Using (22) – (23), we have

$$w'_{2i}(y_i) = w_{2i}(y_i) = w'_{2i}(y_{i+1}) = 0,$$

$$w_{2i-1}(y_{i+1}) = 0,$$

from (39) we find the relation (34). Finally, consider the case $j = 2i$; in this case (36) takes the form:

$$\begin{gathered} w_{2i}(t) = w'_{2i}(y_i)\omega_{2q-3}(t)+ \\ +w_{2i}(y_i)\omega_{2q-2}(t)+ \\ +\sum_{s=q}^{q+k-2} w'_{2i}(x_{s+1})\omega_{2s-1}(t) + \\ +\sum_{s=q}^{q+k-2} w_{2i}(x_{s+1})\omega_{2s}(t) + \\ +w'_{2i}(y_{i+1})\omega_{2q+2k-3}(t)+ \\ +w_{2i}(y_{i+1})\omega_{2q+2k-2}(t). \end{gathered} \quad (40)$$

From (22) - (23) we get

$$w'_{2i}(y_i) = w_{2i}(y_i) = w'_{2i}(y_{i+1}) = 0,$$

$$w_{2i}(y_{i+1}) = 1,$$

and therefore (39) can be represented in the form (35).
This completes the proof.

Corollary 1. If the conditions of Theorem 3 are satisfied, and $k = 2$, then the relations can be given the form

$$\begin{gathered} w_{2i-3}(t)=\grave{u}_{2q-2}(t)+w'_{2i-3}(x_{q+1})\omega_{2q-1}(t)+ \\ +w_{2i-3}(x_{q+1})\omega_{2q}(t). \end{gathered} \quad (41)$$

$$\begin{gathered} w_{2i-2}(t)=\grave{u}_{2q-2}(t)+w'_{2i-2}(x_{q+1})\omega_{2q-1}(t)+ \\ +w_{2i-2}(x_{q+1})\omega_{2q}(t), \end{gathered} \quad (42)$$

$$\begin{gathered} w_{2i-1}(t)=w'_{2i-1}(x_{q+1})\omega_{2q-1}(t) \\ +w_{2i-1}(x_{q+1})\omega_{2q}(t)+\grave{u}_{2q+1}(t), \end{gathered} \quad (43)$$

$$\begin{gathered} w_{2i}(t)=w'_{2i}(x_{q+1})\omega_{2q-1}(t) \\ +w_{2i}(x_{q+1})\omega_{2q}(t)+\grave{u}_{2q+2}(t). \end{gathered} \quad (44)$$

Proof. Putting $k = 2$ in relations (32), (33), (34), (35) we obtain the identities (41), (42), (43), (44), respectively.

Remark 3. In an algorithmic implementation, it is useful to remember that the case $k = 1$ corresponds to mapping χ, in which there are no nodes of grid X between the nodes y_i and y_{i+1}, i.e. $\chi(i) = q$, $\chi(i + 1) = q+1$, so that $y_i = x_q$, $y_{i+1} = x_{q+1}$ (see (15) and (24)); Moreover, if we assume that for $m > n$, expression $\sum_{j=m}^{n} a_j$ equals zero (by definition) then the formulas of Theorems 2 and 3 are also valid in the case of $k = 1$.

Now we assume that $q = \chi(i)$, $q + k = \chi(i + 1)$, $q - k' = \chi(i - 1)$, so that there are nodes x_j, $j = q - 1$, $q - 2, \ldots, q - k + 1$, between nodes y_{i-1} and y_i, and there are nodes x_j, $j = q + 1$, $q + 2$, …, $q + k - 1$, between nodes y_i and y_{i+1}:

$$\begin{gathered} y_{i-1}=x_{q-k'}<x_{q-k'+1}<\ldots<x_{q-2}<x_{q-1}<y_i= \\ =x_q<x_{q+1}<x_{q+2}<\ldots<x_{q+k-1}<x_{q+k} = y_{i+1} \end{gathered} \quad (45)$$

Theorem 4. *If condition (A) is satisfied, and* $t \in (\alpha,\beta)$ *then for any* $i \in \mathbb{Z}$ *the following relations are valid:*

$$w_j(t) = \sum_{s=q'}^{q+k-2}(w'_j(x_{s+1})\omega_{2s-1}(t)+w_j(x_{s+1})\omega_{2s}(t)), \quad (46)$$

where $j \in \{2i-3, 2i-2\}$, $q = \chi(i)$, $q' = \chi(i - 1)$, $k = \chi(i + 1)$.

Proof. The support of the functions w_j, $j = 2i - 3$, $2i - 2$ is located on the segment $[y_{i-1}, y_{i+1}]$. When $t \in (y_i, y_{i+1})$ formula (46) is valid according to Theorem 2. Consider the interval $t \in (y_{i-1}, y_i)$. Replacing in Theorem 2 i by $i - 1$, q by q', and k by $k' \overset{\text{def}}{=} \chi(i) - \chi(i-1)$, we have

$$\begin{gathered} w_j(t) = \\ = \sum_{s'=q'-1}^{q'+k'-1}(w'_j(x_{s'+1})\omega_{2s'-1}(t)+w_j(x_{s'+1})\omega_{2s'}(t)). \end{gathered} \quad (47)$$

Note that, according to the notation (45), the nodes $x_{q'+k'}$ and x_k coincide with the node y_i, and $q'+k'=k$. Therefore, in the sum (47), the term corresponding to the index $s'=q'+k'-1$ coincides with the term in the sum (26) calculated for the index $s = q - 1$. There are no other common terms in these sums. Considering this circumstance and the fact that at the ends of the interval $[y_i, y_{i+1}]$ the corresponding terms are equal to zero, we conclude that the union of the sums (26) and (47) leads to the formula (46).

Remark 4. Introducing the replacement index $i'=i-1$, we get

$$q=\chi(i'+1),\ q'=\chi(i'),\ k=\chi(i'+2)-q. \qquad (48)$$

If we put s'=s+1, then the formula (46) can be written in the following equivalent form

$$w_j(t) = \\ = \sum_{s'=q'+1}^{q+k-1} (w'_j(x_{s'})\omega_{2s'-3}(t)+w_j(x_{s'})\omega_{2s'-2}(t)), \\ j \in \{2i'-1, 2i'\},\ i' \in \mathbb{Z}. \qquad (49)$$

For each $i \in \mathbb{Z}$ we consider $j \in \{2i-1, 2i\}$ and consider

$$q=\chi(i+1), q'=\chi(i), k=\chi(i+2)-q. \qquad (50)$$

Using (49) we have

$$w_j(t) = \\ = \sum_{s=\chi(i)}^{\chi(i+2)} (w'_j(x_s)\omega_{2s-3}(t)+w_j(x_s)\omega_{2s-2}(t)).$$

Since it is obvious that $w'_j(x_{\chi(i)})=w'_j(x_{\chi(i+2)})=0$ and $w_j(x_{\chi(i)})=w_j(x_{\chi(i+2)})=0$, so the previous relation can be written as

$$w_j(t) = \\ = \sum_{s=\chi(i)+1}^{\chi(i+2)-1} (w'_j(x_s)\omega_{2s-3}(t)+w_j(x_s)\omega_{2s-2}(t)). \qquad (51)$$

For each $i \in \mathbb{Z}$, $j \in \{2i-1, 2i\}$ consider the numbers $p_{j,k}$ for every $k \in \mathbb{Z}$ determined by the relations

$$p_{j,2\sigma-3}=w'_j(x_\sigma),\ p_{j,2\sigma-2}=w_j(x_\sigma) \\ \forall\sigma \in \{\chi(i)+1, \dots, \chi(i+2)-1\}, \qquad (52)$$

and the numbers not mentioned in this list $p_{j,k}$ will be considered as equal to zero:

$$p_{j,k}=0\ \forall j \in \mathbb{Z} \\ \forall k \notin \{2\chi(i)-1, 2\chi(i), \dots, 2\chi(i+2)-4\}. \qquad (53)$$

We denote $\boldsymbol{P}$ an infinite matrix, $P=(p_{jk})_{j,k\in Z}$, whose elements are given by equations (52) - (53). Thus, the row of the matrix $\boldsymbol{P}$ with the number $2i$ - 1 is

$$\dots, w'_{2i-1}(x_{\chi(i+2)-1}), w_{2i-1}(x_{\chi(i+2)-1}), 0, 0, \dots, \\ \dots, 0, 0, w'_{2i-1}(x_{\chi(i)+1}), w_{2i-1}(x_{\chi(i)+1}), \dots,$$

and the next row (row with number $2i$) differs from the mentioned one only by the fact that w_{2i-1} should be written instead of w_{2i} everywhere. The numbers of the columns in which these nonzero elements are located are as follows.

$$2\chi(i)-1, 2\chi(i), 2\chi(i)+1, 2\chi(i)+2, \dots \\ \dots, 2\chi(i+2)-5, 2\chi(i+2)-4; \qquad (54)$$

the total number of such columns is $2(\chi(i+2)-\chi(i))-2$.

If i is replaced by $i+1$, then it is necessary to consider rows with numbers $j \in \{2i+1, 2i+2\}$; the sets of their nonzero elements will shift so that their beginning will be in the column with the number $2\chi(i+1)-1$:

$$2\chi(i+1)-1, 2\chi(i+1), 2\chi(i+1)+1, \\ 2\chi(i+1)+2, \dots, 2\chi(i+3)-5, 2\chi(i+3)-4 \qquad (55)$$

The numbers of the common columns in (54) and (55) are as the follows

$$2\chi(i+1)-1, 2\chi(i+1), 2\chi(i+1)+1, \\ 2\chi(i+1)+2, \dots, 2\chi(i+2)-5, 2\chi(i+2)-4.$$

Since the multiplicity of the covering by the supports of the coordinate functions w_j is equal to four, the columns of this matrix contain no more than four nonzero elements (in consecutive four rows), and the matrix itself has an obvious stepped structure.

4 Hermite splines on a more frequent grid

In the previous paragraph, the Hermitian splines were considered on a rarer grid of nodes. It is often required to make the grid more frequent. Let d be some new node added to the grid (1), so that $d \in (x_k, x_{k+1})$. Let us denote s_j – nodes of the new grid which has been constructed:

$$s_j = x_j, \text{where } j \le k, \\ s_{k+1} = d, \\ s_j = x_{j-1}, \text{where } j \ge k+2, \\ S = \{s_j \mid j \in Z\}. \qquad (56)$$

Let $f_j = \varphi(s_j)$, $f'_j = \varphi'(s_j)$. Using the new grid S which has been introduced we construct the functions r_j. Here we use the next formulas that are similar to formulas (5) – (8):

$$r_{2q-1}(t) = \frac{\det(f'_q, f_q, \varphi(t), f_{q+1})}{\det(f'_q, f_q, f'_{q+1}, f_{q+1})}, \qquad (57)$$

for $t \in (s_q, s_{q+1})$,

$$r_{2q-1}(t) = \frac{\det(\varphi(t), f_{q+1}, f'_{q+2}, f_{q+2})}{\det(f'_{q+1}, f_{q+1}, f'_{q+2}, f_{q+2})}, \qquad (58)$$

for $t \in (s_{q+1}, s_{q+2})$,

$$r_{2q}(t) = \frac{\det(f'_q, f_q, f'_{q+1}, \varphi(t))}{\det(f'_q, f_q, f'_{q+1}, f_{q+1})}, \qquad (59)$$

for $t \in (s_{q+1}, s_{q+2})$,

$$r_{2q}(t) = \frac{\det(f'_{q+1}, \varphi(t), f'_{q+2}, f_{q+2})}{\det(f'_{q+1}, f_{q+1}, f'_{q+2}, f_{q+2})}, \qquad (60)$$

for $t \in (s_{q+1}, s_{q+2})$.

5 Biorthogonal system of functionals and their meanings on functions r_j

Over the space $C^1(\alpha, \beta)$ we consider a system of linear functionals $\{g^{(i)}\}_{i\in Z}$, that is defined by the relations:

$$\langle g^{(2q-1)}, u\rangle = u'(x_{q+1}),$$
$$\langle g^{(2q)}, u\rangle = u(x_{q+1}) \quad \forall q \in Z.$$

Using formulas (9) – (12), we have

$$\langle g^{(i)}, \omega_j\rangle = \delta_{i,j}. \qquad (61)$$

Let $\{h^{(i)}\}_{i\in Z}$ be a system of linear functionals.

$$\langle h^{(2q-1)}, u\rangle = u'(s_{q+1}),$$
$$\langle h^{(2q)}, u\rangle = u(s_{q+1}) \quad \forall q \in Z;$$

We obtain in a way similar to relations (61)

$$\langle h^{(i)}, r_j\rangle = \delta_{i,j}.$$

Now we have

$$\langle g^{(i)}, \varphi\rangle =$$
$$= \left(\langle g^{(i)}, [\varphi]_0\rangle, \langle g^{(i)}, [\varphi]_1\rangle, \langle g^{(i)}, [\varphi]_2\rangle, \langle g^{(i)}, [\varphi]_3\rangle\right)^T,$$

$$\langle h^{(i)}, \varphi\rangle =$$
$$= \left(\langle h^{(i)}, [\varphi]_0\rangle, \langle h^{(i)}, [\varphi]_1\rangle, \langle h^{(i)}, [\varphi]_2\rangle, \langle h^{(i)}, [\varphi]_3\rangle\right)^T,$$

We also obtain

$$\langle g^{(2q-1)}, \varphi\rangle = \varphi'_{q+1}, \langle g^{(2q)}, \varphi\rangle = \varphi_{q+1}; \qquad (62)$$

and

$$\langle h^{(2q-1)}, \varphi\rangle = v'_{q+1}, \langle h^{(2q)}, \varphi\rangle = v_{q+1}. \qquad (63)$$

Using the relations

$$\varphi_{q+1} = v_{q+1}, \varphi'_{q+1} = v'_{q+1}, \ q \le k-1,$$

we find from (62) – (63) the formulae

$$\langle g^{(2q-1)}, \varphi\rangle = \langle h^{(2q-1)}, \varphi\rangle,$$
$$\langle g^{(2q)}, \varphi\rangle = h^{(2q)}, \varphi, \text{where } q \le k-1.$$

From relations

$$\varphi_{q+1} = v_{q+2}, \varphi'_{q+1} = v'_{q+2}, \ q \le k,$$

and due to formulas (62) and (63), we have

$$\langle g^{(2q-1)}, \varphi\rangle = \langle h^{(2q+1)}, \varphi\rangle,$$
$$\langle g^{(2q)}, \varphi\rangle = \langle h^{(2q+2)}, \varphi\rangle, \text{where } q \ge k.$$

Let us denote

$$q_{i,j} = \langle g^{(i)}, r_j\rangle. \qquad (64)$$

Theorem 5. *The following relations are valid:*

$$q_{i,j} = \delta_{i,j}, j \le 2k-2,$$
$$q_{i,2k-1} = q_{i,2k} = 0 \quad \forall i \in Z, \qquad (65)$$

$$q_{i,j} = \delta_{i,j-2}, j \ge 2k+1, \forall i \in Z. \qquad (66)$$

Proof. Due to relations

$$r_{2q-1} = \omega_{2q-1}, r_{2q} = \omega_{2q}, \text{where } q \le k-2,$$

we obtain

$$\left\langle g^{(2q'-1)}, r_{2q-1}\right\rangle = \delta_{q,q'}, \left\langle g^{(2q'-1)}, r_{2q}\right\rangle = 0, \quad q \le k-2, \forall q' \in Z. \tag{67}$$

$$\left\langle g^{(2q')}, r_{2q-1}\right\rangle = 0, \left\langle g^{(2q')}, r_{2q}\right\rangle = \delta_{q,q'}, \quad q \le k-2, \forall q' \in Z. \tag{68}$$

For $q \ge k-2$, we have $r_{2q-1} = \omega_{2q-3}, r_{2q} = \omega_{2q-2}$, and then

$$\left\langle g^{(2q'-1)}, r_{2q-1}\right\rangle = \left\langle g^{(2q'-1)}, r_{2q-3}\right\rangle = \delta_{q',q-1}, \quad q \ge k+2, \forall q' \in Z, \tag{69}$$

$$\left\langle g^{(2q'-1)}, r_{2q}\right\rangle = \left\langle g^{(2q'-1)}, \omega_{2q-2}\right\rangle = 0 \quad q \ge k+2, \forall q' \in Z, \tag{70}$$

$$\left\langle g^{(2q')}, r_{2q-1}\right\rangle = \left\langle g^{(2q')}, \omega_{2q-3}\right\rangle = 0 \quad q \ge k+2, \forall q' \in Z, \tag{71}$$

$$\left\langle g^{(2q')}, r_{2q}\right\rangle = \left\langle g^{(2q')}, \omega_{2q-2}\right\rangle = \delta_{q',q-1}, \quad q \ge k+2, \forall q' \in Z. \tag{72}$$

Formulas (67) – (72) can be written briefly in the form of relations if we use notation (64),

$$q_{i,j} = \delta_{i,j}, \text{where } j \le 2k-4, \ \forall i \in Z. \tag{73}$$

$$q_{i,j} = \delta_{i,j-1}, \text{where } j \ge 2k+4, \ \forall i \in Z. \tag{74}$$

Now we find $q_{i,j}$, where

$$\forall i \in Z, \ j = 2k-3, 2k-2, \ldots 2k+3.$$

1. In the case that j=2k-3 we have

$$\left\langle g^{(2p-1)}, r_{2k-3}\right\rangle = 0, \left\langle g^{(2p)}, r_{2k-3}\right\rangle = 0, \tag{75}$$

where

$$((p+1 \le k-1) \vee (p+1 \ge k+1)) \Leftrightarrow ((p \le k-2) \vee (p \ge k)). \tag{76}$$

Now consider the case p=k-1, i.e. we have to find the values:

$$\left\langle g_{(2k-3)}, r_{2k-3}\right\rangle, \left\langle g_{(2k-2)}, r_{2k-3}\right\rangle.$$

When calculating them, it will be necessary to use formulas (57), when q=k-1. Thus, taking into account the equalities $\varphi'_k = f'_k$, $\varphi_k = f_k$, we have

$$q_{2k-3,2k-3} = \left\langle g^{(2k-3)}, r_{2k-3}\right\rangle = \frac{\det(f'_{k-1}, f_{k-1}, \varphi'_k, f_k)}{\det(f'_{k-1}, f_{k-1}, f'_k, f_k)} = 1, \tag{77}$$

$$q_{2k-2,2k-3} = \left\langle g^{(2k-2)}, r_{2k-3}\right\rangle = \frac{\det(f'_{k-1}, f_{k-1}, \varphi_k, f_k)}{\det(f'_{k-1}, f_{k-1}, f'_k, f_k)} = 0, \tag{78}$$

2. For j=2k-2 we have

$$\left\langle g^{(2p-1)}, r_{2k-2}\right\rangle = 0, \left\langle g^{(2p)}, r_{2k-2}\right\rangle = 0, \quad \text{where} ((p \le k-2) \vee (p \ge k)). \tag{79}$$

In the case that p=k-1 we find

$$\left\langle g_{(2k-3)}, r_{2k-2}\right\rangle, \left\langle g_{(2k-2)}, r_{2k-2}\right\rangle.$$

Putting in formula (59) q=k-1, we have

$$q_{2k-3,2k-2} = \left\langle g^{(2k-3)}, r_{2k-2}\right\rangle = \frac{\det(f'_{k-1}, f_{k-1}, f'_k, \varphi'_k)}{\det(f'_{k-1}, f_{k-1}, f'_k, f_k)} = 0, \tag{80}$$

$$q_{2k-2,2k-2} = \left\langle g^{(2k-2)}, r_{2k-2}\right\rangle = \frac{\det(f'_{k-1}, f_{k-1}, f'_k, \varphi_k)}{\det(f'_{k-1}, f_{k-1}, f'_k, f_k)} = 1. \tag{81}$$

3. Now let j=2k-1. In this case we find

$$\left\langle g^{(2p-1)}, r_{2k-1}\right\rangle = 0, \left\langle g^{(2p)}, r_{2k-1}\right\rangle = 0, \tag{82}$$

where $((p \le k-1) \vee (p \ge k+1))$.

Let $p=k$; we calculate

$$\left\langle g^{(2k-1)},r_{2k-1}\right\rangle,\left\langle g^{(2k)},r_{2k-1}\right\rangle.$$

Putting in formula (58) $q=k$ and taking into account relations

$$\varphi_{k+1}=f_{k+2},\varphi'_{k+1}=f'_{k+2}, \quad (83)$$

we get

$$q_{2k-1,2k-1}=\left\langle g^{(2k-1)},r_{2k-1}\right\rangle= =\frac{\det(\varphi'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0, \quad (84)$$

$$q_{2k,2k-1}=\left\langle g^{(2k)},r_{2k-1}\right\rangle= =\frac{\det(\varphi_{k+1},f_{k+1},f'_{k+2},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0. \quad (85)$$

4. Let $j=2k$. We have

$$\left\langle g^{(2p-1)},r_{2k}\right\rangle=0,\left\langle g^{(2p)},r_{2k}\right\rangle=0, \quad (86)$$

where $((p\le k-1)\vee(p\ge k+1))$.

It remains to consider the case $p=k$ and calculate

$$g^{(2k-1)},r_{2k},g^{(2k)},r_{2k}.$$

Putting in formula (60) $q=k$ and taking into account relations (83), we get:

$$q_{2k-1,2k}=\left\langle g^{(2k-1)},r_{2k}\right\rangle= =\frac{\det(f'_{k+1},\varphi'_{k+1},f'_{k+2},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0, \quad (87)$$

$$q_{2k,2k}=\left\langle g^{(2k)},r_{2k}\right\rangle= =\frac{\det(f'_{k+1},\varphi_{k+1},f'_{k+2},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0. \quad (88)$$

5. Considering the case $j=2k+1$ similar to the previous we find

$$\left\langle g^{(2p-1)},r_{2k+1}\right\rangle=0,\left\langle g^{(2p)},r_{2k+1}\right\rangle=0, \quad (89)$$

where $((p\le k-1)\vee(p\ge k+2))$.

Let $p=k$. Substituting $q=k$ into formula (57) and using (83), we find

$$q_{2k-1,2k+1}=\left\langle g^{(2k-1)},r_{2k+1}\right\rangle= =\frac{\det(f'_{k+1},f_{k+1},\varphi'_{k+1},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=1, \quad (90)$$

$$q_{2k,2k+1}=\left\langle g^{(2k)},r_{2k+1}\right\rangle= =\frac{\det(f'_{k+1},f_{k+1},\varphi_{k+1},f_{k+2})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0; \quad (91)$$

Substituting $q=k$ into formula (58), we have

$$q_{2k+2,2k+1}=\left\langle g^{(2k+1)},r_{2k+1}\right\rangle= =\frac{\det(\varphi'_{k+2},f_{k+2},f'_{k+3},f_{k+3})}{\det(f'_{k+2},f_{k+2},f'_{k+3},f_{k+3})}=0, \quad (92)$$

$$q_{2k+2,2k+1}=\left\langle g^{(2k+2)},r_{2k+1}\right\rangle= =\frac{\det(\varphi_{k+2},f_{k+2},f'_{k+3},f_{k+3})}{\det(f'_{k+2},f_{k+2},f'_{k+3},f_{k+3})}=0, \quad (93)$$

because

$$\varphi_{k+2}=f_{k+3},\ \ \varphi'_{k+2}=f'_{k+3}. \quad (94)$$

6. In the case that $j=2k+2$ we find

$$\left\langle g^{(2p-1)},r_{2k+1}\right\rangle=0,\left\langle g^{(2p)},r_{2k+1}\right\rangle=0, \text{ where } ((p\le k-1)\vee(p\ge k+2)). \quad (95)$$

When $p=k$ from equalities (59) with $q=k+1$ we have

$$q_{2k-1,2k+2}=\left\langle g^{(2k-1)},r_{2k+2}\right\rangle= =\frac{\det(f'_{k+1},f_{k+1},f'_{k+2},\varphi_{k+1})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=1, \quad (96)$$

$$q_{2k,2k+2}=\left\langle g^{(2k)},r_{2k+2}\right\rangle= =\frac{\det(f_{k+1},f_{k+1},f'_{k+2},\varphi_{k+1})}{\det(f'_{k+1},f_{k+1},f'_{k+2},f_{k+2})}=0, \quad (97)$$

Here equalities (83) were used.

Now we put $p=k+1$. We find from relation (60), the following relation is considered for $q=k+1$,

$$q_{2k+1,2k+2} = \left\langle g^{(2k+1)}, r_{2k+1} \right\rangle = = \frac{\det\left(f'_{k+2}, \varphi'_{k+2}, f'_{k+3}, f_{k+3}\right)}{\det\left(f'_{k+2}, f_{k+2}, f'_{k+3}, f_{k+3}\right)} = 0, \quad (98)$$

$$q_{2k+2,2k+2} = \left\langle g^{(2k+2)}, r_{2k+2} \right\rangle = = \frac{\det\left(f'_{k+2}, \varphi_{k+2}, f'_{k+3}, f_{k+3}\right)}{\det\left(f'_{k+2}, f_{k+2}, f'_{k+3}, f_{k+3}\right)} = 0. \quad (99)$$

7. Now, considering the case j=2k+3, we get

$$\left\langle g^{(2p-1)}, r_{2k+3} \right\rangle = 0, \left\langle g^{(2p)}, r_{2k+3} \right\rangle = 0, \text{ where } \left((p \le k) \vee (p \ge k+3)\right). \quad (100)$$

Substituting q=k+2 into formula (57) when p=k+1, we have

$$q_{2k+1,2k+3} = \left\langle g^{(2k+1)}, r_{2k+3} \right\rangle = = \frac{\det\left(f'_{k+2}, f_{k+2}, \varphi'_{k+2}, f_{k+3}\right)}{\det\left(f'_{k+2}, f_{k+2}, f'_{k+3}, f_{k+3}\right)} = 1. \quad (101)$$

$$q_{2k+2,2k+3} = \left\langle g^{(2k+2)}, r_{2k+3} \right\rangle = = \frac{\det\left(f'_{k+2}, f_{k+2}, \varphi_{k+2}, f_{k+3}\right)}{\det\left(f'_{k+2}, f_{k+2}, f'_{k+3}, f_{k+3}\right)} = 0. \quad (102)$$

We turn to the case $p = k + 2$. From relation (58), considered for $q = k + 2$, in a similar way, due to the equalities

$$\varphi_{k+3} = f_{k+4}, \quad \varphi'_{k+3} = f'_{k+4},$$

we find

$$q_{2k+3,2k+3} = \left\langle g^{(2k+3)}, r_{2k+3} \right\rangle = = \frac{\det\left(\varphi'_{k+3}, f_{k+3}, f'_{k+4}, f_{k+4}\right)}{\det\left(f'_{k+3}, f_{k+3}, f'_{k+4}, f_{k+4}\right)} = 0, \quad (103)$$

$$q_{2k+4,2k+3} = \left\langle g^{(2k+4)}, r_{2k+3} \right\rangle = = \frac{\det\left(\varphi_{k+3}, f_{k+3}, f'_{k+4}, f_{k+4}\right)}{\det\left(f'_{k+3}, f_{k+3}, f'_{k+4}, f_{k+4}\right)} = 0. \quad (104)$$

Equations (73)–(82), (84)–(93), (95)–(104) established earlier prove the validity of relations (65)–(66).
This concludes the proof.

6 Conclusion

In this paper, we describe the process of removing a group of nodes upon approximation by Hermite splines of the first height. In addition, the process of adding nodes is described here. The proposed results allow us to actively perform simultaneous approximation of the stream of values for the function and its derivative. For this aim we offer (generally speaking, nonpolynomial) splines of the Hermite type. The proposed formulas are quite simple. They lead us to sustainable calculations. They are exact on the components of the generating function. Established calibration relations allow us to obtain the embedded the Hermite spline spaces constructed on the embedded grids. Such relations lead to a number of spline-wavelet decompositions of the mentioned embedded spaces.

References:

[1] L.Setlak, R.Kowalik, MEMS electromechanical microsystem as a support system for the position determining process with the use of the inertial navigation system INS and kalman filter, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.14, paper № 11, 2019, pp. 105-117.

[2] L.Setlak, R. Kowalik, Algorithm controlling the autonomous flight of an unmanned aerial vehicle based on the construction of a glider, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.14, 2019, pp. 56-65.

[3] S.A.Nersisyan, V.M.Staroverov, Gradient estimation for hexagonal grids and its application to classification of instrumentally registered tactile images, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.13, 2018, pp. 123-129.

[4] K.Hyniova, Usage of one-quarter-car active suspension test stand for experimental verification, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.12, 2017, pp. 17-24.

[5] J.Vala, L. Hobst, V.Kozák, Detection of metal fibres in cementitious composites based on signal and image processing approaches, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.10, 2015, pp. 39-46.

[6] V.M.Gomes, A.F.B.A.Prado, H.K. Kuga, Real time orbit determination with different force fields, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.5 (1), 2010, pp. 23-32.

[7] C.-F.Lin, Z.-C. Hong, J.-S. Chern, C.-M. Lin, B.-J. Chang, C.-J. Fong, H.-J. Lin, Structure design for TUUSAT-1A Microsatellite, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.5 (1), 2010, pp. 45-58.

[8] D.P.S. dos Santos, L. Casalino, G.Colasurdo, A.F.B. de Almeida Prado, Optimal Trajectories towards

near-earth-objects using solar electric propulsion (sep) and Gravity Assisted Maneuver, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.4 (3), 2009, pp. 125-135.

[9] Y.F.Hsiao, Y.S. Tarng, K.Y. Kung, Comparison of the grey theory with neural network in the rigidity prediction of linear motion guide, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.4 (1), 2009, pp. 32-41.

[10] A.A. Alugongo, Modeling the nonlinear vibration response of a cracked rotor by time delay and embedding technique, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.3 (8), 2008, pp. 779-788.

[11] V.M. Gomes, A.F.B.A. Prado,Swing-By maneuvers for a cloud of particles with planets of the Solar System, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.3 (11), 2008, pp. 869-878.

[12] V.M. Gomes, A.F.B.A. Prado, H.K.Kuga, Low thrust maneuvers for Artificial Satellites, *WSEAS Transactions on Applied and Theoretical Mechanics*, Vol.3 (10), 2008, pp. 859-868.

[13] D. Černá, Postprocessing Galerkin method using quadratic spline wavelets and its efficiency, *Comp. and Math. with App.*, Vol. 75 (9), 2018, pp. 3186-3200.

[14] J. Chen, W. Tang, M. Xu, Meshless analysis method of ship structures based on a B-spline wavelet. Harbin Gongcheng Daxue Xuebao, *J. of Harbin Engineering Univ.*, Vol. 37 (1), 2016, pp.13-18.

[15] X. Zhang, R.X. Gao, R. Yan, X. Chen, C. Sun, Z. Yang, Analysis of laminated plates and shells using b-spline wavelet on interval finite element, *International J. of Structural Stability and Dynamics*, Vol.17 (6), 2017.

[16] R. Boujelbene, Y. Ben Jemaa, M. Zribi, Toward an optimal B-spline wavelet transform for image compression, *Proceedings of IEEE/ACS International Conference on Computer Systems and Applications, AICCSA*, paper N 7945738, 2017.

[17] C. Wang, Z.H. Gao, J.T. Huang, K. Zhao, J. Li, Smoothing methods based on coordinate transformation in a linear space and application in airfoil aerodynamic design optimization, *Science China Technological Sciences*, Vol. 58 (2), 2015, pp. 297-306.

[18] Ö. Oruç, A numerical procedure based on Hermite wavelets for two-dimensional hyperbolic telegraph equation, *Engineering with Comp*., Vol. 34 (4), 2018, pp.741-755.

[19] E. Ashpazzadeh, B. Han, M. Lakestani, Biorthogonal multiwavelets on the interval for numerical solutions of Burgers' equation, *J. of Comp. and App. Math*, Vol. 317, 2017, pp.510-534.

[20] Z.M. Sulaimanov, B.M. Shumilov, A splitting algorithm for the wavelet transform of cubic splines on a nonuniform grid, *Comp. Math. and Math. Phys*., Vol. 57 (10), 2017, pp.1577-1591.

[21] Yu.K.Dem'yanovich. On embedding and extended smoothness of spline spaces. *Far East J. of Math. Sciences (FJMS)*, Vol. 102, 2017, pp. 2025–2052 .