First Exit and Optimization Problems for a CIR Diffusion Process

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Abstract: Let $\{X(t), t \geq 0\}$ be a CIR diffusion process, and $\tau(x)$ be the first time that $X(t) = 0$ or c, given that $X(0) = x \in (0, c)$. First, we compute the moment-generating function and the expected value of $\tau(x)$. Then, an optimal control problem is considered for $\{X(t), t \geq 0\}$. Finally, we add jumps to the diffusion process and we calculate in a particular case the probability that $X(\tau(x)) = 0$, as well as the expected time needed to leave the interval $(0, c)$. Explicit and exact results are obtained.

Key-Words: First-passage time, Brownian motion, Poisson process, uniform jumps, homing problem, integro-differential equation.

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1 Introduction

We consider the one-dimensional diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}X(t) = a[b - X(t)]\,\mathrm{d}t + \sigma\sqrt{X(t)}\,\mathrm{d}B(t), \quad (1)$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, and a, b and $\sigma > 0$ are constants. This stochastic process is known as a Cox–Ingersoll–Ross (CIR) process. It is important in financial mathematics, where it is used to model the evolution of interest rates.

If the condition $\sigma^2 \leq 2ab$ is satisfied, then the origin cannot be reached in finite time, which is an important feature in the application to interest rates. However, in other applications it is possible to have $X(t) = 0$. In this paper, we assume that

$$\sigma^2 > 2ab, \quad (2)$$

so that the origin is an attainable boundary.

Remark. In fact, it is possible that interest rates will be zero or even negative, as has already been the case in several countries, including Japan.

Assume that $X(0) = x \in (0, c)$. We define the first-passage time

$$\tau(x) = \inf\{t > 0 : X(t) = 0 \text{ or } c\} \quad (3)$$

and we denote its moment-generating function by $M(x; \alpha)$:

$$M(x; \alpha) := E\left[e^{-\alpha\tau(x)}\right], \quad (4)$$

where $\alpha > 0$.

In the next section, we will obtain an exact and explicit expression for the function $M(x; \alpha)$. Moreover, we will also compute the function $m(x) := E[\tau(x)]$, as well as

$$A(x) := E\left[\int_0^{\tau(x)} X(t)\,\mathrm{d}t\right]. \quad (5)$$

The function $A(x)$ represents the expected value of the area between the t-axis and the trajectory of $\{X(t), t \geq 0\}$ in the interval $[0, \tau(x)]$; see, [1].

First-passage problems for CIR or related diffusion processes were studied, in particular, by [2], [3], [4], [5], [6], [7].

In Section 3, we will consider an optimal control problem for the CIR process. This control problem will be a particular *homing problem*. In such problems, the optimizer controls a stochastic process until a given event occurs; for example, until the process hits either of two absorbing boundaries, such as 0 and c in the definition (3) of the random variable $\tau(x)$.

In Section 4, random jumps according to a Poisson process will be added to the CIR process. We will compute the mean time until the jump-diffusion process leaves the interval $(0, c)$, as well as the probability that it will hit the origin before the boundary at c.

2 First-exit Problems for the CIR Process

The function $M(x;\alpha)$ satisfies the Kolmogorov backward equation (see, for instance, [8], and/or, [9])

$$\frac{1}{2}\sigma^2 x M''(x;\alpha) + a(b-x)M'(x;\alpha) = \alpha M(x;\alpha), \tag{6}$$

subject to the boundary conditions $M(0;\alpha) = M(c;\alpha) = 1$. Moreover, we have

$$M(x;\alpha) = E\left[1 - \alpha\tau(x) + \frac{\alpha^2}{2}\tau^2(x) - \cdots\right]. \tag{7}$$

Assuming that the moments of $\tau(x)$ exist (and are finite), we can write that

$$M(x;\alpha) = 1 - \alpha E[\tau(x)] + \frac{\alpha^2}{2}E[\tau^2(x)] - \cdots \tag{8}$$

Substituting the above expression for $M(x;\alpha)$ into Eq. (6), we find that the function $m(x) := E[\tau(x)]$ satisfies the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 x m''(x) + a(b-x)m'(x) = -1. \tag{9}$$

The boundary conditions are $m(0) = m(c) = 0$.

Finally the function $A(x)$ defined in Eq. (5) is a solution of the ODE (see, [1])

$$\frac{1}{2}\sigma^2 x A''(x) + a(b-x)A'(x) = -x, \tag{10}$$

such that $A(0) = A(c) = 0$.

First, we will solve Eq. (6). We rewrite the equation as follows:

$$x M''(x;\alpha) - \left(\frac{2a}{\sigma^2}x - \frac{2ab}{\sigma^2}\right)M'(x;\alpha) - \frac{2\alpha}{\sigma^2}M(x;\alpha) = 0. \tag{11}$$

Let

$$y := \frac{2a}{\sigma^2}x, \tag{12}$$

where we assume that $a \neq 0$, and $N(y;\alpha) := M(x;\alpha)$. We find that the function $N(y;\alpha)$ satisfies the ODE

$$y N''(y;\alpha) - \left(y - \frac{2ab}{\sigma^2}\right)N'(y;\alpha) - \frac{\alpha}{a}N(y;\alpha) = 0, \tag{13}$$

which is a Kummer differential equation. Its general solution is of the form (see, [10])

$$N(y;\alpha) = c_1\Phi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, y\right) + c_2\Psi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, y\right), \tag{14}$$

where c_1 and c_2 are arbitrary constants, and $\Phi(\cdot,\cdot,\cdot)$ and $\Psi(\cdot,\cdot,\cdot)$ are confluent hypergeometric functions of the first and second kind, respectively. Thus,

$$M(x;\alpha) = c_1\Phi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ax}{\sigma^2}\right) + c_2\Psi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ax}{\sigma^2}\right). \tag{15}$$

Next, the boundary conditions $M(0;\alpha) = M(c;\alpha) = 1$ imply that

$$\left.\begin{array}{rcl} c_1\Phi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0\right) + c_2\Psi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0\right) & = & 1, \\ c_1\Phi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2}\right) + c_2\Psi\left(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2}\right) & = & 1. \end{array}\right\} \tag{16}$$

We find that

$$\left.\begin{array}{rcl} c_1 & = & \frac{\Phi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0) - \Psi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0)}{\Phi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2})\Psi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0) - \Psi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2})} \\ \text{and} & & \\ c_2 & = & \frac{\Phi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2}) - 1}{\Phi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2})\Psi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, 0) - \Psi(\frac{\alpha}{a}, \frac{2ab}{\sigma^2}, \frac{2ac}{\sigma^2})}. \end{array}\right\} \tag{17}$$

Proposition 2.1. *The moment-generating function $M(x;\alpha)$ of the random variable $\tau(x)$ is given by Eq. (15), with the constants c_1 and c_2 defined in Eq. (17).*

Now, we turn to Eq. (9). With

$$\delta := \frac{2a}{\sigma^2}, \quad \beta := \frac{2ab}{\sigma^2} \quad \text{and} \quad \theta := -\frac{2}{\sigma^2}, \tag{18}$$

the equation becomes

$$x m''(x) + (\beta - \delta x)m'(x) = \theta, \tag{19}$$

which is a first-order linear ODE for $m'(x)$. We find that

$$m'(x) = x^{-\beta}e^{\delta x}\left[c_0 - \theta\delta^{-\beta}\Gamma(\beta, \delta x)\right], \tag{20}$$

where c_0 is an arbitrary constant and $\Gamma(\cdot,\cdot)$ is the incomplete upper gamma function:

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}\,\mathrm{d}t. \tag{21}$$

We have

$$\Gamma(s,x) = \Gamma(s) - \gamma(s,x), \tag{22}$$

where $\gamma(s,x)$ is the incomplete lower gamma function:

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}\,\mathrm{d}t. \tag{23}$$

Moreover,

$$\gamma(s,x) = x^s\Gamma(s)e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(s+k+1)}. \tag{24}$$

It follows that

$$\Gamma(\beta,\delta x)=\Gamma(\beta)\left[1-(\delta x)^{\beta}e^{-\delta x}\sum_{k=0}^{\infty}\frac{(\delta x)^k}{\Gamma(\beta+k+1)}\right]. \quad (25)$$

Now, integrating the expression obtained for $m'(x)$, we obtain that

$$\begin{aligned} m(x) &= [c_0-\theta\delta^{-\beta}\Gamma(\beta)]\int x^{-\beta}e^{\delta x}\,\mathrm{d}x \\ &\quad + \theta\Gamma(\beta)\int\sum_{k=0}^{\infty}\frac{(\delta x)^k}{\Gamma(\beta+k+1)}\,\mathrm{d}x \\ &\quad + c_{00}, \end{aligned} \quad (26)$$

where c_{00} is a constant to be determined.

Making use of WolframAlpha, we find that

$$\int x^{-\beta}e^{\delta x}\,\mathrm{d}x=-x^{1-\beta}\mathrm{E}_{\beta}(-\delta x), \quad (27)$$

where $\mathrm{E}_n(x)$ is an exponential integral function:

$$\mathrm{E}_n(x):=\int_1^{\infty}\frac{e^{-xt}}{t^n}\,\mathrm{d}t. \quad (28)$$

We have, [11]

$$\mathrm{E}_n(x)=x^{n-1}\Gamma(1-n,x), \quad (29)$$

which implies that

$$\int x^{-\beta}e^{\delta x}\,\mathrm{d}x=-(-\delta)^{\beta-1}\Gamma(1-\beta,-\delta x). \quad (30)$$

Next, integrating term by term (which is allowed because the sum we integrate is uniformly convergent), we have

$$\begin{aligned} \int\sum_{k=0}^{\infty}\frac{(\delta x)^k}{\Gamma(\beta+k+1)}\,\mathrm{d}x &= \sum_{k=0}^{\infty}\frac{\delta^k x^{k+1}}{(k+1)\Gamma(\beta+k+1)} \\ &= \frac{x\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta x\right)}{\Gamma(1+\beta)}, \end{aligned} \quad (31)$$

where ${}_pF_q$ is the generalized hypergeometric function. Hence, we can write that

$$\begin{aligned} m(x) &= (-\delta)^{\beta-1}\Gamma(1-\beta,-\delta x)[\theta\delta^{-\beta}\Gamma(\beta)-c_0] \\ &\quad + \frac{x\theta\Gamma(\beta)\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta x\right)}{\Gamma(1+\beta)} \\ &\quad + c_{00}. \end{aligned} \quad (32)$$

Finally, we deduce from the boundary conditions $m(0)=m(c)=0$ that

$$\begin{aligned} c_0 &= \delta^{-\beta}\theta\Gamma(\beta) \\ &\quad + \frac{\delta c\theta(-\delta)^{-\beta}\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta c\right)}{\beta\gamma(1-\beta,\delta c)} \end{aligned} \quad (33)$$

and

$$c_{00} = -\frac{c\theta\Gamma(1-\beta)\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta c\right)}{\gamma(1-\beta,\delta c)}. \quad (34)$$

Proposition 2.2. *The expected value $m(x)$ of the random variable $\tau(x)$ is given by Eq. (32), with the constants c_0 and c_{00} defined in (33) and (34), respectively. Moreover, the constants δ, β and θ are defined in Eq. (18).*

To conclude this section, we will solve Eq. (10). With the notations introduced in Eq. (18), we rewrite the equation as follows:

$$x\,A''(x)+(\beta-\delta x)\,A'(x)=\theta x. \quad (35)$$

We find that

$$\begin{aligned} A'(x) &= k_1x^{-\beta}e^{\delta x}-\theta\delta^{-\beta-1}x^{-\beta}e^{\delta x}\Gamma(\beta+1,\delta x) \\ &= k_1x^{-\beta}e^{\delta x} \\ &\quad -\theta\delta^{-\beta-1}[\beta x^{-\beta}e^{\delta x}\Gamma(\beta,\delta x)+1], \end{aligned} \quad (36)$$

where k_1 is an arbitrary constant, and the last equation follows from the identity

$$\Gamma(s+1,x)=s\Gamma(s,x)+x^s e^{-x}. \quad (37)$$

Proceeding as above, we can state the following proposition.

Proposition 2.3. *The function $A(x)$ defined in Eq. (5) is given by*

$$\begin{aligned} A(x) &= (-\delta)^{\beta-1}\Gamma(1-\beta,-\delta x)[\theta\delta^{-\beta-1}\Gamma(\beta+1)-k_1] \\ &\quad + \frac{x\beta\theta\Gamma(\beta)\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta x\right)}{\delta\Gamma(1+\beta)} \\ &\quad -\theta\delta^{-\beta-1}x+k_2, \end{aligned} \quad (38)$$

where

$$\begin{aligned} k_1 &= \frac{1}{\Delta}\Bigg[\theta(-\delta)^{-\beta}\delta^{-\beta-1}\Bigg(c\delta^{\beta+1}\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta c\right) \\ &\quad +(-\delta)^{\beta}\Gamma(\beta+1)\gamma(1-\beta,-c\delta)-\delta c\Bigg)\Bigg] \end{aligned} \quad (39)$$

and

$$\begin{aligned} k_2 &= -\frac{1}{\Delta}\Bigg[c\theta\delta^{-\beta-1}\Gamma(1-\beta,0) \\ &\quad \times\left(\delta^{\beta}\,{}_2F_2\left(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1+\beta\end{bmatrix},\delta c\right)-1\right)\Bigg], \end{aligned} \quad (40)$$

in which

$$\Delta:=\gamma(-c\delta) \quad (41)$$

and the constants δ, β and θ are defined in Eq. (18).

3 A homing Problem

In this section, we consider the controlled CIR process $\{X_u(t), t \geq 0\}$ defined by

$$\begin{aligned} \mathrm{d}X_u(t) \;&=\; a\,[b - X_u(t)]\,\mathrm{d}t + B[X_u(t)]\,u[X_u(t)]\,\mathrm{d}t \\ &\quad + \sigma\,\sqrt{X_u(t)}\,\mathrm{d}W(t), \end{aligned} \tag{42}$$

where $B(\cdot) \neq 0$, $u(\cdot)$ is the control variable, which is assumed to be a continuous function, and the stochastic process $\{W(t), t \geq 0\}$ is a standard Brownian motion.

Assume that $X_u(0) = x \in (0, c)$. Our aim is to find the value $u^*[X_u(t)]$ of the control variable that minimizes the expected value of the cost function

$$J(x) = \int_0^{\tau(x)} \left\{ \frac{1}{2}\,Q[X_u(t)]\,u^2[X_u(t)] + R[X_u(t)] \right\} \mathrm{d}t, \tag{43}$$

where $Q(\cdot) > 0$ and $R(\cdot) \neq 0$. To do so, we can make use of dynamic programming. First, we define the value function

$$F(x) = \inf_{\substack{u[X_u(t)] \\ 0 \leq t < \tau(x)}} E[J(x)]. \tag{44}$$

Proceeding as in [12], we obtain the following proposition.

Proposition 3.1. *The value function $F(x)$ satisfies the non-linear ODE*

$$R(x) - \frac{B^2(x)}{2\,Q(x)}\,[F'(x)]^2 + a\,(b - x)\,F'(x) + \frac{1}{2}\,\sigma^2\,x\,F''(x) = 0 \tag{45}$$

for $0 < x < c$, subject to the boundary conditions $F(0) = F(c) = 0$. Moreover, the optimal control is given by

$$u^*(x) = -\frac{B(x)}{Q(x)}\,F'(x). \tag{46}$$

In the case when the relation

$$\sigma^2\,x = \kappa\,B^2(x)/Q(x) \tag{47}$$

holds for a positive constant κ, it is shown in [12], that Eq. (45) can be linearized by defining

$$\Phi(x) = e^{-F(x)/\kappa}. \tag{48}$$

In this paper, we will solve a problem for which Eq. (47) does not hold.

Assume that $a = 0$, $\sigma = 1$ and

$$B(x) = x, \quad Q(x) \equiv 1 \quad \text{and} \quad R(x) = x^2. \tag{49}$$

Remark. Because the function $R(x)$ is positive, the objective is to leave the continuation region $(0, c)$ as soon as possible, while taking the quadratic control costs into account.

Then, Eq. (45) reduces to

$$x^2 - \frac{x^2}{2}\,[F'(x)]^2 + \frac{1}{2}\,x\,F''(x) = 0. \tag{50}$$

With the help of the software program *Maple*, we find that the solution to the above equation that satisfies the boundary condition $F(0) = 0$ can be written as follows:

$$F(x) = \int_0^x -\sqrt{2}\tanh\left(\frac{\sqrt{2}z^2}{2} + C_I\,\sqrt{2}\right)\mathrm{d}z, \tag{51}$$

where C_1 is an arbitrary constant. Let $c = 1$. We find that the only constant C_1 for which $F(1) = 0$ is $C_1 \simeq -0.1652$. Hence,

$$u^*(x) \simeq x\sqrt{2}\tanh\left(\frac{\sqrt{2}x^2}{2} - 0.1652\,\sqrt{2}\right) \tag{52}$$

for $0 < x < 1$. The functions $F(x)$ and $u^*(x)$ are shown in Figure 1 and Figure 2, respectively.

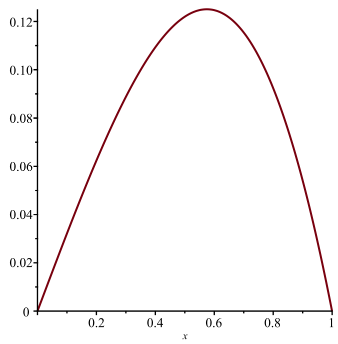

Figure 1: Function $F(x)$ defined in Eq. (51) for x in the interval $[0, 1]$.

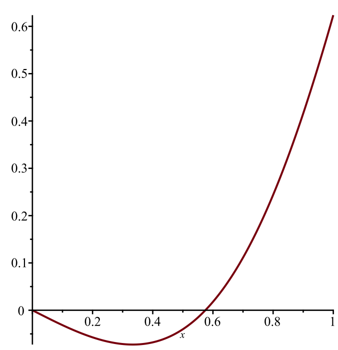

Figure 2: Optimal control $u^*(x)$ defined in Eq. (52) for x in the interval $[0, 1]$.

4 Addition of Random Jumps

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ, which is assumed to be independent of the standard Brownian motion $\{B(t), t \geq 0\}$. In this section, we consider the jump-diffusion process $\{X(t), t \geq 0\}$ defined by

$$X(t) = X(0) + \int_0^t a[b - X(s)]\,\mathrm{d}s + \sigma \int_0^t \sqrt{X(s)}\,\mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i, \quad (53)$$

where $Y_1, Y_2, \ldots$ are independent random variables that are uniformly distributed on the interval $[-x, 0]$:

$$f_{Y_i}(y) = \frac{1}{x} \quad \text{if } -x \leq y \leq 0, \quad (54)$$

for $i = 1, 2, \ldots$ Note that the continuous part of the jump-diffusion process $\{X(t), t \geq 0\}$ is the CIR process defined in Eq. (1).

We can show (see, [13], and/or, [14]) that the moment-generating function of the random variable $\tau(x)$ defined in Eq. (3) now satisfies the integro-differential equation (IDE)

$$\frac{1}{2}\sigma^2 x M''(x;\alpha) + a(b-x)M'(x;\alpha) + \lambda\left\{\frac{1}{x}\int_{-x}^0 M(x+y;\alpha)\,\mathrm{d}y - M(x;\alpha)\right\} = \alpha M(x;\alpha). \quad (55)$$

Similarly, the mean $m(x)$ of $\tau(x)$ satisfies

$$\frac{1}{2}\sigma^2 x m''(x) + a(b-x)m'(x) + \lambda\left\{\frac{1}{x}\int_{-x}^0 m(x+y)\,\mathrm{d}y - m(x)\right\} = -1. \quad (56)$$

The boundary conditions are $m(0) = m(c) = 0$.

Let

$$p(x) := P[X(\tau(x)) = 0]. \quad (57)$$

This function is a solution of the IDE

$$\frac{1}{2}\sigma^2 x p''(x) + a(b-x)p'(x) + \lambda\left\{\frac{1}{x}\int_{-x}^0 p(x+y)\,\mathrm{d}y - p(x)\right\} = 0 \quad (58)$$

such that $p(0) = 1$ and $p(1) = 0$. We will calculate the functions $m(x)$ and $p(x)$ in the case when $a = 0$.

We have

$$\int_{-x}^0 m(x+y)\,\mathrm{d}y = \int_0^x m(z)\,\mathrm{d}z \quad (59)$$

(and similarly with the function $p(x)$). Differentiating Eq. (56), we find that the function $m(x)$ satisfies the linear third-order ODE

$$\frac{1}{2}\sigma^2 x m'''(x) + \sigma^2 m''(x) - \lambda m'(x) = -\frac{1}{x}. \quad (60)$$

Likewise, we have

$$\frac{1}{2}\sigma^2 x p'''(x) + \sigma^2 p''(x) - \lambda p'(x) = 0. \quad (61)$$

Let us choose $\sigma = \sqrt{2}$, $\lambda = 1$ and $c = 1$. We find that

$$m(x) = d_1 I_0\left(2\sqrt{x}\right) + d_2 K_0\left(2\sqrt{x}\right) + 2\ln\left(2\sqrt{x}\right) + d_3, \quad (62)$$

where $I_0(\cdot)$ and $K_0(\cdot)$ are modified Bessel functions. The constants d_1, d_2 and d_3 can be determined as follows: we impose the conditions $m(0) = m(1) = 0$ and $m(0.5) = r$. Then, we find that r must be equal to (approximately) 0.3281 in order for the function $m(x)$ thus obtained to satisfy the IDE in (56). The expression for $m(x)$ is rather long and will therefore not be reproduced here.

When there are no jumps, the function $m_0(x)$ that corresponds to $m(x)$ satisfies the second-order linear ODE

$$x m_0''(x) = -1. \quad (63)$$

The solution for which $m_0(0) = m_0(1) = 0$ is

$$m_0(x) = -x\ln(x) \quad \text{for } 0 \leq x \leq 1. \quad (64)$$

The functions $m(x)$ and $m_0(x)$ are displayed in Figure 3. We can see the effect of the jumps on the expected value $m(x)$.

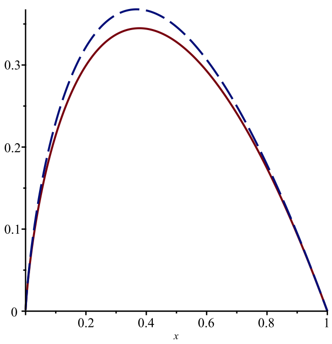

Figure 3: Functions $m(x)$ (full line) and $m_0(x) = -x\ln(x)$ for x in the interval $[0, 1]$.

To obtain the function $p(x)$, we must solve the ODE

$$x p'''(x) + 2p''(x) - p'(x) = 0. \quad (65)$$

The solution such that $p(0) = 1$, $p(0.5) = \rho$ and $p(1) = 0$ is

$$p(x) = \frac{I_0(2) - I_0(2\sqrt{x})}{I_0(2) - 1}, \tag{66}$$

which is valid if and only if $\rho \approx 0.5576$. In the absence of jumps, the corresponding function $p_0(x)$ satisfies the simple ODE

$$x p_0''(x) = 0. \tag{67}$$

The solution for which $p_0(0) = 1$ and $p_0(1) = 0$ is $p_0(x) = 1 - x$. See Figure 4.

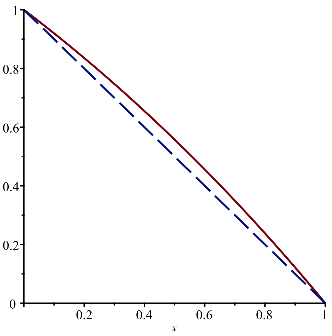

Figure 4: Functions $p(x)$ (full line) and $p_0(x) = 1-x$ for x in the interval $[0, 1]$.

Finally, in Figure 5 we present the functions $p(x)$ and $p_0(x)$ when the parameter λ is equal to 10, instead of 1. We observe the much greater effect of the jumps on the probability of absorption at the origin.

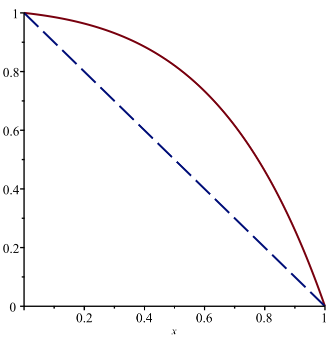

Figure 5: Functions $p(x)$ (full line) and $p_0(x) = 1-x$ for x in the interval $[0, 1]$, when $\lambda = 10$.

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