On the Oscillation in Multidimensional Odd Competitive Systems

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Abstract: The paper considers a multidimensional odd competitive model, which is a generalization to a multidimensional (more than two dimensions) case of both the P. Verhulst logistic model and the A. Lotka–V. Volterra competition model.

Key-Words: - Population models, Multidimensional competition, Oscillation, Modeling, Simulation

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1 Introduction

When modeling periodic processes, conservative "predator-prey" like systems, represented by even skew-symmetric matrices, are known sources of oscillations. Nevertheless, it is not always natural to use such tools; for example, in the economy, here no one clearly eats anyone, but everyone competes with each other.

Odd skew-symmetric matrices are known to be degenerate. However, an addition with a degenerate matrix consisting of "1" elements can remove this degeneracy, and we get an odd competition matrix. We can investigate such a system analytically, but with very strong assumptions about the shape of the matrix. The main conclusion of the study is that $(n–1)/2$ oscillations with different frequencies occur in the proposed odd n-dimensional system.

Numerical experiments show that if the assumptions in which an analytical solution to the system is available are significantly weakened, oscillations remain. Moreover, such oscillations occur if the differential equations model is replaced by a discrete analog implemented in the environment of cellular automata.

It is known that the trajectory evolution ends in a stable node in one, [1], and two dimensional, [2], models of competition. Starting from 3 dimensions, a stationary point can have the type of center or focus; there may be oscillations of the trajectory around the stationary point. This occurs when a non-transitive order relates the coefficients of the double standard matrix, similar to the game "rock, paper, scissors." For example, in the three-dimensional case, there are populations A, B, and C. A is tolerant of B and intolerant of C; B is tolerant of C and intolerant of A; and finally, C is tolerant of A and intolerant of B. In this paper, we will investigate multidimensional ($n \geq 3, n$ is odd) competition systems for the oscillation occurrence.

2 A Bit of Linear Algebra and Matrix Theory

Let $1 > d > 0$. We will consider square $n \times n$ matrices:

$$\left\| \begin{array}{cccc} 1 & 1+d & ... & 1-d \\ 1-d & 1 & ... & 1+d \\ ... & ... & ... & ... \\ 1+d & 1-d & ... & 1 \end{array} \right\|, \tag{1}$$

where n is an odd, and device-like even matrix:

$$\left\| \begin{array}{cccc} 1 & 1+d & ... & 1+d \\ 1-d & 1 & ... & 1-d \\ ... & ... & ... & ... \\ 1-d & 1+d & ... & 1 \end{array} \right\|, \tag{2}$$

where n is even.

These seemingly similar matrices actually have very different properties, as we will see further.

Along with the indicated matrices (1) and (2), we will consider the associated skew-symmetric matrices:

$$\left\| \begin{array}{cccc} 0 & d & ... & -d \\ -d & 0 & ... & d \\ ... & ... & ... & ... \\ d & -d & ... & 0 \end{array} \right\|, \tag{3}$$

n is odd, and

$$\left\| \begin{array}{cccc} 0 & d & ... & d \\ -d & 0 & ... & -d \\ ... & ... & ... & ... \\ -d & d & ... & 0 \end{array} \right\|, \tag{4}$$

n is even.

Let us prove that the determinant of the matrix (2) equals to d^n. We will prove it by induction. For $n = 2$, this is true: suppose this formula is true up to the determinant of the matrix (2) of dimension $n-2$. Let us calculate the determinant of the n-dimensional matrix. For this, let us subtract from its last row the first. The determinant will not change from this, and its last line takes the form $(-d, 0, ..., 0, -d)$. Let us decompose the determinant by this line. Note that the algebraic complement of the last element in the string is a determinant of matrix (1), of $n-1$ dimension. Let's denote its size via Δ_{n-1}. The minor corresponding to the first element of the line has the following form:

$$\left\| \begin{matrix} 1+d & 1-d & ... & 1-d & 1+d \\ 1 & 1+d & ... & 1+d & 1-d \\ ... & ... & ... & ... & ... \\ 1-d & 1+d & ... & 1 & 1+d \end{matrix} \right\|.$$

Let's lower the first line down, changing it sequentially with the second,..., $n-1$. This will not change the sign of the determinant, since the number of exchanges is even. Note that in the algebraic complement, the determinant of this minor comes with a minus sign. So we need to compute:

$$A_{n,1} = -\begin{vmatrix} 1 & 1+d & ... & 1-d \\ 1-d & 1 & ... & 1+d \\ ... & ... & ... & ... \\ 1+d & 1-d & ... & 1+d \end{vmatrix}.$$

Note that this determinant differs from the determinant of the matrix (1) of dimension $n-1$ only by $+d$ in the last diagonal element; therefore, due to the linearity of the determinant as a column function,

$$A_{n,1} = -\Delta_{n-1} - \begin{vmatrix} 1 & ... & 1+d & 0 \\ 1-d & ... & 1-d & 0 \\ ... & ... & ... & ... \\ 1+d & ... & 1-d & d \end{vmatrix} =$$

$$= -\Delta_{n-1} - d \cdot d^{n-2}.$$

We decomposed the last term into the last column and used the induction hypothesis. Finally, we calculate our determinant:

$$\Delta_n = -dA_{n,1} - dA_{n,n} =$$

$$= -d(-\Delta_{n-1} - d^{n-1}) - d\Delta_{n-1} = d^n,$$

which is what was required to be proved.

Note that the same reasoning with the same result d^n, is also applicable to the matrix determinant (4). Moreover, if A – a non-degenerate skew-symmetric even matrix, and B differs from it by the fact that 1 is added to each of its elements, then due to the linearity determinant, as functions of a column, recalling Kramer's rule, we have:

$$|B| = |A| + |B|\,(\bar{y}, e), \tag{5}$$

where e is a vector of units $(1, ..., 1)$, and $\bar{y}$ is a solution for the equation $By = e$. Next, note that $|B|\,(\bar{y}, e) = |A|\,(\bar{x}, e)$, where $\bar{x}$ is the solution to the equation $Ax = e$ – in the matrix has a unit column that can be subtracted from the rest, not changing the determinant values. But by virtue of the fact that the matrix A skew-symmetric, we conclude: $(x, Ax) = -(xA, x) = 0$, $\forall x$, whence follows $(\bar{x}, e) = (\bar{x}, A\bar{x}) = 0$, and hence $|B| = |A|$.

Note that for odd matrices, everything is completely different. As it is known, the determinant of the odd skew-symmetric matrix is zero: $|A| = |-A| = -|A| = 0$, the determinant of the same matrix (1), as we have to find out next, is $n^2 d^{n-1}$.

Now let us move on to calculating the determinant of the matrix (1). Let's choose in the matrix (1) j-th column and add all the others to it. The determinant of the matrix will not change from this, so we get:

$$\begin{vmatrix} 1 & 1+d & ... & 1-d \\ 1-d & 1 & ... & 1+d \\ ... & ... & ... & ... \\ 1+d & 1-d & ... & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & ... & n & ... & 1-d \\ 1-d & ... & n & ... & 1+d \\ ... & ... & ... & ... & ... \\ 1+d & ... & n & ... & 1 \end{vmatrix} =$$

$$= n \begin{vmatrix} 1 & ... & 1 & ... & 1-d \\ 1-d & ... & 1 & ... & 1+d \\ ... & ... & ... & ... & ... \\ 1+d & ... & 1 & ... & 1 \end{vmatrix} = \tag{6}$$

$$= n \begin{vmatrix} 0 & ... & 1 & ... & -d \\ -d & ... & 1 & ... & d \\ ... & ... & ... & ... & ... \\ d & ... & 1 & ... & 0 \end{vmatrix}. \tag{7}$$

The resulting formula is an analog of (5), so in this case $|A| = 0$. Obviously, determinants (6) and (7) at any $0 \le j \le n$ are equal to each other, and all are equal to $\frac{1}{n}\Delta_n$, where Δ_n is the desired determinant of the matrix (1). When we prove that $\Delta_n > 0$, from here it will follow that if denoted by D matrix (1), then the solution of the equation $Dx = e$ is the vector $\bar{x} = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$. Consider now the determinant of the form (7) at $j = n$:

$$\Delta = \begin{vmatrix} 0 & d & ... & d & 1 \\ -d & 0 & ... & -d & 1 \\ ... & ... & ... & ... & ... \\ d & -d & ... & -d & 1 \end{vmatrix} \tag{8}$$

Let us try to calculate the determinant (8) using the bordered minor formula:

$$\Delta = \begin{vmatrix} & & 1 \\ & A_{n-1} & ... \\ d & ... & 1 \end{vmatrix} =$$

$$= |A_{n-1}| - (d, ..., -d)\tilde{A}_{n-1}\begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix} =$$

$$= d^{n-1} + d\sum_{j,k=1}^{n-1}(-1)^k A_{j,k} =$$

$$= d^{n-1} + \sum_{k=1}^{n-1}(-1)^k d\sum_{j=1}^{n-1} A_{j,k}.$$

Here A_{n-1} is a well-known even matrix of type (4), whose determinant $|A_{n-1}| = d^{n-1}$ was calculated previously; $\tilde{A}_{n-1}$ is an adjugate matrix with A_{n-1}, i.e., such that in place of each element $a_{i,j}$ is its cofactor $A_{i,j}$; and finally $A_{j,k}$ – already mentioned cofactors of the elements of the matrix A_{n-1}.

Note that the line $\left\{(-1)^k d\right\}_{k=1}^{n-1}$ is a sum of all rows of matrix (4) of dimension $n-1$, hence, remembering the property of the products of the determinant strings on their cofactors, conclude:

$$\sum_{k=1}^{n-1}(-1)^k d\sum_{j=1}^{n-1} A_{j,k} = \sum_{k=1}^{n-1}\sum_{l=1}^{n-1} a_{l,k}\sum_{j=1}^{n-1} A_{j,k} =$$

$$= \sum_{l=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} a_{l,k}A_{j,k} =$$

$$= \sum_{l=1}^{n-1}\sum_{j=1}^{n-1}\delta_{l,j}\,|A_{n-1}| = (n-1)d^{n-1}.$$

From this, we get the value of the determinant of the matrix (8):

$$\Delta = nd^{n-1}.$$

Having calculated the determinant (8), and knowing that the desired determinant of the matrix (1) is n times more, we can finally write the determinant value of the odd matrix (1):

$$\Delta_n = n^2 d^{n-1}.$$

We now calculate the eigenvalues of the matrix (1). Note that this matrix is a circulant – a special kind of Toeplitz matrix that is fully determined by its first row. Subsequent rows are obtained from the previous ones by a cyclic shift to the right by one position. From the linear algebra course, [3], it is known that the circulant is diagonalized by the discrete Fourier transform, which is given by the following Vandermonde matrix:

$$\begin{Vmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \varepsilon & \varepsilon^2 & ... & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & ... & \varepsilon^{2(n-1)} \\ ... & ... & ... & ... & \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & ... & \varepsilon^{(n-1)^2} \end{Vmatrix},$$

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ – is n-th degree root of 1. In this case, the eigenvalues of the matrix (1) can be determined by the formula:

$$\lambda_j = 1+(1+d)\varepsilon_j+(1-d)\varepsilon_j^2+...+(1-d)\varepsilon_j^{n-1}, \quad (9)$$

where $\varepsilon_j = \cos\frac{2\pi j}{n} + i\sin\frac{2\pi j}{n}$, $0 \le j \le n-1$.

Let's try to calculate these eigenvalues. The easiest way to do this is at j =0, then ε_0 =1 and $\lambda_0 = n$ is the only real eigennumber. Let now $j > 0$, we will calculate λ_j by formula (9). We will use the known property roots of the n-th degree of 1, following from De Moivre's formula:

$$\varepsilon_j^k = \varepsilon_1^{jk} = \cos\frac{2\pi jk}{n} + i\sin\frac{2\pi jk}{n}.$$

First, note that all members associated with 1 disappear from the sum of formula (9). Indeed, by the formula of the geometric progression sum

$$\sum_{k=0}^{n-1}\varepsilon_j^k = \frac{1-\varepsilon_j^n}{1-\varepsilon_j} = 0.$$

Secondly, all members associated with cosines are reduced. Indeed, at odd k, $0 < k < n$ terms $d\cos\frac{2\pi jk}{n}$ are included in the sum of (9) with a plus sign, and with even ones with a minus sign. Each k, $0 < k < \frac{n}{2}$ matches to l, $\frac{n}{2} < l < n$, such that $l = n-k$, so

$$\cos\frac{2\pi jk}{n} = \cos\left(-\frac{2\pi jk}{n}\right) =$$

$$= \cos\left(-\frac{2\pi jk}{n} + 2\pi j\right) = \cos\frac{2\pi jl}{n},$$

but k and l have opposite parity since n is odd, and therefore are included in the sum (9) with opposite signs. By virtue of the above, all cosines in (9) are mutually destroyed, and all the sines will double, on the contrary, as odd functions. Given what has been said, rewrite (9) as follows:

$$\lambda_j = 2id\sum_{k=1}^{\frac{n-1}{2}} \sin\left(\frac{2\pi jk}{n}\right) =$$

$$= 2id\frac{\sin\left(\frac{\frac{n-1}{2}+1}{2}\cdot\frac{2\pi j}{n}\right)\sin\left(\frac{n-1}{4}\cdot\frac{2\pi j}{n}\right)}{\sin\left(\frac{\pi j}{n}\right)} =$$

$$= 2id\frac{\sin\left(\frac{\pi j}{2}+\frac{\pi j}{2n}\right)\sin\left(\frac{\pi j}{2}-\frac{\pi j}{2n}\right)}{\sin\frac{\pi j}{n}} =$$

$$= \frac{2id}{\sin\frac{\pi j}{n}}\left(\sin\frac{\pi j}{2}\cos\frac{\pi j}{2n}+\cos\frac{\pi j}{2}\sin\frac{\pi j}{2n}\right)\times$$

$$\times\left(\sin\frac{\pi j}{2}\cos\frac{\pi j}{2n}-\cos\frac{\pi j}{2}\sin\frac{\pi j}{2n}\right) =$$

$$= id\frac{\sin^2\frac{\pi j}{2}\cos^2\frac{\pi j}{2n}-\cos^2\frac{\pi j}{2}\sin^2\frac{\pi j}{2n}}{\sin\frac{\pi j}{2n}\cos\frac{\pi j}{2n}}.$$

The latter expression behaves differently depending on the parity of j. At the even j, $\sin\frac{\pi j}{2} = 0$ and $\cos^2\frac{\pi j}{2} = 1$, so $\lambda_j = -id\tan\frac{\pi j}{2n}$. At the odd j, $\sin^2\frac{\pi j}{2} = 1$ and $\cos\frac{\pi j}{2} = 0$, so $\lambda_j = id\cot\frac{\pi j}{2n}$.

Note that every even l, $0 < l < n$, corresponds to an odd j, $0 < j < n$ such that $j = n - l$ and vice versa. From here, we conclude:

$$\lambda_j = id\cot\left(\frac{\pi j}{2n}\right) = id\cot\left(\frac{\pi(n-l)}{2n}\right) =$$

$$= id\cot\left(\frac{\pi}{2}-\frac{\pi l}{2n}\right) = id\tan\left(\frac{\pi l}{2n}\right) = -\lambda_l.$$

Pairs of purely imaginary eigenvalues of the matrix (1) are, as expected, complex conjugates. Now we can write a single formula for pairs of complex conjugate eigenvalues:

$$\lambda_{k_{1,2}} = \mp id\tan\left(\frac{\pi k}{n}\right), 1 \le k \le \frac{n-1}{2}. \quad (10)$$

Further, on the one hand, as shown above, the determinant of the matrix (1) is n^2d^{n-1}, with the other being the product of all eigenvalues: $nd^{n-1}\prod_{k=1}^{\frac{n-1}{2}}\tan^2\left(\frac{\pi k}{n}\right)$. Equating these expressions, we get a curious identity: $n = \prod_{k=1}^{\frac{n-1}{2}}\tan^2\left(\frac{\pi k}{n}\right)$, or:

$$2n+1 = \prod_{k=1}^{n}\tan^2\left(\frac{\pi k}{2n+1}\right).$$

This equality is a special case of Eulerian decomposition of a sine into a product [4]. When $n = 1$, it is well-known: $\tan\frac{\pi}{3} = \sqrt{3}$.

The eigenvector corresponding to $\lambda_0 = n$ is the vector e, with "1" components. Invariant two-dimensional subspaces, corresponding to pairs of imaginary eigenvalues $\lambda_{k_{1,2}}$ (10), are linear shells of the following vector pairs:

$$\begin{cases} \left(1, \cos\left(\frac{2\pi k}{n}\right), ..., \cos\left(\frac{2\pi(n-1)k}{n}\right)\right), \\ \left(1, \sin\left(\frac{2\pi k}{n}\right), ..., \sin\left(\frac{2\pi(n-1)k}{n}\right)\right), \\ 1 \le k \le \frac{n-1}{2}. \end{cases}$$

3 Odd Competitive System Analysis

We will consider the n-dimensional system of competition equations (n – odd, $n > 1$):

$$\frac{dx_i}{dt} = \alpha_i x_i\left(1-\sum_{j=1}^{n} m_{i,j}\frac{x_j}{x_j^*}\right), m_{k,k} = 1. \quad (11)$$

Here α_i is the Malthusian factor of the population i, the environment capacity x_i^*, that is the maximum population size, which would be established in this model if there were no other populations in it. A matrix $M = \|m_{i,j}\|_{i,j=1}^n$ – is a competition matrix. Its components $m_{i,j}$ are the double standard factors, their meaning in the subject area of the model – comparison of inter-population competition with the intra-population one. So $m_{i,j}$ shows how many times the competition of population j with population i is stronger ($m_{i,j} > 1$, j is intolerant for i), or vice versa, – weaker ($m_{i,j} < 1$, j is tolerant for i) than the competition within the population j itself. Since the competition within populations acts here as a measurement standard, so all diagonal elements are equal to 1.

Let us make a few assumptions for simplicity:

- Get rid of environment capacities in (11) by moving to the relative numbers of populations

$X_i = \frac{x_i}{x_i^*}$. This substitution is reversible, so if we get any results then by making a reverse substitution, we will interpret them in terms of the initial system (11) with environment capacities.

- Assume the same Malthusian factors: $\alpha_i = \alpha$, $1 \le i \le n$. This is a much more restrictive assumption; however the author is not able to get any meaningful results without it. Having made this assumption, we can get rid of α in the equations by replacing time $t = \alpha\tau$.
- We assume that the matrix $M = \|m_{i,j}\|_{i,j=1}^n$ in (11) is of the form (1), described by the formula (12), because we showed above that it produces the oscillation. $M = \|d_{i,j}\|_{i,j=1}^n$, where

$$d_{i,j} = \begin{cases} 1-(-1)^{j-i}d, & i<j; \\ 1, & i=j; \\ 1+(-1)^{i-j}d, & i>j. \end{cases} \tag{12}$$

Here $1 > d > 0$.

As a result of the assumptions and transformations made, we get the following system of equations:

$$\dot{X}_i = X_i \left(1 - \sum_{j=1}^n d_{i,j} X_j\right). \tag{13}$$

In the previous section it was shown that the determinant of the matrix $D = \|d_{i,j}\|_{i,j=1}^n$ is $n^2 d^{n-1}$ >0, so the solution of the linear equations system

$$1 - \sum_{j=1}^n d_{i,j} X_j = 0$$

exists, is unique, equals $\bar{X} = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$, according to the previous section. The vector $\bar{X}$ is the stationary point of the system (13). We examine the system (13) in a small vicinity of this stationary point. Let $X_i = \bar{X}_i + x_i$, where x_i are small. Then, neglecting the higher orders of smallness, we get:

$$\dot{x}_i = -\bar{X}_i \sum_{j=1}^n d_{i,j} x_j = -\frac{1}{n} \sum_{j=1}^n d_{i,j} x_j,$$

and after another time replacement $\tau = \frac{1}{n} t$:

$$\dot{x}_i = -\sum_{j=1}^n d_{i,j} x_j. \tag{14}$$

The resulting system (14) is a linear homogeneous system of differential equations with constant coefficients. To solve this system, we are to find the eigenvalues and eigenvectors of the matrix $-D$.

We found eigenvalues and invariant subspaces for the matrix (1), $D = \|d_{i,j}\|_{i,j=1}^n$ in the previous section. In the equations (14) the same matrix appears, but with a minus sign. Everything remains true to it with the following correction: the only real eigennumber $-n$ and determinant $-n^2 d^{n-1}$ now become negative; the rest, purely imaginary eigenvalues, are still determined by the formula (10). Also remain invariant for $-D$, all found for D invariant subspaces.

The trajectory of the system (13) in the small vicinity of the stationary point $\bar{X}$ behaves as follows: negative real eigennumber "pulls" it into the affine hyperplane $\{X : X = \bar{X} + x, (x, e) = 0\}$. This hyperplane passes through n points

$$(1.0, ..., 0), (0.1, ..., 0), \dots, (0, ..., 1).$$

It is a direct sum of $(n-1)/2$ two-dimensional affine subspaces, in which oscillations occur with frequencies:

$$d \tan\left(\frac{\pi k}{n}\right), 1 \le k \le \frac{n-1}{2}.$$

Recall that we twice made a time replacement: $t = \alpha\tau$ and $\tau = \frac{1}{n} t$. Returning to the original time, we get the oscillation frequencies:

$$\frac{\alpha d}{n} \tan\left(\frac{\pi k}{n}\right), 1 \le k \le \frac{n-1}{2}. \tag{15}$$

Note that at $n \to \infty$ is true:

$$\frac{\alpha d}{n} \tan \frac{\pi(n-1)}{2n} = \frac{\alpha d}{n} \frac{\cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \underset{n\to\infty}{\longrightarrow} \frac{2\alpha d}{\pi}. \tag{16}$$

The frequency of the highest harmonic of the odd n-dimensional systems tends to $\frac{2\alpha d}{\pi}$, at $n \to \infty$. The limit frequencies of the previous harmonics will be less in 2, 3, 4, 5,... times.

Returning to the original variables $x_i = x_i^* X_i$, we get: stationary point $\bar{x}$ of system (1) has components $\bar{x}_i = \frac{x_i^*}{n}$, and the oscillation hyperplane passes through the points of environment capacities:

$$(x_1^*, 0, ..., 0), (0, x_2^*, 0, ..., 0), \dots, (0, ..., 0, x_n^*).$$

This hyperplane is the boundary of tolerance that divides tolerant and intolerant regions in the phase space of the system (11).

The equation $1 - \sum_{i=1}^{n} d_{i,j} \frac{x_i}{x_i^*} = 0$ defines the stationary point of the system (11). Its solution is:

$$(\bar{x}_1, ..., \bar{x}_n), \;\; \bar{x}_i = \frac{x_i^*}{n}. \tag{17}$$

The invariant subspaces in the original variables will look like this: $(x_1^*, x_2^*, ..., x_n^*)$ – one-dimensional subspace corresponding to a single real eigenvalue, and two-dimensional subspaces corresponding to the conjugate imaginary pairs:

$$\begin{cases} \left(x_1^*, x_2^* \cos\frac{2\pi k}{n}, ..., x_n^* \cos\frac{2\pi(n-1)k}{n}\right), \\ \left(x_1^*, x_2^* \sin\frac{2\pi k}{n}, ..., x_n^* \sin\frac{2\pi(n-1)k}{n}\right), \\ \qquad 1 \le k \le \frac{n-1}{2}. \end{cases} \tag{18}$$

4 Simulation Experiments

The 7-dimensional competition model was built in the AnyLogic, [5, 6], simulation system using its System Dynamics tools.

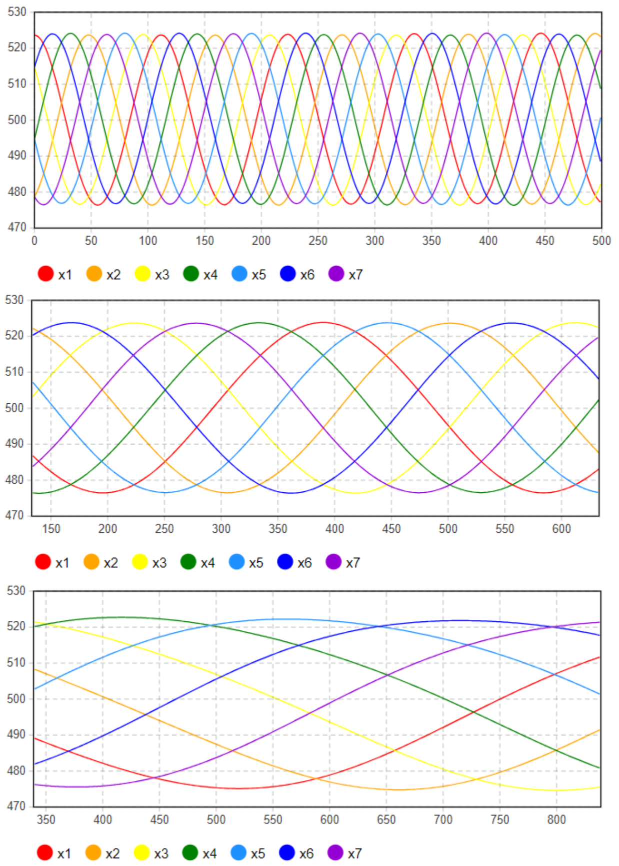

Figure 1: Oscillations of Three Different Frequencies in the Small Neighborhood of the Stationary Point

According to the theory described above, three types of oscillations with different frequencies are possible in such a system: $3 = (7 - 1)/2$. Each of these types can be distinguished separately by setting initial conditions in the corresponding invariant two-dimensional subspace.

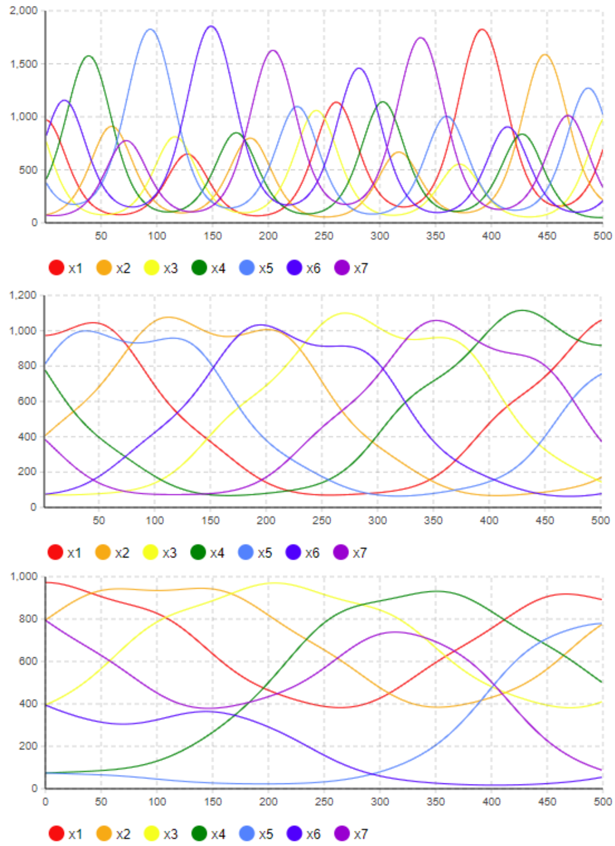

Figure 2: Oscillations of Three Different Frequencies in the Large Neighborhood of the Stationary Point

By analogy with the “predator-prey” model, [2, 7], it is interesting to consider the potentials of individual populations

$$C(x_i) = \frac{1}{\alpha_i}\left(\ln x_i - \frac{n x_i}{x_i^*}\right), \tag{19}$$

where $n = 7$, and the potential of the entire system, which is their sum

$$C = \sum_{i=1}^{n} \frac{1}{\alpha_i}\left(\ln x_i - \frac{n x_i}{x_i^*}\right). \tag{20}$$

It is also interesting to check, based on the simulation experiments, whether the potential of the entire system is constant or not.

Let us start by highlighting the different types of oscillations. As initial conditions, we take the coordinates of a stationary point

$$(\bar{x}_1, ..., \bar{x}_n), \bar{x}_i = \frac{x_i^*}{n},$$

to which we add vectors of one of the invariant subspaces from (18) that are small in value.

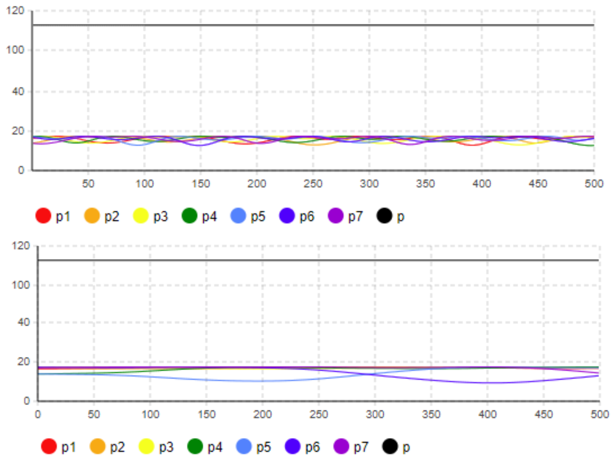

Figure 3: Potentials in the Large Vicinity of the Stationary Point, the Highest and the Lowest Frequencies

As a result, we get three types of oscillations with different frequencies, which are shown in Fig. 1. It can be seen that in the small neighborhood of the stationary point, the oscillations are indistinguishable from harmonic ones.

Further, we show the results of experiments with large deviations of the trajectory from the stationary point.

In this case, the two-dimensional subspaces lose their invariance; even if we take the initial data in such a space, the remaining oscillation frequencies still contribute to the trajectory. The harmonic nature of the oscillations is also disrupted (Fig. 2).

When deviations of the trajectory from the stationary point are small, the values of the populations' potentials (19) are approximately equal to their value at the stationary point and practically do not change.

It is interesting to consider the dynamics of the populations' potentials (19) and the potential of the entire system (20), when the deviations from the stationary point are sufficiently large. We see in Fig. 3, that the system potential remains constant (we took the highest and lowest frequencies).

Finally, let us take arbitrary initial data. In addition, we let the Malthusian factors be different.

We see that the potential of the entire system (20) remains constant even with unequal Malthusian coefficients and arbitrary (non-owned to invariant subspaces) initial data. Whereas the potentials of individual populations fluctuate in Fig. 3 and Fig. 4.

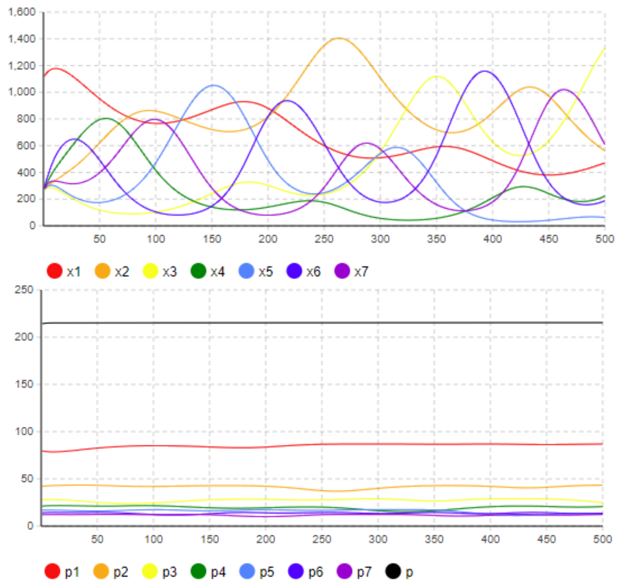

Figure 4: Oscillation and Potentials in the Large Vicinity with Different Malthusian Factors

In contrast to Fig. 3, where the potentials of the populations are approximately the same, in Fig. 4 they differ in magnitude markedly. This is because the Malthusian factors differ, see (19).

5 Simulating Multidimensional Odd Competition by Cellular Automata

In the work [8], a description of the cell automatic analog of the multidimensional competition model was given. Let us quote from there the rules of automata dynamics.

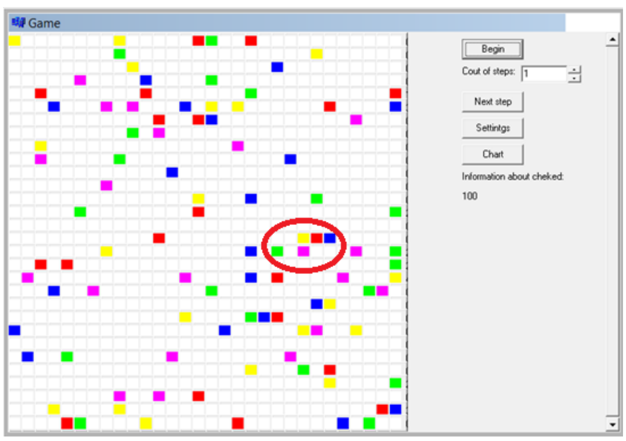

Figure 5: Working Area of 5-Dimensional Cellular Automata. Five different agents are outlined

For any cell, we will enter a parameter P_j –

competitive pressure force of the j-th population. It equals the sum of the agents of j-th type in the Moore vicinity of that cell.

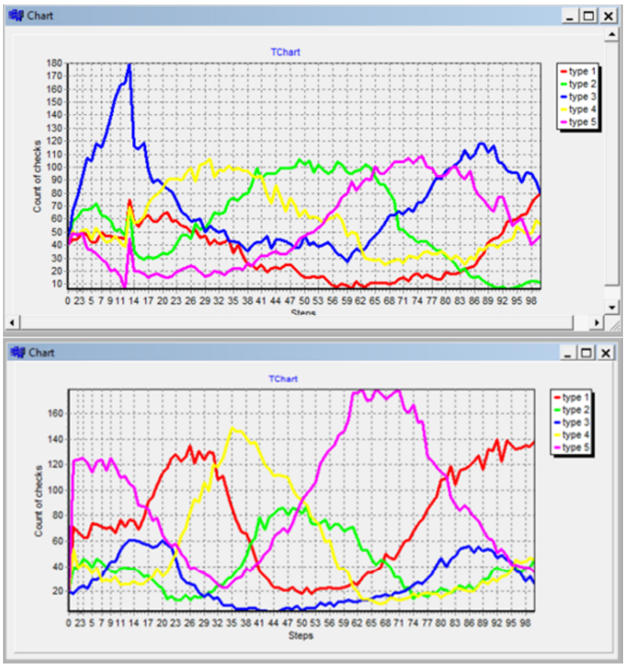

Figure 6: Oscillation in 5-Dimensional Cellular Automata

The cellular automata field has a square, toroidal-closed shape (Fig. 5). Each cell either contains an agent of one of the five types, or remains empty.

At each simulation step, for each agent:

1. The competitive pressure K on the cell in which this agent (of some type i) is located is calculated. In this case, this means the sum of the products of the double standard factor $m_{i,j}$ (attitude of the agent j towards the currently processed agent i) by P_j, where j runs through all types of agents located in the Moore vicinity. If it turns out to be greater than or equal to K_i, – the death threshold, then the agent dies (and the subsequent rules do not apply to it). Here K_i indicates resistance to the competition for the i-th population – a parameter describing the minimum level of competitive pressure that the agent cannot withstand.

2. If there is at least one free cell near the agent, then it passes into a random free cell.

3. The age of an agent increases by one. If an agent reaches the reproduction age, its age is reset to zero, – and a descendant is created in the abandoned cell of the same type with zero age.

It can be shown that the reproduction age is reciprocal to the Malthusian coefficient, i.e., $\alpha = 1/3 = 0.33$. In our case, $d = 0.3$. Hence, the short oscillation period is approximately 100, and the long one is approximately 430, according to the formula for the highest frequency, following from (15): $\approx \frac{2\alpha d}{\pi}$. This is what we see in the top half of Fig. 6.

Now increase the Malthusian factor; reduce the reproduction age to 2, then $\alpha = 1/2 = 0.5$. Now the shorter oscillation period should decrease to about 70, which we see at the bottom of Fig. 6.

6 Conclusion

The analysis of the odd n-dimensional system (11) with the double standard matrix of the type (1) shows that there are $(n-1)/2$ oscillations with frequencies given by (15) near the stationary point (17), and two-dimensional invariant subspaces are given by (18).

We can relax the rigid assumptions in numerical experiments. For example, you can allow the Malthusian factors to be different (Fig. 4). We see that the oscillations remain.

Striving for the limit of the highest frequency (16) is interesting. This means that at large dimensions, oscillations with approximately the same frequencies are noticeable in competitive systems. For example, the cycles of Kitchin (3-4 years), Zhuglyar (7-11 years), Kuznets (15-25 years), and Kondratiev (45-60 years) are noticeable in the economy, [9] and [10]. The ratios between periods are about the same as in 9 and 11 dimensional systems of our type. Perhaps the cyclical nature of the economy is due, among other things, to competition, with some a bit smaller (related parties) and others – larger (competitors).

Simulation experiments show that in odd competitive systems of the form (11) with the double standard matrix of the type (1), the potential of the system (20) conserves on the trajectory. At the same time, the populations exchange part of their potential (19) with each other (Fig. 3 and Fig. 4).

Odd five-dimensional competition equations are implemented in the environment of cellular automata, [8]. The adequacy of this implementation is proved by the existence of such a subtle effect as oscillations and a good correspondence between the theoretical and observed oscillation periods.

As a side result, an interesting identity is obtained. It turns out that any odd number is a complete square, although not of integers, but still of very familiar quantities: tangents of the first quadrant angles.

$$2n+1 = \prod_{k=1}^{n} \tan^2\left(\frac{\pi k}{2n+1}\right).$$

For example, a New Year's theme:

$$2025 = \prod_{k=1}^{1012} \tan^2\left(\frac{\pi k}{2025}\right) = \prod_{k=1}^{22} \tan^4\left(\frac{\pi k}{45}\right).$$

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