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One – Sided Approximation in $L_p(X)$

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Abstract: - The aim of this research to study the approximation of functions in the space- L_p by the "algebraic polynomial" in terms of the" average modulus" of the k-order also, we will estimate the degree of the (O-S- A), (means one – sided approximation) in term of averaged modulus where all the results which number is eleven we need to prove the main theorem that (the degree of best (O-S- A) of f by trigonometric polynomials of order n in $L_p(X)$, $(\widetilde{E_n}(f)_p)$) is less than or equal to (Averaged modulus of smoothness of f of order-k, $(\tau_k\left(f,\frac{1}{n}\right)_p)$) have been proven, It has also been proven the converse theorem for the main theorem in this research.

Key-Words: - Modulus of continuity, local of smoothness, trigonometric polynomials, average modulus, degree of best approximation, periodic functions.

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1 Introduction

(O-S- A) was studied with unity space in in L_1-space and quadrature formulae, [1]. Also the problem of the uniqueness of elements of the best approximations in the $L_p[a,b]$ space has been studied and the problem of the best approximation, the best (α,β)-approximation of continuous functions and the problem of the (O-S- A), of continuously differentiable functions have been analyzed, [2]. On the other hand, (O-S- A) was presented in L_p −norm and the difference degree between the function and the polynomials used in the research has been obtained, for more information, [3]. Also, some researchers got the (O-S- A) of the form W_∞^1 of differentiable functions by" algebraic polynomials" in(L_1 −space), [4]. Moreover, authors studied polynomials of the (O-S- A) to a step function on [-1,1], and they proved that polynomials are obtained by Hermite interpolation at the zeros of some quasi-orthogonal Jacobi polynomial, [5]. After that, in 2016 a study, obtained, the (O-S- A) of functions of several variables, by HAAR Polynomials by modulus of continuity $\omega_1(f,x)$, [6].

In the same year, two researchers studied positive factors for the(O-S-A) of the infinite functions in the weighted _space_ $L_{p,\alpha}(X)$ and provided an estimate of the degree of the (O-S- A) in terms of the mean continuity coefficient, [7]. Now in this paper we will prove the degree of best (O-S- A) of f by trigonometric polynomials of order n in $L_p(X)$ less than or equal to the integral modulus of f of order , and the Converse theorem. While most of the previous studies are about the relationship between the function and its best approximation and the amount of difference between them, they were able to prove that the difference between the function and its best approximation goes to zero when n go to infinity. So, regarding the topic of this paper, we need the following definition:
Let $L_p(Y)$ [8] is the space of all bounded functions with the norm:

$$\|g(y)\|_{L_p} = \|g(y)\|_p = \left(\int_Y |g(y)|^p\right)^{\frac{1}{p}} < \infty \ ,$$
$$Y = [a,b] \ , 1 \le p < \infty.$$

2 Main Results

In this paper we will obtain the degree of the best (O-S- A) of periodic bounded function in $L_p(X)$ − space $X = [0, 2\pi]$. Also, we will estimate the degree of the best (O-S- A) in term of averaged modulus. Before we state our main results, we need the following notes and lemmas.

Integral modulus of f of order k, $\delta \in \left[0,\frac{b-a}{k}\right]$ is defined by:

$$\omega_k(f,\delta)_p = \sup_{0\le h\le\delta}\left(\int_a^{b-kh}\left|\Delta_h^k f(x)\right|^p dx\right)^{1/p} \quad (1)$$

The local of smoothness for f of order k at point $x \in [a,b]$, $\delta \in \left[0,\frac{b-a}{k}\right]$ is defined by:

$$\omega_k(f,x,\delta)_p = \sup_{|h|<\delta}\{\|\Delta_h^k f(\mathrm{t})\|_p\, t: t, t+kh \in [x-\tfrac{k\delta}{2}, x+\tfrac{k\delta}{2}] \cap [a,b]\} \quad (2)$$

where $\Delta_h^k f(x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f(x-\frac{k\delta}{2}+ih)$, $x \mp \frac{kh}{2} \in X$, is the difference of a function f of order k with step h at a point x. Averaged modulus of smoothness of f of order- k is defined by:

$$\tau_k(f,\delta)_p = \|\omega_k(f,\cdot,\delta)\|_p,\ p \in [1,\infty),\ k \in N \quad (3)$$

Now,
Let $f \in L_p(X)$, $X = [0, 2\pi]$, f is bounded 2π – periodic function, then the degree of best (O-S- A) of f by trigonometric polynomials of order n in $L_p(X)$ is defined by:

$$\widetilde{E_n}(f)_p = \underbrace{\inf}_{n\in N}\left\{\begin{matrix}\|\mathrm{p}_n - \mathrm{q}_n\|_p : \mathrm{q}_n(x), \mathrm{p}_n(x) \in T_n, \\ \mathrm{q}_n(x) \le f(x) \le \mathrm{p}_n(x)\end{matrix}\right\} \quad (4)$$

where T_n is the set of all real trigonometric polynomials of order n. Also, the degree of best approximation of a function $f \in L_p(X)$ is define by:

$$E_p = \underbrace{\inf}_{\mathrm{p}_n \in T_n} \|f - \mathrm{p}_n\|_p.$$

Lemma1:

Let $f \in L_p(X)$, $X = [0, 2\pi]$, then

$$E_n(f)_p \le \widetilde{E_n}(f)_p \quad (5)$$

Proof:

Consider $\mathrm{q}_n, \mathrm{p}_n$ be the best (O-S- A) of , were $\mathrm{q}_n(x) \le f(x) \le \mathrm{p}_n(x)$

$E_p(f) = \inf\{\|f - \mathrm{p}_n\|_p : \mathrm{p}_n \in T_n\}$

$= \underbrace{\inf}_{n\in N}\left\{\int_X (|(f-\mathrm{p}_n)(x)|^p\, \mathrm{d}x)^{1/p} : \mathrm{p}_n \in T_n\right\}$

$\le \underbrace{\inf}_{n\in N}\left\{(\int_X (|(p_n - q_n)(x)|^p\, \mathrm{d}x))^{\frac{1}{p}} : \mathrm{p}_n \in T_n\right\}$

$= \underbrace{\inf}_{n\in N}\|p_n - q_n\|_p = \widetilde{E_n}(f)_p$ ■

Lemma 2:

Let $f \in L_p(X)$, $X = [0, 2\pi]$. Then

$$\widetilde{E_n}(f)_p \le C_p E_p(f) \quad (6)$$

C_p is a constant depending on p

Proof:

Consider $p^*(x) \in T_n$ is the best approximation of $f \in L_p(X)$ and $s_1, s_2 \in T_n$ are the best (O-S- A) of f such that $s_2(x) \le f(x) \le s_1(x)$

$\widetilde{E_n}(f)_p = \|s_1(x) - s_2(x)\|_p$

$\widetilde{E_n}(f)_p = \|s_1(x) - s_2(x) + f(x) - f(x) - p^*(x) + p^*(x)\|_p$

$= \left(\int_{\mathrm{x}}|[s_1(x) - s_2(x) + f(x) - f(x) - p^*(x) + p^*(x)]|^p dx\right)^{1/p}$

$\le \left(\int_{\mathrm{x}}|[(s_1(x) - s_2(x))]|^p dx\right)^{1/p} + \left(\int_{\mathrm{x}}|[(f(x) - p^*(x))]|^p dx\right)^{1/p} + \left(\int_{\mathrm{x}}|[(f(x) - p^*(x))]|^p dx\right)^{1/p} \le \|f - p^*\|_p + \|f - p^*\|_p + \|s_1 - s_2\|_p$

$\le 2E_n(f)_p + \|s_1 - s_2\|_p \le C_p E_n(f)_p$ ■

Lemma3:

Let $f, g, \varphi \in L_p(X)$, be 2π – periodic functions, C_p is constant depends on p. if $|[f(x) - g(x)]| \le \varphi(x)$ Then

$$\widetilde{E_n}(f)_p \le C_p(\widetilde{E_n}(g)_p + 2\widetilde{E_n}(\varphi)_p + \|\varphi\|_p) \quad (7)$$

Proof:

Let p_n^* is the best approximation of f and g_n^* is the best approximation of φ:

$E_n(f)_p = \underbrace{\inf}_{p_n \in T_n}\|f - p_n\|_p = \|f - p_n^*\|_p$

$= \left(\int_{\mathrm{x}}|f - p_n^*|^p dx\right)^{1/p}$

$= (\int_{\mathrm{x}}|f + g + \varphi + g_n^* - g - \varphi - g_n^* - p_n^*|^p d\, x)^{1/p}$

$\le \left(\int_X |g - p_n^*|^p d\, x\right)^{1/p} + \left(\int_X |(f - g)|^p d\, x\right)^{1/p} + \left(\int_X (\varphi - g_n^*)^p d\, x\right)^{1/p} + \left(\int_{\mathrm{x}}|(\varphi - g_n^*)|^p d\, x\right)^{1/p}$

$\le \|(g - p_n^*)\|_p + \|(f - g)\|_p + \|(\varphi - g_n^*)\|_p + \|(\varphi - g_n^*)\|_p$

$\le E_n(g)_p + \|\varphi\|_p + 2\widetilde{E_n}(\varphi)_p + \widetilde{E_n}(\varphi)_p$

$\le C_p\widetilde{E_n}(g)_p + \|\varphi\|_p + 2\widetilde{E_n}(\varphi)_p$ by (6). Then

$E_n(f)_p \le C_p\widetilde{E_n}(g)_p + \|\varphi\|_p + 2\widetilde{E_n}(\varphi)_p$ ■

Lemma4:

Let $f \in L_p(X)$, $X = [0, 2\pi]$, then

$$\omega(f,\delta)_p = \omega_1(f,x)_p \le \delta\,\|\bar{f}\|_p \quad (8)$$

where $\bar{f}$ first derivative of f

Proof:

$\omega(f,\delta)_p = \sup_{0\le h\le\delta} \|\Delta_h^1 f(.)\|_p =$

$\sup_{0\le h\le\delta} \left(\int_0^{2\pi} |\Delta_h^1 (f)(x)|^p\, dx\right)^{1/p}$

$= \sup_{0\le h\le\delta} \left(\int_0^{2\pi} |[(f)(x+h) - (f)(x)]|^p\, dx\right)^{1/p}$

$= \sup_{0\le h\le\delta} \left(\int_0^{2\pi} \left|\int_x^{x+h} (\bar{f})(t)dt\right|^p dx\right)^{1/p}$

$= \sup_{0\le h\le\delta} \int_x^{x+h} \left(\int_0^{2\pi} |(\bar{f})(t)|^p dx\right)^{1/p} dt$

$\le \int_x^{x+h} \|\bar{f}(.)\|_p\, dt \quad \le \quad h \ \|\bar{f}(.)\|_p \le \ \delta\ \|\bar{f}\|_p$ ■

Lemma5:

Let $f \in L_p(X)$, $X = [0,2\pi]$, with $\omega_k(f,x,h)_p$ is a function of x , then

$$\tau_1(\omega_k(f,x,h),\delta)_p \le \tau_1\left(f, h + \frac{\delta}{k}\right)_p \qquad (9)$$

Proof:

Let $g(x) = \omega_k(f,x,h),\delta)_p$ (10)

$\omega_1(g,x,\delta)_p = \sup\{ |\Delta_\theta^1 g(t)| : t, t+\theta \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$

$= \sup\{ |(g(t+\theta) - g(t))| : t, t+\theta \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\} \le \sup\{ (g)(t) : t \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\}$.

Then by (2) and (10) we get:

$\omega_1(g,x,\delta)_p \le \sup\{ |\Delta_m^k (f)(s)| :$ s, s + $km \in [x - \frac{\delta}{2} - \frac{kh}{2}, x + \frac{\delta}{2} - \frac{kh}{2}]\}$

$= \omega_k(f,x,h + \frac{k}{2})_p$.

By taking the norm tow sided we get that

$\|\omega_1(g,x,\delta)_p\|_p \le \| \omega_k(f,x,h + \frac{k}{2})_p \|_p$.

By (3) we get:

$\tau_1(g,x,h,\delta)_p \le \tau_k(f,x,h + \frac{k}{2})_p$.

From (10) we have:

$\tau_1(\omega_k(f,x,h),\delta)_p \le \tau_k\left(f,x,h + \frac{k}{2}\right)_p$ ■

Lemma6:

Let $f \in L_p(X)$, $\bar{f}$ exists, then:

$$\tau_k(f,\delta)_p \le \tau_{k-1}\left(\bar{f}, \frac{\delta}{k-1}\delta\right)_p \qquad (11)$$

Proof:

since $\Delta_h^k[(f)(t)] = \Delta_h^{k-1} \Delta_h^1[(f)(t)]$

$= \Delta_h^{k-1}([(f)(t+h) - (f)(t)])$

$= \Delta_h^{k-1}(\int_0^h [(f)^-(u+t)]\, du)$, $h > o$

$|\Delta_h^k[(f)(t)]| \le \int_0^h |\Delta_h^{k-1}(f)^-(u+t)| du$.

Taking the supremum and integral both sides we get

$\sup\{ |\Delta_h^k[(f)(t)]]| : t , t + k\,h \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}]\} \le \sup\{ \int_0^h |\Delta_h^{k-1}(f\partial_n)^-(u+t)du|\} \cap$ [a, b]

$\omega_k(f,x;\delta)_p \le h\omega_{k-1}(\bar{f},x;\delta)_p \le \delta\omega_{k-1}(\bar{f},x;\delta)_p$,

$\|\omega_{k-1}(f,x;\delta)\|_p \le \delta\|\omega_{k-1}(\bar{f},x;\delta)\|_p$

$\tau_k(f,\delta)_p \le \delta\, \tau_{k-1}(\bar{f}, \frac{k}{k-1}\delta)_p$ ■

Lemma7:

Let $f \in L_p(X)$, $X = [0, 2\pi]$, $\delta \ge 0$,then

$$\tau_1(f,\delta)_p \le \delta\, \|\bar{f}\|_p \qquad (12)$$

Proof:

$\omega_1(f,x,\delta)_p = \sup\{|[f(s_1) - f(s_2)]| : s_1, s_2 \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\} = \sup\{\left|\int_{s_2}^{s_1}(f)^-(t)\, dt\right| : s_1, s_2 \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]\} \le \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} |(f)^-(t)|\, dt = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |(f)^-(x+t)|\, dt$.

Then

$\|\omega_1(f,x,\delta)_p\|_p \le \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \|(f)^-(x+t)\|_p\, dt$.

Then $\tau_1(f,\delta)_p \le \delta\, \|\bar{f}\|_p$ ■

Lemma8:

For $f \in L_p(X)$, $C(k)$is constant depends on k.

$$\text{Then } \tau_k(f,\delta)_p \le C(k)\,\delta^k\, \|f^{(k)}\|_p \qquad (13)$$

Proof:

From (11), we get:

$\tau_k(f,\delta)_p \le \delta\,\tau_{k-1}(\bar{f}, \frac{k}{k-1}\delta)_p$

$\le \delta\delta\, \tau_{k-2}((f)^=, \frac{k-1}{k-2}\delta)_p \le \delta\delta\delta\, \tau_{k-3}((f)^{\equiv}, \frac{k-2}{k-3}\delta)_p$

$\le \delta^{k-1}\tau_1(f^{(k-1)}, 2\delta)_p$,

where $(f)^{\equiv}$,is the third derivative of f. From (12), we get:

$\tau_k(f,\delta)_p \le \delta^{k-1}\tau_1(f^{(k-1)}, 2\delta)_p \le 2\,\delta^k\, \|f^{(k)}\|_p \le$ c$(k)\,\delta^k\, \|f^{(k)}\|_p$

Lemma 9:
For every natural number k and δ , $0 < \delta < 2\pi/k$, there exists a function $f_{k,\delta} \in L_p(X)$, $X = [a, b]$ with the properties:

- **i.** $\left|f(x) - f_{k,\delta}(x)\right| \le C_1(k)\, \omega_k(f, x, 2\delta)_p$
- **ii.** $\left\|f(.) - f_{k,\delta}(.)\right\|_p \le C_1(k\, \omega_k(f, \delta)_p$
- **iii.** $\left\|f_{k,\delta}^{(r)}\right\|_p \le C_2(k) \frac{1}{\delta} \omega_r(f, \delta)_p$, $r = 1,2,\cdots,k$.

Proof:
i. Define the function $f_{k,\delta}$, [9], such that:

$f_{k,\delta}(x) = (-\delta)^{-k} \underbrace{\int_0^{\delta} \cdots}_{k-time} \int_0^{\delta} \{\binom{k}{0} - (f)(x + \frac{k}{k}(t_1, \ldots + t_k) + \binom{k}{1}(f)(x + \frac{k-1}{k}(t_1, \ldots + t_k)) + \ldots \ldots + (-1)^k \binom{k}{k-1} + (f)(x + \frac{1}{k}(t_1, \ldots + t_k))\} dt_1 \ldots dt_k$.

$|f(x) - f_{k,\delta}(x)| \le \frac{1}{\delta^k} \int_0^{\delta} \ldots \int_0^{\delta} \left|\Delta^k_{(t_1, \ldots + t_k)/k}(f)(x)\right| dt_1 \ldots dt_k \le \omega_k(f, x; 2\delta)_p$.

ii. $\|f - f_{k,\delta}\|_p \le \frac{1}{\delta^k} \int_0^{\delta} \ldots \int_0^{\delta} \left\|\Delta^k_{(t_1, \ldots + t_k)/k}(f)(x)\right\|_p dt_1 \ldots dt_k \le c_1\, \omega_k(f, \delta)_{p,\partial_n}$ .Now , for the derivative of order r of the function $f_{k,\delta}$,we have:

iii. $f_{k,\delta}^{(r)}(x) = \frac{1}{(-\delta)^k} \underbrace{\int_0^{\delta} \cdots}_{k-time} \int_0^{\delta} \{\binom{k}{0}\binom{k}{k}^r (-1)^1 \Delta^r_{\frac{k}{k}\delta}(f\partial_n)(x + \frac{k}{k}\sum_{i=1}^{k-r} t_i) + \binom{k}{1}\binom{k}{k-1}^r (-1)^2 \Delta^r_{\frac{(k-1)\delta}{k}}(f)(x + \frac{(k-1)}{k}(\sum_{i=1}^{k-r} t_i)) + \ldots.. + (-1)^k \binom{k}{k-1}\binom{k}{1}^r \Delta^r_{\frac{\delta}{k}}(f)(x + \frac{1}{k}(\sum_{i=1}^{k-r} t_i)\} \prod_{i-1}^{k} dt_i$

$\left\|f_{k,\delta}^{(r)}\right\|_p \le \frac{1}{\delta^k} \int_0^{\delta} \ldots \int_0^{\delta} \{\|\Delta^r_{\delta} f(x + (\sum_{i=1}^{k-r} t_i)\|_p + \binom{k}{1}\binom{k}{k-1}^r \left\|\Delta^r_{\frac{(k-1)\delta}{k}} f[x + \frac{k-1}{k}(\sum_{i=1}^{k-r} t_i)]\right\|_p + \ldots + \binom{k}{k-1} k^r \left\|\Delta^r_{\frac{\delta}{k}} f\left(x + \frac{1}{k}(\sum_{i=1}^{k-r} t_i)\right)\right\|_p\} \prod_{i-1}^{k} dt_I$

$\left\|f_{k,\delta}^{(r)}\right\|_p \le \frac{1}{\delta^k} \{ \omega_r(f, \delta)_p + \binom{k}{1}\binom{k}{k-1}^r \omega_r(f, \frac{(k-1)\delta}{k})_p + \cdots\cdots + \binom{k}{k-1} k^r \omega_r(f, \frac{\delta}{k})_p \} \le C_2(k) \frac{1}{\delta^r} \omega_r(f, \delta)_p$ ■

Lemma.10:
Consider the function f where: $f \in L_p(X), X = [0, 2\pi], 1 \le p < \infty$, C is a constant we have:

$$\widetilde{E_n}(f)_p \le C\, \tau_1(f, \frac{1}{n})_p \tag{14}$$

Where $C = (\pi + 1)(4C_n + 1)$.

Proof:
Let $x_i = i\pi/n$, $i = 0,1,2,\cdots 2n$, $y_i = \frac{x_{i-1}+x_i}{2}$, $i = 1,2,\cdots 2n$, $y_{2n+1} = y_1$
define the following 2π – periodic functions γ_n and β_n with the following specifications:

$$\gamma_n(x) = \begin{cases} sup\{(f)(t) : t \in [x_{i-1}, x_i]\}, \text{for } x = y_i,\ i = 1,2,\cdots 2n \\ max\{\gamma_n(y_i), \gamma_n(y_{i+1})\}, \quad for\ x = x_i,\ i = 1,2,\cdots 2n \\ \gamma_n(0) = \gamma_n(2\pi) \end{cases}$$

$$\beta_n(x) = \begin{cases} inf\{(f)(t) : t \in [x_{i-1}, x_i]\}, x = y_i,\ i = 1,2,\cdots 2n \\ min\{\beta_n(y_i), \beta_n(y_{i+1})\} for\ x = x_i,\ i = 1,2,\cdots 2n \\ \beta_n(0) = \beta_n(2\pi) \end{cases}$$

$\gamma(x)$ and $\beta(x)$ are continuous functions for $x \in [x_{i-1}, y_i]$, $x \in [y_i, x_i]$, respectively, $i = 1,2,\cdots 2n$.
$\gamma_n(x)$ and $\beta_n(x)$ are differentiable functions on [0, 2 π] except eventually at the points x_i , $i = 0, \ldots 2n$ and y_i , $i = 1, \ldots n$

We note $\gamma_n(x) \le f(x) \le \beta_n(x)$, $x \in [0, 2\pi]$
Since $\beta_n(x)$ is differentiable function in (y_i, x_i) , there exist $x \in (y_i, x_i)$, such that:

$$\beta_n^-(x) = \frac{\beta_n(y_{i+1}) - \beta_n(y_i)}{y_{i+1} - y_i},\ i = 1,2,\ldots,2n$$

Let $x_i = /n$, $x_{i-1} = \frac{(i-1)\pi}{n}$, $x_{i+1} = \frac{(i+1)\pi}{n}$, $i = 0, \ldots 2n$
$y_i = \frac{(x_{i-1} + x_i)}{2}$, $y_{i+1} = \frac{(x_i + x_{i+1})}{2}$, $i = 1,2,\cdots 2n$
, then

$$y_{i+1} - y_i = \frac{(x_i + x_{i+1})}{2} - \frac{(x_{i-1} + x_i)}{2} = \frac{\frac{i\pi}{n} + \frac{(i+1)\pi}{n}}{2} - \frac{\frac{(i-1)\pi}{n} + \frac{i\pi}{n}}{2} = \pi/n$$

$$\beta_n^-(x) = \frac{\beta_n(y_{i+1}) - \beta_n(y_i)}{y_{i+1} - y_i} = \frac{\beta_n(y_{i+1}) - \beta_n(y_i)}{\pi/n},$$

Let $n/\pi = k_n$
$\beta_n^-(x) = k_n[\beta_n(y_{i+1}) - \beta_n(y_i)]$
$|\beta_n^-(x)| \le k_n|\beta_n(y_{i+1}) - \beta_n(y_i)| \le k_n\, sup\{|[(f)(y_{i+1}) - (f)(y_i)]|\}$
$\le k_n\, sup\{\|\Delta_h^1 f(.)\|_p\} \le k_n\, \omega_1(f, x, \frac{\pi}{n})_p$

By taking the norm tow sided we get the following:

$\|\beta_n^-(.)\ \|_p \le \left\| C_n\, \omega_1 \left(f\,,x\,,\ \frac{\pi}{n}\right)_{p,\partial_n} \right\|_p =$ $C_n\tau_1\left(f\,,\ \frac{\pi}{n}\right)_p$. which means

$$\|\beta_n^-(\,.\,)\ \|_p \le C_n\tau_1\left(f\,,\ \frac{\pi}{n}\right)_p \qquad (15)$$

Also:
Since $\gamma_n(x)$ is differentiable function in $(y_i\,,x_i)$, there exist $x \in (y_i\,,x_i)$, such that:
$\gamma_n^-(\,x) = \frac{\gamma_n(y_{i+1})-\gamma_n(y_i)}{y_{i+1}-y\,\mathrm{i}} = \frac{\gamma_n(y_{i+1})-\gamma_n(y_i)}{\pi/n}$, $i = 1,2,\ldots,2n$.
$\gamma_n^-(\,x) = \frac{(\gamma_n(y_{i+1})-\gamma_n(y_i))n}{\pi}$.

Let $n/\pi = C_n$
$\gamma_n^-(\,x) = C_n[\gamma_n(y_{i+1}) - \gamma_n(y_i)]$
$|\gamma_n^-(\,x)\ | \le C_n|\gamma_n(y_{i+1}) - \gamma_n(y_i)| \le$
$C_n sup\{|[(f)(y_{i+1}) - (f)(y_i)]|\}$
$= C_n\ sup\ \{\left\|\Delta_h^1 f(.)\ \right\|_p\} \le k_n\omega_1\left(f\,,x\,,\ \frac{\pi}{n}\right)_p$

By taken the norm tow sided we get the following:
$\|\gamma_n^-(\,.\,)\ \|_p \le \left\|C_n\,\omega_1\left(f\,,x\,,\ \frac{\pi}{n}\right)_{p,\partial_n}\right\|_p$

$$\|\gamma_n^-(\,.\,)\ \|_p\ \le C_n\tau_1\left(f\,,\ \frac{\pi}{n}\right)_p \qquad (16)$$

On the other hand:
$0 \le \gamma_n(x) - \beta_n(x) = sup\{(f)(t):\ t \in [x_{i-1}, x_i]\} - inf\{(f)(t):\ t \in [x_{i-1}, x_i]\}$
$\le sup\{(f)(x_i) - (f)(x_{i-1})\} = \omega_1\left(f\,,x\,,\ \frac{\pi}{n}\right)_p$

By taking the norm for both sides we get the following:

$$\left\|\gamma_n(x) - \beta_n(x)\right\|_p \le \left\|\ \omega_1\left(f\,,x\,,\frac{\pi}{n}\right)_{p,\partial_n}\right\|_p \le \tau_1\left(f\,,\frac{\pi}{n}\right)_p \qquad (17)$$

Using (15), (16) and (17) we have:
$\widetilde{E_n}(f)_p = inf_{n\in N}\|\beta_n - \gamma_n\|_p \le \|\beta_n - \gamma_n\|_p =$
$\|\beta_n - \beta_n^- + \beta_n^- + \gamma_n^- - \gamma_n^- - \gamma_n\|_p \le \|-\beta_n^-\|_p +$
$\|\beta_n^-\|_p + \|\gamma_n^-\|_p + \|-\gamma_n^-\|_p + \|\beta_n - \gamma_n\|_p \le$
$2C_n\,\tau_1\left(f\,,\pi\frac{1}{\mathrm{n}}\right)_p + 2C_n\,\tau_1\left(f\,,\pi\frac{1}{\mathrm{n}}\right)_p +$
$\tau_1\left(f\,,\pi\frac{1}{n}\right)_p$.

Then by using:
$\left(\tau_k\left(f\,,\mu\delta\right)_p \le 2(\mu+1)^k\tau_k\left(f\,,\delta\right)_p\right)$, [10], for $=1$), we get:
$\widetilde{E_n}(f)_p = 4C_n(\pi+1)\tau_1\left(f\,,\frac{1}{n}\right)_p + 4C_n(\pi+1)$
$\tau_1\left(f\,,\frac{1}{n}\right)_p + 2(\pi+1)\tau_1\left(f\,,\frac{1}{n}\right)_p$
$= 8C_n(\pi+1)\tau_1\left(f\,,\frac{1}{n}\right)_p + 2(\pi+1)\tau_1\left(f\,,\frac{1}{n}\right)_p =$
$(\pi+1)(\,4C_n\ +\ 1)\ \tau_1\left(f\,,\frac{1}{n}\right)_p$ ■

Lemma11:
Let $f \in L_p(X)$, $X = [\,0,2\pi]$, $g_n(x) = \omega_k\left(f\,,x\,,n^{-1}\right)_p$, then
$\widetilde{E_n}(g_n)_p \le$

$$\mathrm{C}\ \tau_k\left(f\,,\frac{1}{n}\right)_p \qquad (18)$$

C is a constant.

Proof:
From (10) letting $h = \delta = \frac{1}{n}$, we have, $\tau_1\left(g_n\,,\frac{1}{n}\right)_p \le \tau_k\left(f,\frac{1}{n}\right)_p$ from (14) get
$\widetilde{E_n}(g_n)_p \le \mathrm{C}\ \tau_1\left(g_n\,,\frac{1}{n}\right)_p \le \mathrm{C}\ \tau_k\left(f,\frac{1}{n}\right)_p$ ■

3 Main Results: Direct Theorem:

Theorem12:
Let $f \in L_p(X)$,for every natural number k, there is a constant C (p,k) *Depends on* p *and* k such that:

$$\widetilde{E_n}(f)_p \le \mathrm{C}\ (p,k)\tau_k\left(f\,,\frac{1}{n}\right)_p \qquad (19)$$

Proof:
Applying (7) for the functions , $f_{k,\delta}$ and $\varphi(x) = \omega_k(f\,,2\mathrm{n}-1\,)_p$
$\widetilde{E_n}(f)_p \le C_p\left(\widetilde{E_n}(f_{k\,,\delta}\,)_p\ +\ 2\widetilde{E_n}(\,\varphi\,)_p + 2\|\varphi(x)\|_p\right) = \left(\widetilde{E_n}(f_{k\,,\delta}\,)_p\ +\ 2\widetilde{E_n}(\,\varphi\,)_p + 2\left\|\omega_k(f\,,2\mathrm{n}-1\,)_p\right\|_p\right)$.

Using (18) with $\frac{1}{n} = \delta$ we get:

$$\widetilde{E_n}(f)_p \le \mathrm{C}_1\,\widetilde{E_n}(f_{k\,,\delta}\,)_p + 2\mathrm{C_p}\tau_k(\,f\,,\ \delta\,)_p, 2\mathrm{C_p}\ \tau_k(f\,,\delta\,)_p \qquad (20)$$

From lemma (9, iii, for $r = k$), we get

$$\widetilde{E_n}(f_{k\,,\delta}\,)_p \le \mathrm{C}\ (k)\left\|f_{k\,,\delta}^{(k)}\right\|_p \le \mathrm{C}_1(k)\ \omega_k\,(\,f\,,\delta)_p \le \mathrm{C}_2(k)\ \tau_k\,(\,f\,,\delta)_p \qquad (21)$$

From (20) and (21), we get:
$\widetilde{E_n}(f\,)_{p,\partial_n} \le \mathrm{C}_2(k)\tau_k\,(\,f\,,\delta)_{p,\partial_n}$ + $2\mathrm{C_p}\ \tau_k\,(\,f\,,\delta)_{p,\partial_n}$
$+\ 2\mathrm{C_p}\tau_k\,(\,f\,,\delta)_p \le [4\mathrm{C_p} + \mathrm{C}_2\,(k)]\ \tau_k\,(\,f\,,\delta)_p \le \mathrm{C}\ (p,k)\ \ \tau_k\,(\,f\,,\delta)_p$ ■

Converse theorem:

Theorem13:

Let $f \in L_p(X)$, then there is C(k), such that:

$\tau_k(f\,,\ n-1)_p \leq \frac{C(k)}{n^k} \sum_{s=0}^{n} (s+1)^{k-1} \widetilde{E_n}(f)_p$

Proof:

Let $\theta_n, \vartheta_n, \in \square_n$ are trigonometric polynomials $\widetilde{E_n}(f)_p = \underbrace{inf}_{n\in N} \|\theta_n - \vartheta_n\|_p$, $\vartheta_n(x) \leq f(x) \leq \theta_n(x)$, $x \in [0, 2\pi]$

If $\Delta_h^k(f)$ (t) ≥ 0, then:

$\Delta_h^k(f)(t) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} (f)(t+(k-m)h)$
$\leq \sum_{i=0}^{k/2} \binom{k}{2i} \theta_n(t+(k-2i)h) - \sum_{i=0}^{k-1/2} \binom{k}{2i+1} \vartheta_n(t+(k-2i-1)h)$
$= \Delta_h^k \theta_n(t) - \left\{ \sum_{i=0}^{(k-1)/2} \binom{k}{2i+1} \{ \theta_n(t+(k-2i-1)h) + \vartheta_n(t+(k-2i-1)h) \} \right\}$
$= \Delta_h^k \theta_n(t) - \sum_{i=0}^{(k-1)/2} \binom{k}{2i+1} \{ \theta_n(t+(k-2i-1)h) + \vartheta_n(t+(k-2i-1)h) - [\theta_n(x) - \vartheta_n(x)]\} + \sum_{i=0}^{(k-1)/2} \binom{k}{2i+1} [\theta_n(x) - \vartheta_n(x)]$
$\leq \Delta_h^k \theta_n(t) - 2^k \{ \omega_1(\theta_n - \vartheta_n, x; k\delta)_{p,\partial_n} + [\theta_n(x) - \vartheta_n(x)]\}$

$$\left|\Delta_h^k(f\partial_n)(t)\right| \leq \left|\Delta_h^k \theta_n(t)\right| + 2^k[\omega_1(\theta_n - \vartheta_n, x; k\delta)_{p,\partial_n} + |\theta_n(x) - \vartheta_n(x)|] \quad (22)$$

Now if $\Delta_h^k(f)$ (t) ≤ 0, then in the same way, we obtain:

$$\left|\Delta_h^k(f)(t)\right| \leq \left|\Delta_h^k \vartheta_n(t)\right| + 2^k[\omega_1(\theta_n - \vartheta_n, x; k\delta)_p + [\theta_n(x) - \vartheta_n(x)] \quad (23)$$

Equations (22) and (23), we get:

$\omega_k(f, x, \delta)_p \leq \omega_k(\theta_n, x, \delta)_p + \omega_k(\vartheta_n, x, \delta)_p + 2^k[\omega_1(\theta_n - \vartheta_n, x, \delta)_p + |\theta_n(x) - \vartheta_n(x)|]$.

Taking the norm for the both sides:

$\tau_k(f, \delta)_p \leq \tau_k(\theta_n, \delta)_p + \tau_k(\vartheta_n, \delta)_p + 2^k[\tau_1(\theta_n - \vartheta_n, k\delta)_p + \widetilde{E_n}(f)_p]$

From (12), we get:

$\tau_1(\theta_n - \vartheta_n, k\delta)_p \leq k\delta \,\|(\theta_n - \vartheta_n)^-\|_p$

Using the Bernstein inequality for θ_n and $\vartheta_n \in \square_n$. Then:

$\tau_1(\theta_n - \vartheta_n, k\delta)_p \leq nk\,\delta\|\theta_n - \vartheta_n\|_p = k\delta\, n\, \widetilde{E_n}(f)_p$.

Then

$\tau_k(f, \delta)_p \leq \tau_k(\theta_n, \delta)_p + \tau_k(\vartheta_n, \delta)_p + 2^k(k\delta n + 1)\widetilde{E_n}(f)_p$

By using method of Salem Steckin, [9], with $\tau_k(f+g, \delta)_p \leq \tau_k(f, \delta)_p + \tau_k(g, \delta)_p$, [10]. Let us set $n = 2^{s_0}$

$$\tau_k(f, \delta)_p \leq \sum_{i=1}^{s_0}[\tau_k(\theta_{2^i} - \theta_{2^{i-1}}, \delta)_p + \tau_k(\vartheta_{2^i} - \vartheta_{2^{i-1}}, \delta)_p] + \tau_k(\theta_1 - \theta_0, \delta)_p + \tau_k(\vartheta_1 - \vartheta_0, \delta)_p + 2^k(k\delta n + 1)\widetilde{E_n}(f)_p \quad (24)$$

Now

$\tau_k(\theta_{2^i} - \theta_{2^{i-1}}, \delta)_p \leq k\delta^k \left\|(\theta_{2^i} - \theta_{2^{i-1}})^k\right\|_p \leq k\delta^k 2^{ik} \left\|\theta_{2^i} - \theta_{2^{i-1}}\right\|_p = k\delta^k 2^{ik} \left\|\theta_{2^i} - \theta_{2^{i-1}} - f + f\right\|_p$
$\leq k\delta^k 2^{ik}\left[\left\|\theta_{2^i} - f\right\|_p + \left\|\theta_{2^{i-1}} - f\right\|_p\right] \leq k\delta^k 2^{ik}\left[\left\|\theta_{2^i} - \vartheta_{2^i}\right\|_p + \left\|\theta_{2^{i-1}} - \vartheta_{2^{i-1}}\right\|_p\right]$
$= k\delta^k 2^{ik}\left[\widetilde{E_{2^i}}(f)_p + \widetilde{E_{2^{i-1}}}(f)_p\right]$

Then

$$\tau_k(\theta_{2^i} - \theta_{2^{i-1}}, \delta)_p \leq 2k\delta^k 2^{ik}\widetilde{E_{2^{i-1}}}(f)_p \quad (25)$$

$\tau_k(\vartheta_{2^i} - \vartheta_{2^{i-1}}, \delta)_p \leq k\delta^k \left\|(\vartheta_{2^i} - \vartheta_{2^{i-1}})^k\right\|_p \leq k\delta^k 2^{ik}\left\|\vartheta_{2^i} - \vartheta_{2^{i-1}}\right\|_p$
$\leq k\delta^k 2^{ik}\left[\left\|\vartheta_{2^i} - f\right\|_p + \left\|\vartheta_{2^{i-1}} - f\right\|_p\right] \leq k\delta^k 2^{ik}\left[\left\|\theta_{2^i} - \vartheta_{2^i}\right\|_p + \left\|\theta_{2^{i-1}} - \vartheta_{2^{i-1}}\right\|_p\right]$

Then

$$\tau_k(\vartheta_{2^i} - \vartheta_{2^{i-1}}, \delta)_p \leq 2k\delta^k 2^{ik}\widetilde{E_{2^{i-1}}}(f)_p \quad (26)$$

from (24), (25) and (26), we get:

$\tau_k(f, \delta)_p \leq 4k\delta^k \sum_{i=1}^{s_0} 2^{ik}\widetilde{E_{2^{i-1}}}(f)_p + 2k\delta^k\widetilde{E_0}(f)_p + 2^k(k\delta n + 1)\widetilde{E_n}(f)_p$

$\tau_k(f, \delta)_p \leq 4^{k+1}k\delta^k \sum_{s=0}^{n}(s+1)^{k-1}\widetilde{E_s}(f)_p + 2^k(k\delta n + 1)\widetilde{E_n}(f)_p$, taking $\delta = n^{-1}$, we get:

$\tau_k(f, \delta)_p \leq 4^{k+1}kn^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\widetilde{E_s}(f)_p + 2^k(k+1)\widetilde{E_n}(f)_p \leq 2^{3k+1}n^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\widetilde{E_s}(f)_p$

$$\tau_k(f, \delta)_p \leq \frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}\widetilde{E_s}(f)_p \quad (27)\ ■$$

The following corollary characterize the best (O-S- A) in L_p –space by averaged moduli of smoothness $\tau_k(f, \delta)_p$ and moduli of smoothness $\omega_k(f, \delta)_p$ in $L_p(X)$.

Corollary14:

Let $f \in L_p(X)$, $X = [0, 2\pi]$, p≥ 1 ,then

$\tau_k(f,\delta)_p = O(\delta^{\alpha})$ iff $\widetilde{E_n}(f)_p = O(n^{-\alpha})$

Proof:

Let $\widetilde{E_n}(f)_{p,} = O(n^{-\alpha})$ by (27) we get:
$\tau_k(f,\frac{1}{n})_p = \frac{C(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}\widetilde{E_n}(f)_p = \frac{C(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}O(n^{-\alpha})$,put: $n=\frac{1}{\delta}$
$\tau_k(f,\delta)_p = \frac{C(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}O(\delta^{\alpha})$, then $\tau_k(f,\delta)_p = O(\delta^{\alpha})$.
Now let $\tau_k(f,\delta)_p = O(\delta^{\alpha})$, from(19)
$\widetilde{E_n}(f)_p \le C(p,k)\tau_k(f,\delta)_p = C(p,k)\,O(\delta^{\alpha})$, taking $\delta=\frac{1}{n}$.Then $\widetilde{E_n}(f)_p = O(n^{-\alpha})$ ■

4 Discussion and Conclusion

Through this research, we got the degree of the best (O-S- A) of periodic bounded function in $L_p(X)-$space. $X=[0,2\pi]$. Also, we estimate the degree of the best (O-S- A) in term of averaged modulus. As well as the relationship between the degree of the best (O-S- A), of the function f and averaged modulus of order k.

5 Future Work

Our future work will be ((O-S- A), of the function f) by q-Bernstein-Kantorovich Operator on the Sobolev space which is Hilbert space.

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