A Valid Transport Related SVEIHR Stochastic Epidemic Model with Coverage and Time Delays

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Abstract: The ability of people to move freely between cities is thought to be a major factor in accelerating the spread of infectious diseases. To investigate this issue, we propose a SEVIHR stochastic epidemic model, which emphasizes the effects of transport related infections and media coverage. At the same time, the time delay caused by the information time difference is considered. Firstly, we study the existence and uniqueness of the global positive solution of the model by means of Lyapunov function and stopping time, and obtain sufficient conditions for the extinction and persistence of the disease. Secondly, in order to control the spread of the disease in time and effectively, appropriate control strategies are formulated according to the stochastic optimal theory. Finally, the extinction and persistence of the disease were simulated by MATLAB.

Key-Words: stochastic epidemic model, media coverage, transport related infection, time delay, extinction and persistence, optimal control.

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1 Introduction

In recent years, there have been frequent outbreaks of infectious diseases around the world, such as COVID-19, HIV/AIDS, dengue fever, etc[1],[2],[3] The outbreak of infectious diseases has threatened the life and health of people all over the world and caused huge economic losses, which has aroused the wide attention of scholars all over the world, and carried out a series of work such as modeling research and practical investigation. Through lots of studies and analyses, the researchers have revealed the mechanism of disease transmission, the effectiveness of interventions, and the impact on population health. The infectious disease model and its related contents studied by mathematicians and infectious disease scientists have deepened our understanding of infectious disease. Among them, the study of some complex infectious disease models has simulated the extinction and persistence of diseases, providing valuable advice for disease prevention and control, alleviating the impact of infectious diseases, and ensuring public health security. Therefore, mathematical modeling is one of the effective means to study infectious disease control strategy. It can both help people understand infectious diseases and help people actively prevent them.

When a disease breaks out, the movement of people between cities by different modes of transportation plays a crucial role in the spread of the disease. In order to study the impact of transportation on the control of infectious diseases, many researchers have established different models of transport related infectious diseases for study[4],[5],[6]. In reference [7], the SEIVR infectious disease model related to transportation was considered. On this basis, the inpatient individuals were added and the SEVIHR infectious disease model related to transportation was established:

$$\begin{cases} \frac{\mathrm{d}S_i(t)}{\mathrm{d}t} = [\Lambda + rS_i(t)(1-\alpha S_i(t)) \\ \qquad -\frac{\beta_1 S_i(t)I_i(t)}{N_i} - \frac{\mu_1\delta S_j(t)I_i(t)}{N_j} \\ \qquad -(w+d_1)S_i(t) + \gamma R_i(t) \\ \qquad +\upsilon V_i(t) - \delta(S_i(t)-S_j(t)), \\ \frac{\mathrm{d}V_i(t)}{\mathrm{d}t} = wS_i(t) - \delta(V_i(t)-V_j(t)) \\ \qquad -(\upsilon+d_1)V_i(t) - \frac{\beta_2 V_i(t)I_i(t)}{N_i} \\ \qquad -\frac{\mu_2\delta V_j(t)I_i(t)}{N_j}, \\ \frac{\mathrm{d}E_i(t)}{\mathrm{d}t} = \frac{\beta_1 S_i(t)I_i(t)}{N_i} + \frac{\mu_1\delta S_j(t)I_i(t)}{N_j} \\ \qquad +\frac{\beta_2 V_i(t)I_i(t)}{N_i} + \frac{\mu_2\delta V_j(t)I_i(t)}{N_j} \\ \qquad -(\theta+d_1)E_i(t) - \delta(E_i(t) \\ \qquad -E_j(t)), \\ \frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \theta E_i(t) - (d_1+\varphi+f_h)I_i(t) \\ \qquad -\delta(I_i(t)-I_j(t)), \\ \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} = f_h I_i(t) + (-1)^{i+1}g_1 I_2(t) - (f_r \\ \qquad +d_1+d_2)H_i(t), \\ \frac{\mathrm{d}R_i(t)}{\mathrm{d}t} = \varphi I_i(t) + f_r H_i(t) - (\gamma \\ \qquad +d_1)R_i(t) - \delta(R_i(t)-R_j(t)), \end{cases} \tag{1}$$

where $i = 1,2$ and $j = i-(-1)^i$ $(i = 1,2)$ ($i = 1$ represents city A, $i = 2$ represents city B). Λ is the birth rate(birth rates in both cities are expressed as average birth rates), r is the average growth rate of the two cities, α is the inverse of carrying capacity, β_1 (β_2) is the infection rates of I_i to S_i (V_i), w is the average vaccination rate of the two cities, υ is the rate of loss of vaccine immunity. d_1 and d_2 are natural mortality and disease mortality, θ is the proportion of E_i who become infectious individuals I_i, $\mu_1(\mu_2)$ is the infection rate of the infected individuals in the local area to the susceptible individuals (vaccinated individuals) in the nonnative area. f_h is the hospitalization rate, f_r is the recovery rate. Due to the differences in regional medical water, we assume that the medical level of place A is higher than that of place B, and patients suitable for treatment in place A will be transferred to place A, and the transfer rate is set at g_1. And the recovered population will return to the susceptible population with a transfer rate of γ due to loss of antibodies. Because there is normal communication between cities, we assume that δ is the transmission rate between two cities. φ represents the recovery rate of the infectious individuals without hospitalization. And divide the population into S_i means susceptible individuals, V_i means vaccinated individuals, E_i means exposed individuals, I_i means infectious individuals, H_i means inpatient individuals, R_i means recovered individuals, N_i means total urban population.

Nowadays, with the rapid development of the Internet, people's life is surrounded by all kinds of information, so people can get information about diseases and corresponding preventive measures through media reports, [8], [9]. The following function was proposed in literature [10] as the dynamic change of infection rate under the influence of media coverage: $\beta(I) = \mu e^{-mI}$, where μ indicates the usual contact rate, m is the are the coefficient of the influence infection rate of media coverage, which represents how media reports affect transmission. Since media coverage and vigilance are not inherently deterministic factors that cause transmission, it is reasonable to assume that $m > 0$. When m is $m > 0$ and relatively small, $\beta(I)$ approaches the parameter μ. However, the increase of m means that the media has carried out more comprehensive reports to the public, increasing the public's awareness of the infectious disease and taking preventive measures to further reduce the infection rate. At the same time, it is worth noting that from the time the danger of infectious disease transmission is recognized and publicized, people usually do not respond immediately, which leads to a certain time delay, thus affecting the spread of the disease to a certain extent[11], [12]. In reference [13], the extinction and persistence of asymptomatic infection models with media coverage were studied. In reference [14], the global asymptotic stability of disease free homeostasis was established in a spatially heterogeneous environment with media coverage.

Due to the delay in updating the number of cases, the accuracy of media reports will be affected to a certain extent, and then the infection rate will fluctuate accordingly, so the spread of infectious diseases will be limited to a certain extent. At the same time,white noise[15],[16],[17]in nature will also affect the spread of diseases, under the influence of the above factors, we study the following random model:

$$
\left\{
\begin{aligned}
&\mathrm{d}S_1(t) = [\Lambda + rS_1(1-\alpha S_1) - \delta(S_1 - S_2) - \tfrac{\beta_1 S_1 I_1}{N_1} e^{-m_1 I_1(t-\tau_1)} \\
&\quad - \tfrac{\mu_1 \delta S_2 I_1}{N_2} e^{-m_2 I_1(t-\tau_1)} + \upsilon V_1 + \gamma R_1 - (w+d_1)S_1]\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}B_1(t), \\
&\mathrm{d}V_1(t) = [wS_1 - (\upsilon + d_1)V_1 - \delta(V_1 - V_2) - \tfrac{\beta_2 V_1 I_1}{N_1} e^{-m_3 I_1(t-\tau_1)} \\
&\quad - \tfrac{\mu_2 \delta V_2 I_1}{N_2} e^{-m_4 I_1(t-\tau_1)}]\mathrm{d}t + \sigma_2 V_1(t)\mathrm{d}B_2(t), \\
&\mathrm{d}E_1(t) = [\tfrac{\beta_1 S_1 I_1}{N_1} e^{-m_1 I_1(t-\tau_1)} + \tfrac{\mu_1 \delta S_2 I_1}{N_2} e^{-m_2 I_1(t-\tau_1)} - (\theta + d_1)E_1 + \tfrac{\beta_2 V_1 I_1}{N_1} e^{-m_3 I_1(t-\tau_1)} \\
&\quad + \tfrac{\mu_2 \delta V_2 I_1}{N_2} e^{-m_4 I_1(t-\tau_1)} - \delta E_1 + \delta E_2)]\mathrm{d}t + \sigma_3 E_1(t)\mathrm{d}B_3(t), \\
&\mathrm{d}I_1(t) = [\theta E_1 - (d_1 + \varphi + f_h)I_1 - \delta(I_1 - I_2)]\mathrm{d}t + \sigma_4 I_1(t)\mathrm{d}B_4(t), \\
&\mathrm{d}H_1(t) = [f_h I_1 + g_1 I_2 - (f_r + d_1 + d_2)H_1]\mathrm{d}t + \sigma_5 H_1(t)\mathrm{d}B_5(t), \\
&\mathrm{d}R_1(t) = [\varphi I_1 + f_r H_1 - (\gamma + d_1)R_1 - \delta(R_1 - R_2)]\mathrm{d}t + \sigma_6 R_1(t)\mathrm{d}B_6(t), \\
&\mathrm{d}S_2(t) = [\Lambda + rS_2(1-\alpha S_2) - \delta(S_2 - S_1) - \tfrac{\beta_1 S_2 I_2}{N_2} e^{-m_1 I_2(t-\tau_2)} \\
&\quad - \tfrac{\mu_1 \delta S_1 I_2}{N_1} e^{-m_2 I_2(t-\tau_2)} + \upsilon V_2 + \gamma R_2 - (w+d_1)S_2]\mathrm{d}t + \sigma_1 S_2(t)\mathrm{d}B_1(t), \\
&\mathrm{d}V_2(t) = [wS_2 - (\upsilon + d_1)V_2 - \delta(V_2 - V_1) - \tfrac{\beta_2 V_2 I_2}{N_2} e^{-m_3 I_2(t-\tau_2)} \\
&\quad - \tfrac{\mu_2 \delta V_1 I_2}{N_1} e^{-m_4 I_2(t-\tau_2)}]\mathrm{d}t + \sigma_2 V_2(t)\mathrm{d}B_2(t), \\
&\mathrm{d}E_2(t) = [\tfrac{\beta_1 S_2 I_2}{N_2} e^{-m_1 I_2(t-\tau_2)} + \tfrac{\mu_1 \delta S_1 I_2}{N_1} e^{-m_2 I_2(t-\tau_2)} - (\theta + d_1)E_2 + \tfrac{\beta_2 V_2 I_2}{N_2} e^{-m_3 I_2(t-\tau_2)} \\
&\quad + \tfrac{\mu_2 \delta V_1 I_2}{N_1} e^{-m_4 I_2(t-\tau_2)} + \delta E_1 - \delta E_2]\mathrm{d}t + \sigma_3 E_2(t)\mathrm{d}B_3(t), \\
&\mathrm{d}I_2(t) = [\theta E_2 - (d_1 + \varphi + f_h)I_2 - \delta(I_2 - I_1)]\mathrm{d}t + \sigma_4 I_2(t)\mathrm{d}B_4(t), \\
&\mathrm{d}H_2(t) = [f_h I_2 - g_1 I_2 - (f_r + d_1 + d_2)H_2]\mathrm{d}t + \sigma_5 H_2(t)\mathrm{d}B_5(t), \\
&\mathrm{d}R_2(t = [\varphi I_2 + f_r H_2 - (\gamma + d_1)R_2 - \delta(R_2 - R_1)]\mathrm{d}t + \sigma_6 R_2(t)\mathrm{d}B_6(t),
\end{aligned}
\right. \tag{2}
$$

while $B_i(t)$ $(i = 1, \cdots, 6)$ are autonomous and independent standard Brownian motions, σ_i $(i = 1, \cdots, 6)$ are the frequencies or intensities of the standard Gaussian white noises. m_i $(i = 1, 2, 3, 4)$ are the coefficient of the influence infection rate of media coverage. The delay of $I_1(I_2)$ people count is $\tau_1(\tau_2)$.

In the second part of this paper, we prove the existence and uniqueness of the global positive solution of model (2) and obtain the sufficient conditions for extinction. In the third part, we discuss disease persistence and get the sufficient conditions for disease persistence. In the fourth part, we use the optimal control theory to establish appropriate control strategies, so as to control the spread of the disease better and faster under the limited resources. In the fifth part, we use Milstein's high order method to simulate the results of this paper.

2 The Existence and Extinction of Disease

2.1 The Existence and Uniqueness of Positive Solution

Theorem 2.1. For any initial condition $(S_1(0), V_1(0), E_1(0), I_1(0), H_1(0), R_1(0), S_2(0), V_2(0), E_2(0), I_2(0), H_2(0), R_2(0)) \in R_+^{12}$. The stochastic model (2) admits a unique solution $(S_1(t), V_1(t), E_1(t), I_1(t), H_1(t), R_1(t), S_2(t), V_2(t), E_2(t), I_2(t), H_2(t), R_2(t))$ for $t \geq -[\tau_1 \vee \tau_2]$ and the solution will remain in R_+^{12} with probability one.

Proof. The proof of this theorem is similar to reference [18]. Therefore, we omit this proof.

2.2 Extinction

For the convenience of expression in this paper, we define $Z(t) = (Z_1(t), Z_2(t))$, where $Z_i(t) = (S_i(t), V_i(t), E_i(t), I_i(t), H_i(t), R_i(t))$ $(i = 1, 2)$. And define the following symbols $\langle Q(t)\rangle = \frac{1}{t}\int_0^t Q(s)\mathrm{d}s$, where $Q(t)$ is any integral function defined on $[0, \infty]$.

Lemma 2.1. Let $Z(t)$ be the solution of model (2) give any initial value $Z(0)$. Then for $i \in 1, 2$, we have

$$\lim_{t\to\infty}\frac{S_i(t)}{t} = 0,\ \lim_{t\to\infty}\frac{V_i(t)}{t} = 0,\ \lim_{t\to\infty}\frac{E_i(t)}{t} = 0,$$
$$\lim_{t\to\infty}\frac{I_i(t)}{t} = 0,\ \lim_{t\to\infty}\frac{H_i(t)}{t} = 0,\ \lim_{t\to\infty}\frac{R_i(t)}{t} = 0.$$

lemma 2.2. Let $Z(t)$ be the solution of model (2) give any initial value $Z(0) \in R_+^{12}$. Then for $i \in 1, 2$, we have

$$\lim_{t\to\infty}\frac{\int_0^t S_i(r)\mathrm{d}B_1(r)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t V_i(r)\mathrm{d}B_2(r)}{t} = 0,$$
$$\lim_{t\to\infty}\frac{\int_0^t E_i(r)\mathrm{d}B_3(r)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t I_i(r)\mathrm{d}B_4(r)}{t} = 0,$$
$$\lim_{t\to\infty}\frac{\int_0^t H_i(r)\mathrm{d}B_5(r)}{t} = 0,\ \lim_{t\to\infty}\frac{\int_0^t R_i(r)\mathrm{d}B_6(r)}{t} = 0.$$

Theorem 2.2. Let $Z(t)$ be the solution of model (2) for any initial value $Z(0) \in R_+^{12}$. Thus, in the case of $R_0^{jp} < 1$, the following property holds:

$$\lim_{t\to\infty}\sup\frac{\ln(I_1(t) + I_2(t))}{t} \leq P(R_0^{jp} - 1) < 0\ a.s.$$

and have $\lim_{t\to\infty} I_1(t) = 0, \lim_{t\to\infty} I_2(t) = 0\ a.s$, where $R_0^{jp} = \frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{d_1+\varphi+f_h} - \frac{(\sigma_3\wedge\sigma_4)^2}{2(d_1+\varphi+f_h)}$, $P = d_1 + \varphi + f_h$

Proof. Let $G_1(t) = E_1(t) + E_2(t) + I_1(t) + I_2(t)$. According to It$\hat{o}'$s formula and model (2), we can get $\mathrm{d}G_1(t) = LG_1\mathrm{d}t + \sigma_3(E_1 + E_2)\mathrm{d}B_3(t) + \sigma_4(I_1 + I_2)\mathrm{d}B_4(t)$, where

$$\begin{aligned}
LG_1 &= \frac{\beta_1 S_1 I_1}{N_1}e^{-m_1 I_1(t-\tau_1)} + \frac{\mu_1\delta S_2 I_1}{N_2}e^{-m_2 I_1(t-\tau_1)}\\
&+ \frac{\beta_2 V_1 I_1}{N_1}e^{-m_3 I_1(t-\tau_1)} + \frac{\mu_2\delta V_2 I_1}{N_2}e^{-m_4 I_1(t-\tau_1)}\\
&+ \frac{\beta_1 S_2 I_2}{N_2}e^{-m_1 I_2(t-\tau_2)} + \frac{\mu_1\delta S_1 I_2}{N_1}e^{-m_2 I_2(t-\tau_2)}\\
&+ \frac{\beta_2 V_2 I_2}{N_2}e^{-m_3 I_2(t-\tau_2)} + \frac{\mu_2\delta V_1 I_2}{N_1}e^{-m_4 I_2(t-\tau_2)}\\
&- (\theta + d_1)E_2 - (\theta + d_1)E_1 + \theta(E_1\\
&+ E_2) - (d_1 + \varphi + f_h)(I_1 + I_2)\\
&\leq \beta_1 I_1 + \mu_1\delta I_1 + \beta_2 I_1 + \mu_2\delta I_1 + \beta_1 I_2\\
&+ \mu_1\delta I_2 + \beta_2 I_2 + \mu_2\delta I_2 - d_1(E_1 + E_2)\\
&- (d_1 + \varphi + f_h)(I_1 + I_2)\\
&= (\beta_1 + \delta\mu_1 + \beta_2 + \delta\mu_2)(I_1 + I_2) - d_1(E_1\\
&+ E_2) - (d_1 + \varphi + f_h)(I_1 + I_2).
\end{aligned}$$

Define differentiable mapping G_2, $G_2 = \ln[E_1(t) + E_2(t) + I_1(t) + I_2(t)]$. On the basis of It$\hat{o}'$s formula and model (2), we have

$$\begin{aligned}
\mathrm{d}G_2 &= LG_2\mathrm{d}t + \sigma_3\frac{E_1 + E_2}{I_1 + I_2 + E_1 + E_2}\mathrm{d}B_3(t)\\
&+ \sigma_4\frac{I_1 + I_2}{I_1 + I_2 + E_1 + E_2}\mathrm{d}B_4(t),
\end{aligned}$$

where

$$\begin{aligned}
LG_2 &\leq \frac{1}{I_1 + I_2 + E_1 + E_2}[\beta_1 I_1 + \mu_1\delta I_1 + \beta_2 I_1\\
&+ \mu_2\delta I_1 + \beta_1 I_2 + \mu_1\delta I_2 + \beta_2 I_2 + \mu_2\delta I_2\\
&- d_1(E_1 + E_2) - (d_1 + \varphi + f_h)(I_1 + I_2)]\\
&- \frac{1}{2}(\sigma_3 \wedge \sigma_4)^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{I_1+I_2+E_1+E_2}((\beta_1+\delta\mu_1+\beta_2 \\
&+ \delta\mu_2)(I_1+I_2)-d_1(E_1+E_2)-(d_1+\varphi \\
&+ f_h)(I_1+I_2))-\frac{1}{2}(\sigma_3\wedge\sigma_4)^2 \\
&\leq (\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)-(d_1+\varphi+f_h) \\
&- \frac{1}{2}(\sigma_3\wedge\sigma_4)^2.
\end{aligned}$$

Therfore, we have

$$\begin{aligned}
dG_2 &\leq [(\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)-\frac{1}{2}(\sigma_3\wedge\sigma_4)^2 \\
&- (d_1+\varphi+f_h)]\mathrm{d}t+\frac{\sigma_3(E_1+E_2)}{I_1+I_2+E_1+E_2}\mathrm{d}B_3(t) \\
&+ \sigma_4\frac{I_1+I_2}{I_1+I_2+E_1+E_2}\mathrm{d}B_4(t) \\
&\leq [(\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)-\frac{1}{2}(\sigma_3\wedge\sigma_4)^2 \\
&- (d_1+\varphi+f_h)]dt+\sigma_3\mathrm{d}B_3(t)+\sigma_4\mathrm{d}B_4(t).
\end{aligned}$$

Integrate the above formula from 0 to t and multiply by $\frac{1}{t}$, we can get the following inequality

$$\begin{aligned}
&\frac{\ln(I_1(t)+I_2(t)+E_1(t)+E_2(t))}{t} \\
\leq\ &\frac{\ln(I_1(0)+I_2(0)+E_1(0)+E_2(0))}{t}+(\beta_1 \\
+\ &\delta\mu_1+\beta_2+\delta\mu_2)-(d_1+\varphi+f_h)-\frac{1}{2}(\sigma_3\wedge\sigma_4)^2 \\
+\ &\frac{1}{t}\int_0^t\sigma_3\mathrm{d}B_3(s)+\frac{1}{t}\int_0^t\sigma_4\mathrm{d}B_4(s),
\end{aligned}$$

using the result of Lemma 2.1 and Lemma 2.2, and according to $R_0^{jp}<1$ we can get the following conclusion

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\ln(I_1(t)+I_2(t)+E_1(t)+E_2(t))}{t} \\
\leq\ &(\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)-\frac{1}{2}(\sigma_3\wedge\sigma_4)^2 \\
-\ &(d_1+\varphi+f_h) \\
\leq\ &P(R_0^{jp}-1) \\
\leq\ &0.
\end{aligned}$$

where $P=d_1+\varphi+f_h$. Then, we deduce that

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{\ln(I_1(t))}{t} \\
\leq\ &\lim_{t\to\infty}\frac{\ln(I_1(t)+I_2(t)+E_1(t)+E_2(t))}{t}<0, \\
&\lim_{t\to\infty}\sup\frac{\ln(I_2(t))}{t} \\
\leq\ &\lim_{t\to\infty}\frac{\ln(I_1(t)+I_2(t)+E_1(t)+E_2(t))}{t}<0.
\end{aligned}$$

The aforementioned results lead to the conclusion that

$$\lim_{t\to\infty}I_1(t)=\lim_{t\to\infty}I_2(t)=0\ \ a.s.$$

3 Persistence

Theorem 3.1 For any initial value $Z(0)\in R_+^{12}$, $Z(t)$ is the solution of model (2). Therefore, if the condition $R_0^p>1$ is hold, the disease in the two cities persists in the mean. Moreover, the following hold:
(i) $\lim_{t\to\infty}\inf\langle I_1(t)+I_2(t)\rangle\geq\kappa(R_0^p-1)=\underline{I}>0\ \ a.s.$
(ii) $\underline{S}\leq\lim_{t\to\infty}\langle S_1+S_2\rangle\leq\bar{S}$,
(iii) $\underline{V}\leq\lim_{t\to\infty}\langle V_1+V_2\rangle\leq\bar{V}$,
(iv) $\underline{E}\leq\lim_{t\to\infty}\langle E_1+E_2\rangle\leq\bar{E}$,
(v) $\underline{H}\leq\lim_{t\to\infty}\langle H_1+H_2\rangle\leq\bar{H}$,
(vi) $\underline{R}\leq\lim_{t\to\infty}\langle R_1+R_2\rangle\leq\bar{R}$,

where $R_0^p=\frac{d_1+\theta+\frac{\sigma_3^2}{2}}{\delta+\theta}$, $\underline{S}=\frac{\theta+d_1}{w}\bar{E}-\frac{\beta_1+\delta\mu_1}{w}\underline{I}$, $\bar{S}=\frac{2\Lambda}{w+d_1-r}+\frac{2\upsilon N^*}{w+d_1-r}+\frac{\gamma}{w+d_1-r}\lim_{t\to\infty}\inf\langle R_1+R_2\rangle$, $\underline{V}=\frac{w}{\upsilon+d_1}\bar{S}-\frac{\beta_2+\delta\mu_2}{\upsilon+d_1}\underline{I}$, $\bar{V}=\frac{w}{\upsilon+d_1}\underline{S}$, $\underline{E}=\frac{e^{-(m_1\vee m_2\vee m_3\vee m_4)N^*}}{N^*}\frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I}$, $\bar{H}=\frac{f_h+\varphi}{d_1+d_2}\underline{I}$, $\bar{E}=\frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I}$, $\underline{H}=\frac{f_h}{d_1+d_2}\underline{I}-\frac{\gamma+d_1}{d_1+d_2}\bar{R}$, $\bar{R}=\frac{f_h+\varphi}{\gamma+d_1}\underline{I}$. $\underline{R}=\frac{w+d_1-r}{\gamma}\underline{S}-\frac{\upsilon}{\gamma}\bar{V}-\frac{2\Lambda}{\gamma}$,$\kappa=\frac{\delta+\theta}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta}$.

Proof. According to Theorem 2.2, we can get $G_3(t)=E_1(t)+E_2(t)$, and using Itô′s formula to get $\mathrm{d}G_3(t)=LG_3\mathrm{d}t+\sigma_3(E_1+E_2)\mathrm{d}B_3(t)$, where

$$\begin{aligned}
LG_3 &= \frac{\beta_1S_1I_1}{N_1}e^{-m_1I_1(t-\tau_1)}+\frac{\mu_1\delta S_2I_1}{N_2}e^{-m_2I_1(t-\tau_1)} \\
&+ \frac{\beta_2V_1I_1}{N_1}e^{-m_3I_1(t-\tau_1)}+\frac{\mu_2\delta V_2I_1}{N_2}e^{-m_4I_1(t-\tau_1)} \\
&- (\theta+d_1)E_1+\frac{\beta_1S_2I_2}{N_2}e^{-m_1I_2(t-\tau_2)} \\
&+ \frac{\mu_1\delta S_1I_2}{N_1}e^{-m_2I_2(t-\tau_2)}+\frac{\beta_2V_2I_2}{N_2}e^{-m_3I_2(t-\tau_2)} \\
&+ \frac{\mu_2\delta V_1I_2}{N_1}e^{-m_4I_2(t-\tau_2)}-(\theta+d_1)E_2 \\
&\leq -(\theta+d_1)(E_1+E_2)+(\beta_1+\beta_2+\mu_1\delta \\
&+ \mu_2\delta)I_1+(\beta_1+\beta_2+\mu_1\delta+\mu_2\delta)I_2 \\
&= (\beta_1+\beta_2+\mu_1\delta+\mu_2\delta)(I_1+I_2) \\
&- (\theta+d_1)(E_1+E_2).
\end{aligned}$$

We define $G_4=\ln[E_1(t)+E_2(t)]$, we can obtain

$$\begin{aligned}
\mathrm{d}G_4 &\leq [\frac{1}{E_1+E_2}((\beta_1+\beta_2+\mu_1\delta+\mu_2\delta)(I_1+I_2) \\
&- (\theta+d_1)(E_1+E_2))-\frac{\sigma_3^2}{2}]\mathrm{d}t+\sigma_3\mathrm{d}B_3(t) \\
&\leq [(\delta+\theta)+(\beta_1+\beta_2+\mu_1\delta+\mu_2\delta)(I_1+I_2) \\
&- (\theta+d_1+\frac{\sigma_3^2}{2})]\mathrm{d}t+\sigma_3\mathrm{d}B_3(t),
\end{aligned}$$

integrate the above formula from 0 to t, we have

$$\begin{aligned} &\frac{\ln(E_1(t)+E_2(t))-\ln(E_1(0)+E_2(0))}{t} \\ \leq\ & (\delta+\theta)+(\beta_1+\beta_2+\mu_1\delta+\mu_2\delta) < I_1+I_2 > \\ -\ & (\theta+d_1+\frac{\sigma_3^2}{2})+\frac{1}{t}\int_0^t \sigma_3 \mathrm{d}B_3(t), \end{aligned}$$

on the basis of [19], we can get

$$\begin{aligned} \langle I_1+I_2\rangle \ \geq\ & -\frac{(\delta+\theta)}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta} \\ +\ & \frac{\ln(E_1(0)+E_2(0))-\ln(E_1(t)+E_2(t))}{t(\beta_1+\beta_2+\mu_1\delta+\mu_2\delta)} \\ +\ & \frac{(\theta+d_1+\frac{\sigma_3^2}{2})}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta} \\ +\ & \frac{1}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta}\cdot\frac{1}{t}\int_0^t \sigma_3 \mathrm{d}B_3(t). \end{aligned}$$

Therefore

$$\begin{aligned} & \lim_{t\to\infty}\langle I_1+I_2\rangle \\ \geq\ & -\frac{(\delta+\theta)}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta}+\frac{(\theta+d_1+\frac{\sigma_3^2}{2})}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta} \\ =\ & \frac{(\delta+\theta)}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta}(\frac{d_1+\theta+\frac{\sigma_3^2}{2}}{\delta+\theta}-1) \\ =\ & \kappa(R_0^p-1), \end{aligned}$$

where $\kappa=\frac{\delta+\theta}{\beta_1+\beta_2+\mu_1\delta+\mu_2\delta}$.
Since $R_0^p>1$, we can get the following results:

$$\lim_{t\to\infty}\inf\langle I_1(t)+I_2(t)\rangle \geq \kappa(R_0^p-1)=\underline{I}>0 \ \ a.s.$$

Define differentiable mapping W_1, $W_1=S_1(t)+S_2(t)$. On the basis of Itô′s formula and model (2), we have $\mathrm{d}W_1(t)=LW_1\mathrm{d}t+\sigma_1(S_1+S_2)\mathrm{d}B_1(t)$, where

$$\begin{aligned} LW_1 \ =\ & \Lambda+rS_1(1-\alpha S_1)-\frac{\beta_1 S_1 I_1}{N_1}e^{-m_1I_1(t-\tau_1)} \\ -\ & \frac{\mu_1\delta S_2 I_1}{N_2}e^{-m_2I_1(t-\tau_1)}-(w+d_1)S_1+\gamma R_1 \\ +\ & vV_1-\delta(S_1-S_2)+\Lambda+rS_2(1-\alpha S_2) \\ -\ & \frac{\beta_1 S_2 I_2}{N_2}e^{-m_1I_2(t-\tau_2)}-\frac{\mu_1\delta S_1 I_2}{N_1}e^{-m_2I_2(t-\tau_2)} \\ -\ & (w+d_1)S_2+\gamma R_2+vV_2-\delta(S_2-S_1) \\ \leq\ & 2\Lambda-(w+d_1-r)(S_1+S_2)+\gamma(R_1+R_2) \\ +\ & v(V_1+V_2). \end{aligned}$$

Integrate over the above equation from 0 to t and multiply by $\frac{1}{t}$

$$\begin{aligned} &\frac{(S_1(t)+S_2(t))-(S_1(0)+S_2(0))}{t} \\ \leq\ & 2\Lambda-(w+d_1-r)\langle S_1+S_2\rangle+\gamma\langle R_1+R_2\rangle \\ +\ & v(V_1+V_2)+\frac{\sigma_1}{t}\int_0^t (S_1+S_2)dB_1(s), \end{aligned}$$

then, using the result of Lemma 2.1 and Lemma 2.2 we can obtain

$$\begin{aligned} \lim_{t\to\infty}\langle S_1+S_2\rangle \ \leq\ & \frac{2\Lambda}{w+d_1-r}+\frac{2vN^*}{w+d_1-r} \\ +\ & \frac{\gamma}{w+d_1-r}\lim_{t\to\infty}\inf\langle R_1+R_2\rangle \\ \doteq\ & \bar{S}. \end{aligned}$$

where N^* is the maximum environmental capacity.

Define differentiable mapping W_2, $W_2=E_1(t)+E_2(t)+V_1(t)+V_2(t)$. On the basis of Itô′s formula and model (2), we have $\mathrm{d}W_2(t)=LW_2\mathrm{d}t+\sigma_2(V_1+V_2)\mathrm{d}B_2(t)+\sigma_3(E_1+E_2)\mathrm{d}B_3(t)$, where

$$\begin{aligned} LW_2(t) \ =\ & w(S_1+S_2)-(v+d_1)(V_1+V_2)-(\theta \\ +\ & d_1)(E_1+E_2)+\frac{\beta_1 S_1 I_1}{N_1}e^{-m_1I_1(t-\tau_1)} \\ +\ & \frac{\mu_1\delta S_2 I_1}{N_2}e^{-m_2I_1(t-\tau_1)}+\frac{\beta_1 S_2 I_2}{N_2}e^{-m_1I_2(t-\tau_2)} \\ +\ & \frac{\mu_1\delta S_1 I_2}{N_1}e^{-m_2I_2(t-\tau_2)} \\ \leq\ & w(S_1+S_2)-(\theta+d_1)(E_1+E_2)+(\beta_1 \\ +\ & \mu_1\delta)(I_1+I_2). \end{aligned}$$

Integrate from 0 to t

$$\begin{aligned} &\frac{\sum_{i=1}^2[(E_i(t)+V_i(t))-(E_i(0)+V_i(0))]}{t} \\ \leq\ & w\langle S_1+S_2\rangle-(\theta+d_1)\langle E_1+E_2\rangle+(\beta_1 \\ +\ & \mu_1\delta)\langle I_1+I_2\rangle+\frac{\sigma_2}{t}\int_0^t (V_1+V_2)\mathrm{d}B_2(s) \\ +\ & \frac{\sigma_3}{t}\int_0^t (E_1+E_2)\mathrm{d}B_3(s). \end{aligned}$$

So, using the result of Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned} \lim_{t\to\infty}\langle S_1+S_2\rangle \ \geq\ & \frac{\theta+d_1}{w}\lim_{t\to\infty}\sup\langle E_1+E_2\rangle \\ -\ & \frac{\beta_1+\delta\mu_1}{w}\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\ =\ & \frac{\theta+d_1}{w}\bar{E}-\frac{\beta_1+\delta\mu_1}{w}\underline{I}\doteq\underline{S}. \end{aligned}$$

Therefore, we can get

$$\underline{S} \leq \lim_{t\to\infty} \langle S_1 + S_2 \rangle \leq \bar{S},$$

where $\underline{S} = \frac{\theta+d_1}{w}\bar{E} - \frac{\beta_1+\delta\mu_1}{w}\underline{I}$, $\bar{S} = \frac{2\Lambda}{w+d_1-r} + \frac{2\upsilon N^*}{w+d_1-r} + \frac{\gamma}{w+d_1-r}\lim_{t\to\infty}\inf\langle R_1 + R_2\rangle$.

Define differentiable mapping W_3, $W_3 = V_1(t) + V_2(t)$. On the basis of Itô's formula and model (2), we have $\mathrm{d}W_3(t) = LW_3\mathrm{d}t + \sigma_2(V_1+V_2)\mathrm{d}B_2(t)$, where

$$\begin{aligned} LW_3(t) &= w(S_1+S_2) - (\upsilon+d_1)(V_1+V_2) \\ &- \frac{\beta_2 V_1 I_1}{N_1}e^{-m_3 I_1(t-\tau_1)} - \frac{\mu_2\delta V_2 I_1}{N_2}e^{-m_4 I_1(t-\tau_1)} \\ &- \frac{\beta_2 V_2 I_2}{N_2}e^{-m_3 I_2(t-\tau_2)} - \frac{\mu_2\delta V_1 I_2}{N_1}e^{-m_4 I_2(t-\tau_2)} \\ &\leq w(S_1+S_2) - (\upsilon+d_1)(V_1+V_2), \end{aligned}$$

integrate from 0 to t and using the result of Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned} \lim_{t\to\infty}\langle V_1+V_2\rangle &\leq \frac{w}{\upsilon+d_1}\lim_{t\to\infty}\inf\langle S_1+S_2\rangle \\ &= \frac{w}{\upsilon+d_1}\underline{S} \doteq \bar{V}. \end{aligned}$$

On the other hand

$$\begin{aligned} LW_3(t) &\geq w(S_1+S_2) - (\upsilon+d_1)(V_1+V_2) \\ &- \frac{\beta_2 V_1 I_1}{N_1} - \frac{\mu_2\delta V_2 I_1}{N_2} - \frac{\beta_2 V_2 I_2}{N_2} \\ &- \frac{\mu_2\delta V_1 I_2}{N_1} \\ &\geq w(S_1+S_2) - (\upsilon+d_1)(V_1+V_2) \\ &- (\beta_2+\delta\mu_2)(I_1+I_2) \end{aligned}$$

The same can be obtained

$$\begin{aligned} &\lim_{t\to\infty}\langle V_1+V_2\rangle \\ \geq\ & \frac{w}{\upsilon+d_1}\lim_{t\to\infty}\sup\langle S_1+S_2\rangle \\ -\ & \frac{\beta_2+\delta\mu_2}{\upsilon+d_1}\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\ =\ & \frac{w}{\upsilon+d_1}\bar{S} - \frac{\beta_2+\delta\mu_2}{\upsilon+d_1}\underline{I} \doteq \underline{V}. \end{aligned}$$

So, we can obtain

$$\underline{V} \leq \lim_{t\to\infty}\langle V_1+V_2\rangle \leq \bar{V},$$

where $\underline{V} = \frac{w}{\upsilon+d_1}\bar{S} - \frac{\beta_2+\delta\mu_2}{\upsilon+d_1}\underline{I}$, $\bar{V} = \frac{w}{\upsilon+d_1}\underline{S}$.

Define differentiable mapping W_4, $W_4 = E_1(t) + E_2(t)$. On the basis of Itô's formula and model (2), we have $\mathrm{d}W_4(t) = LW_4\mathrm{d}t + \sigma_3(E_1+E_2)\mathrm{d}B_3(t)$, where

$$\begin{aligned} & LW_4(t) \\ =\ & \frac{\beta_1 S_1 I_1}{N_1}e^{-m_1 I_1(t-\tau_1)} + \frac{\mu_1\delta S_2 I_1}{N_2}e^{-m_2 I_1(t-\tau_1)} \\ +\ & \frac{\beta_2 V_1 I_1}{N_1}e^{-m_3 I_1(t-\tau_1)} + \frac{\mu_2\delta V_2 I_1}{N_2}e^{-m_4 I_1(t-\tau_1)} \\ +\ & \frac{\beta_1 S_2 I_2}{N_2}e^{-m_1 I_2(t-\tau_2)} + \frac{\mu_1\delta S_1 I_2}{N_1}e^{-m_2 I_2(t-\tau_2)} \\ +\ & \frac{\beta_2 V_2 I_2}{N_2}e^{-m_3 I_2(t-\tau_2)} + \frac{\mu_2\delta V_1 I_2}{N_1}e^{-m_4 I_2(t-\tau_2)} \\ -\ & (\theta+d_1)(E_1+E_2) \\ \leq\ & (\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)(I_1+I_2) \\ -\ & (\theta+d_1)(E_1+E_2), \end{aligned}$$

then

$$\begin{aligned} \mathrm{d}W_4(t) &\leq [(\beta_1+\delta\mu_1+\beta_2+\delta\mu_2)(I_1+I_2) - (\theta \\ &+ d_1)(E_1+E_2)]\mathrm{d}t + \sigma_3(E_1+E_2)\mathrm{d}B_3(t), \end{aligned}$$

integrate from 0 to t and using the result of Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned} & \lim_{t\to\infty}\langle E_1+E_2\rangle \\ \leq\ & \frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\ =\ & \frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I} \\ \doteq\ & \bar{E}. \end{aligned}$$

On the other hand

$$\begin{aligned} LW_4(t) &\geq \frac{\beta_1 S_1 I_1}{N_1}e^{-m_1 N^*} + \frac{\mu_1\delta S_2 I_1}{N_2}e^{-m_2 N^*} \\ &+ \frac{\beta_2 V_1 I_1}{N_1}e^{-m_3 N^*} + \frac{\mu_2\delta V_2 I_1}{N_2}e^{-m_4 N^*} \\ &+ \frac{\beta_1 S_2 I_2}{N_2}e^{-m_1 N^*} + \frac{\mu_1\delta S_1 I_2}{N_1}e^{-m_2 N^*} \\ &+ \frac{\beta_2 V_2 I_2}{N_2}e^{-m_3 N^*} + \frac{\mu_2\delta V_1 I_2}{N_1}e^{-m_4 N^*} \\ &- (\theta+d_1)(E_1+E_2) \\ &\geq \frac{e^{-(m_1\vee m_2\vee m_3\vee m_4)N^*}}{N^*}(\beta_1+\delta\mu_1+\beta_2 \\ &+ \delta\mu_2)(I_1+I_2) - (\theta+d_1)(E_1+E_2). \end{aligned}$$

Since people's immunity is one of the factors to resist infectious diseases, and there are always people with low immunity in the population, people with low immunity are usually classified as susceptible groups, so we assume that $S_i > 1$. Also, let's assume that $V_i > 1$. After the integral, can be obtained

$$\begin{aligned}
&\lim_{t\to\infty}\langle E_1+E_2\rangle \\
\geq\ & \frac{e^{-(m_1\vee m_2\vee m_3\vee m_4)N^*}}{N^*}(\frac{\beta_1+\delta\mu_1+\beta_2}{\theta+d_1} \\
+\ & \frac{\delta\mu_2}{\theta+d_1})\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\
=\ & \frac{e^{-(m_1\vee m_2\vee m_3\vee m_4)N^*}}{N^*}\frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I} \\
\doteq\ & \underline{E}.
\end{aligned}$$

So, we can get

$$\underline{E}\leq\lim_{t\to\infty}\langle E_1+E_2\rangle\leq\bar{E},$$

where $\underline{E} = \frac{e^{-(m_1\vee m_2\vee m_3\vee m_4)N^*}}{N^*}\frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I}$, $\bar{E}=\frac{\beta_1+\delta\mu_1+\beta_2+\delta\mu_2}{\theta+d_1}\underline{I}$.

Define differentiable mapping W_5, $W_5 = \sum_{i=1}^{2}(H_i(t)+R_i(t))$. On the basis of It$\hat{o}'$s formula and model (2), we have $dW_5(t)=LW_5dt+\sigma_5(H_1+H_2)dB_5(t)+\sigma_6(R_1+R_2)d6_5(t)$, where

$$\begin{aligned}
LW_5 &= (f_h+\varphi)(I_1+I_2)-(d_1+d_2)(H_1+H_2) \\
&- (\gamma+d_1)(R_1+R_2) \\
&\leq (f_h+\varphi)(I_1+I_2)-(d_1+d_2)(H_1+H_2).
\end{aligned}$$

After integrating, using the result of lemma 2.1 and lemma 2.2, we can get

$$\begin{aligned}
\lim_{t\to\infty}\langle H_1+H_2\rangle &\leq \frac{f_h+\varphi}{d_1+d_2}\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\
&= \frac{f_h+\varphi}{d_1+d_2}\underline{I} \\
&\doteq \bar{H},
\end{aligned}$$

the same can be obtained

$$\lim_{t\to\infty}\langle R_1+R_2\rangle\leq\frac{f_h+\varphi}{\gamma+d_1}\underline{I}\doteq\bar{R}.$$

Take the limit to get

$$\begin{aligned}
\lim_{t\to\infty}\langle H_1+H_2\rangle &\geq \frac{f_h}{d_1+d_2}\lim_{t\to\infty}\inf\langle I_1+I_2\rangle \\
&- \frac{\gamma+d_1}{d_1+d_2}\lim_{t\to\infty}\inf\langle R_1+R_2\rangle \\
&= \frac{f_h}{d_1+d_2}\underline{I}-\frac{\gamma+d_1}{d_1+d_2}\bar{R} \\
&\doteq \underline{H}.
\end{aligned}$$

According to the W_1, we have

$$\begin{aligned}
LW_1 &\leq 2\Lambda+r(S_1+S_2)-(w+d_1)(S_1+S_2) \\
&+ \upsilon(V_1+V_2)+\gamma(R_1+R_2),
\end{aligned}$$

then, $dW_5(t)$ can be obtained after integration

$$\begin{aligned}
\gamma\langle R_1+R_2\rangle &\geq (w+d_1-r)\langle S_1+S_2\rangle-\upsilon\langle V_1 \\
&+ V_2\rangle-2\Lambda-\frac{\sigma_1}{t}\int_0^t dB_1(t) \\
&+ \frac{(S_1(t)+S_2(t))-(S_1(0)+S_2(0))}{t},
\end{aligned}$$

using the result of Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned}
\lim_{t\to\infty}\langle R_1+R_2\rangle &\geq \frac{w+d_1-r}{\gamma}\lim_{t\to\infty}\inf\langle S_1+S_2\rangle \\
&- \frac{\upsilon}{\gamma}\lim_{t\to\infty}\sup\langle V_1+V_2\rangle-\frac{2\Lambda}{\gamma} \\
&= \frac{w+d_1-r}{\gamma}\underline{S}-\frac{\upsilon}{\gamma}\bar{V}-\frac{2\Lambda}{\gamma} \\
&\doteq \underline{R}.
\end{aligned}$$

Then, we can get

$$\begin{aligned}
\underline{H} &\leq \lim_{t\to\infty}\langle H_1+H_2\rangle\leq\bar{H}, \\
\underline{R} &\leq \lim_{t\to\infty}\langle R_1+R_2\rangle\leq\bar{R},
\end{aligned}$$

where $\underline{H}=\frac{f_h}{d_1+d_2}\underline{I}-\frac{\gamma+d_1}{d_1+d_2}\bar{R}$, $\bar{H}=\frac{f_h+\varphi}{d_1+d_2}\underline{I}$, $\underline{R}=\frac{w+d_1-r}{\gamma}\underline{S}-\frac{\upsilon}{\gamma}\bar{V}-\frac{2\Lambda}{\gamma}$, $\bar{R}=\frac{f_h+\varphi}{\gamma+d_1}\underline{I}$.

The proof completes here.

4 Optimal Control

For infectious diseases, understanding how to control and eliminate/eradicate infectious diseases is one of the main goals of mathematical epidemiology and public health[20]. To this end, we studied the impact of vaccination and aggressive treatment on reducing disease transmission. At the same time, we actively call for a reduction in the movement of people everywhere. To this effect, we introduce into the model (2) a set of time dependent control variables $u(t)=(u_1(t),u_2(t),u_3(t),u_4(t))$, where
(a) $u_1(t)$ represents the infection rate of infectious individuals (I_i) to susceptible individuals (S_i).
(b) $u_2(t)$ represents the infection rate of local infected individuals to nonnative susceptible individuals,
(c) $u_3(t)$ represents the recovery rate, the cure of disease,
(d) $u_4(t)$ represents the vaccination of susceptible individuals.

A stochastic control system with control variables

$u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ is as follows:

$$\begin{cases} dS_i(t) = [\Lambda + rS_i(1-\alpha S_i) + \gamma R_i + \upsilon V_i \\ \quad -\frac{u_1(t)S_iI_i}{N_i}e^{-m_1I_i(t-\tau_i)} - (u_4(t) \\ \quad +d_1)S_i - \frac{u_2(t)\delta S_jI_i}{N_j}e^{-m_2I_i(t-\tau_i)} \\ \quad -\delta(S_i - S_j)]dt + \sigma_1 S_i dB_1(t), \\ dV_i(t) = [u_4(t)S_i - \frac{\beta_2V_iI_i}{N_i}e^{-m_3I_i(t-\tau_i)} \\ \quad -\frac{\mu_2\delta V_jI_i}{N_j}e^{-m_4I_i(t-\tau_i)} - (\upsilon \\ \quad +d_1)V_i - \delta(V_i - V_j)]dt \\ \quad +\sigma_2V_idB_2(t), \\ dE_i(t) = [\frac{u_1(t)S_iI_i}{N_i}e^{-m_1I_i(t-\tau_i)} - (\theta \\ \quad +d_1)E_i + \frac{u_2(t)\delta S_jI_i}{N_j}e^{-m_2I_i(t-\tau_i)} \\ \quad +\frac{\beta_2V_iI_i}{N_i}e^{-m_3I_i(t-\tau_i)} - \delta(E_i \\ \quad -E_j + \frac{\mu_2\delta V_jI_i}{N_j}e^{-m_4I_i(t-\tau_i)}]dt \\ \quad +\sigma_3E_idB_3(t), \\ dI_i(t) = [\theta E_i - (d_1+\varphi+f_h)I_i - \delta(I_i \\ \quad -I_j)]dt + \sigma_4I_idB_4(t), \\ dH_i(t) = [f_hI_i + (-1)^{i+1}g_1I_2 - (u_3(t) \\ \quad +d_1+d_2)H_i]dt + \sigma_5H_idB_5(t), \\ dR_i(t) = [\varphi I_i + u_3(t)H_i - (\gamma+d_1)R_i \\ \quad -\delta(R_i - R_j)]dt + \sigma_6R_idB_6(t), \end{cases} \tag{3}$$

where $i = 1, 2$ and $j = i - (-1)^i$ $(i = 1, 2)$.

Our control problem is to minimize the number of symptomatic individuals as well as minimizing the cost of treatment via minimization of the following cost functional, thus our objective function is as follows

$$J(u_1, u_2, u_3, u_4) = E\int_0^{t_f}[\sum_{i=1}^{2}(A_1I_i + A_2H_i) \tag{4}$$

$$+\frac{1}{2}(B_1u_1(t)^2+B_2u_2(t)^2+B_3u_3(t)^2+B_4u_4(t)^2)]dt,$$

where t_f is the final time, $A_i,\ B_j\ (i = 1, 2;\ j = 1, 2, 3, 4)$ are balancing cost factors. In order to optimize our control strategy, we need to save as much money as possible while controlling the spread of the disease. Thus, we seek to find an optimal control, $u^* = (u_1^*, u_2^*, u_3^*, u_4^*)$, such that the optimal control function

$$J(u^*) = \inf_U J(u), \tag{5}$$

where $U = \{(u_1, u_2, u_3, u_4) \in (L^4(0, t_f))^2 \mid a_i \leq u_i(t) \leq b_i,\ t \in [0, t_f]\}$ is the control set, and $a_i, b_i\ (i = 1, 2, 3, 4)$ are fixed positive.

4.1 Existence of an Optimal Control

Using the results obtained in reference [21], it is possible to prove the existence of the solution of model (3) in the finite time interval of the given control in the admissible control set U. And there are the following results.

Theorem 4.1 Given any control $(u_1, u_2, u_3, u_4) \in U$, there exists a bounded solution to model (3).

Since the state variables and the controls are uniformly bounded, existence of an optimal control follows boundedness of solutions and their derivatives of the model (3) for a finite time interval. The boundedness and convexity of the target functional provide sufficient compactness for the existence of the optimal control [22], [23]. Thus, with the objective functional J in Eq. (4) subject to the control set U, there exists an optimal control $u^* \in U$ such that $J(u^*) = \min_{u\in U} J(u)$.

4.2 Specific Description of Optimal Control

Pontryagin's Maximum Principle is a necessary condition for optimal control. This principle converts (3) and (4) into a problem of minimizing pointwise a Hamiltonian H, with respect to $u = (u_1, u_2, u_3, u_4)$. Firstly, we obtain the Hamiltonian from the target functional (4) and the stochastic system (3), and thus obtain the optimality condition. So, by processing the Hamiltonian function, we can obtain

$$\begin{aligned} & H(t) \\ = & \sum_{i=1}^{2}(A_1I_i + A_2H_i) + \frac{1}{2}(B_1u_1(t)^2 + B_2u_2(t)^2 \\ + & B_3u_3(t)^2 + B_4u_4(t)^2) + \sum_{i=1}^{2}\{\lambda_{S_i}[\Lambda + rS_i(1 \\ - & \alpha S_i) - \frac{u_1(t)S_iI_i}{N_i}e^{-m_1I_i(t-\tau_i)} - (u_4(t) \\ + & d_1)S_i\frac{u_2(t)\delta S_jI_i}{N_j}e^{-m_2I_i(t-\tau_i)} + \gamma R_i + \upsilon V_i \\ - & \delta(S_i - S_j)] + \lambda_{V_i}[u_4(t)S_i - \frac{\beta_2V_iI_i}{N_i}e^{-m_3I_i(t-\tau_i)} \\ - & \frac{\mu_2\delta V_jI_i}{N_j}e^{-m_4I_i(t-\tau_i)} - (\upsilon + d_1)V_i - \delta(V_i - V_j)] \\ + & \lambda_{E_i}[\frac{u_1(t)S_iI_i}{N_i}e^{-m_1I_i(t-\tau_i)} + \frac{u_2(t)\delta S_jI_i}{N_j}e^{-m_2I_i(t-\tau_i)} \\ + & \frac{\beta_2V_iI_i}{N_i}e^{-m_3I_i(t-\tau_i)} + \frac{\mu_2\delta V_jI_i}{N_j}e^{-m_4I_i(t-\tau_i)} - (\theta \\ + & d_1)E_i - \delta(E_i - E_j)] + \lambda_{I_i}[\theta E_i - (d_1 + \varphi + f_h)I_i \\ - & \delta(I_i - I_j)] + \lambda_{H_i}[f_hI_i + (-1)^{i+1}g_1I_2 - (u_3(t) \\ + & d_1 + d_2)H_i] + \lambda_{R_i}[\varphi I_i + u_3(t)H_i - (\gamma + d_1)R_i \\ - & \delta(R_i - R_j)]\} + \sigma_1(S_1q_1 + S_2q_2) + \sigma_2(V_1q_3 \\ + & V_2q_4) + \sigma_3(E_1q_5 + E_2q_6) + \sigma_4(I_1q_7 + I_2q_8) \\ + & \sigma_5(H_1q_9 + H_2q_{10}) + \sigma_6(R_1q_{11} + R_2q_{12}), \end{aligned}$$

where the $\lambda_{S_i}, \lambda_{V_i}, \lambda_{E_i}, \lambda_{I_i}, \lambda_{H_i},\ \lambda_{R_i}\ \ (i = 1, 2)$ are the associated adjoints for the state variables

$S_i, V_i, E_i, I_i, H_i, R_i$. The sys of equations is found by taking the appropriate partial derivatives of the Hamiltonian with respect to the associated state variable.

Given the optimal control $u^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ and the corresponding state solutions S_i^*, V_i^*, E_i^*, I_i^*, H_i^*, R_i^*, $(i = 1, 2)$ of system (3) that minimizes $J(u_1, u_2, u_3, u_4)$ over U, there exists $\lambda_{S_i}, \lambda_{V_i}, \lambda_{E_i}, \lambda_{I_i}, \lambda_{H_i}, \lambda_{R_i}$ $(i = 1, 2)$ satisfying the adjoint system

$$
\begin{aligned}
\frac{d\lambda_{S_1}}{dt} &= (\lambda_{S_1} - \lambda_{E_1})e^{-m_1 I_1(t-\tau_1)}\frac{u_1(t)I_1}{N_1} + \sigma_1 q_1 + (\lambda_{S_1} \\
&- \lambda_{V_1})u_4(t) + (\lambda_{S_2} - \lambda_{E_2})e^{-m_2 I_2(t-\tau_2)}\frac{u_2(t)\delta I_2}{N_1} \\
&+ \delta(\lambda_{S_1} - \lambda_{S_2}) + \lambda_{S_1}(d_1 - r(1 - 2\alpha S_1)), \\
\frac{d\lambda_{S_2}}{dt} &= (\lambda_{S_2} - \lambda_{E_2})e^{-m_1 I_2(t-\tau_2)}\frac{u_1(t)I_2}{N_2} + \sigma_1 q_2 + (\lambda_{S_2} \\
&- \lambda_{V_2})u_4(t) + (\lambda_{S_1} - \lambda_{E_1})e^{-m_2 I_1(t-\tau_1)}\frac{u_2(t)\delta I_1}{N_2} \\
&+ \delta(\lambda_{S_2} - \lambda_{S_1}) + \lambda_{S_2}(d_1 - r(1 - 2\alpha S_2)), \\
\frac{d\lambda_{V_1}}{dt} &= (\lambda_{V_1} - \lambda_{E_1})e^{-m_3 I_1(t-\tau_1)}\frac{\beta_2 I_1}{N_1} + (\lambda_{V_2} \\
&- \lambda_{E_2})e^{-m_4 I_2(t-\tau_2)}\frac{\mu_2\delta I_2}{N_1} + (\lambda_{V_1} - \lambda_{S_1})\upsilon \\
&+ d_1\lambda_{V_1} + \delta(\lambda_{V_1} - \lambda_{V_2}) + \sigma_2 q_3, \\
\frac{d\lambda_{V_2}}{dt} &= (\lambda_{V_2} - \lambda_{E_2})e^{-m_3 I_2(t-\tau_1 2)}\frac{\beta_2 I_2}{N_2} + (\lambda_{V_1} \\
&- \lambda_{E_2 1})e^{-m_4 I_1(t-\tau_1)}\frac{\mu_2\delta I_1}{N_2} + d_1\lambda_{V_2} \\
&+ (\lambda_{V_2} - \lambda_{S_2})\upsilon + \delta(\lambda_{V_2} - \lambda_{V_1}) + \sigma_2 q_4, \\
\frac{d\lambda_{E_1}}{dt} &= (\lambda_{E_1} - \lambda_{E_2})\delta + (\lambda_{E_1} - \lambda_{I_1})\theta + \lambda_{E_1}d_1 \\
&+ \sigma_3 q_5, \\
\frac{d\lambda_{E_2}}{dt} &= (\lambda_{E_2} - \lambda_{E_1})\delta + (\lambda_{E_2} - \lambda_{I_2})\theta + \lambda_{E_2}d_1 \\
&+ \sigma_3 q_6, \\
\frac{d\lambda_{I_1}}{dt} &= -A_1 + \sigma_4 q_7 + (\lambda_{I_1} - \lambda_{I_2})\delta + \lambda_{I_1}d_1 \\
&+ (\lambda_{S_1} - \lambda_{E_1})(u_1(t)e^{-m_1 I_1(t-\tau_1)}\frac{S_1}{N_1} \\
&+ \delta u_2(t)e^{-m_2 I_1(t-\tau_1)}\frac{S_2}{N_2}) + (\lambda_{I_1} - \lambda_{R_1})\varphi \\
&+ (\lambda_{I_1} - \lambda_{H_1})f_h + (\lambda_{V_1} \\
&- \lambda_{E_1})(\beta_2 e^{-m_3 I_1(t-\tau_1)}\frac{V_1}{N_1} \\
&+ \delta\mu_2 e^{-m_4 I_1(t-\tau_1)}\frac{V_2}{N_2}) - \chi_{[0,T-\tau_1]}\{[\lambda_{E_1}(t+\tau_1) \\
&- \lambda_{S_1}(t+\tau_1)](u_1(t)m_1 e^{-m_1 I_1(t-\tau_1)}\frac{S_1 I_1}{N_1} \\
&+ e^{-m_2 I_1(t-\tau_1)}\frac{u_2(t)m_2\delta S_2 I_1}{N_2}) \\
&+ [\lambda_{E_1}(t+\tau_1) \\
&- \lambda_{V_1}(t+\tau_1)](e^{-m_3 I_1(t-\tau_1)}\frac{m_3\beta_2 V_1 I_1}{N_1} \\
&+ e^{-m_4 I_1(t-\tau_1)}\frac{m_4\delta\mu_2 V_2 I_1}{N_2})\} \\
\frac{d\lambda_{I_2}}{dt} &= -A_1 + \sigma_4 q_8 + (\lambda_{I_2} - \lambda_{I_1})\delta + \lambda_{I_1}d_1 \\
&+ (\lambda_{S_2} - \lambda_{E_2})(u_1(t)e^{-m_1 I_2(t-\tau_2)}\frac{S_2}{N_2} \\
&+ \delta u_2(t)e^{-m_2 I_2(t-\tau_2)}\frac{S_1}{N_1}) + (\lambda_{I_2} - \lambda_{R_2})\varphi \\
&+ (\lambda_{I_2} - \lambda_{H_2})f_h + (\lambda_{H_2} - \lambda_{H_1})g_1 + (\lambda_{V_2} \\
&- \lambda_{E_2})(\beta_2 e^{-m_3 I_2(t-\tau_2)}\frac{V_2}{N_2} \\
&+ \delta\mu_2 e^{-m_4 I_2(t-\tau_2)}\frac{V_1}{N_1}) - \chi_{[0,T-\tau_2]}\{[\lambda_{E_2}(t+\tau_2) \\
&- \lambda_{S_2}(t+\tau_2)](u_1(t)m_1 e^{-m_1 I_2(t-\tau_2)}\frac{S_2 I_2}{N_2} \\
&+ u_2(t)\delta m_2 e^{-m_2 I_2(t-\tau_2)}\frac{S_1 I_2}{N_1}) \\
&+ [\lambda_{E_2}(t+\tau_2) \\
&- \lambda_{V_2}(t+\tau_2)](e^{-m_3 I_2(t-\tau_2)}\frac{m_3\beta_2 V_2 I_2}{N_2} \\
&+ e^{-m_4 I_2(t-\tau_2)}\frac{m_4\delta\mu_2 V_1 I_2}{N_1})\}, \\
\frac{d\lambda_{H_1}}{dt} &= -A_2 + (\lambda_{H_1} - \lambda_{R_1})u_3(t) + \lambda_{H_1}(d_1 + d_2) \\
&+ \sigma_5 q_9, \\
\frac{d\lambda_{H_2}}{dt} &= -A_2 + (\lambda_{H_2} - \lambda_{R_2})u_3(t) + \lambda_{H_2}(d_1 + d_2) \\
&+ \sigma_5 q_{10}, \\
\frac{d\lambda_{R_1}}{dt} &= (\lambda_{R_1} - \lambda_{S_1})\gamma + \lambda_{R_1}d_1 + (\lambda_{R_1} - \lambda_{R_2})\delta \\
&+ \sigma_6 q_{11}, \\
\frac{d\lambda_{R_2}}{dt} &= (\lambda_{R_2} - \lambda_{S_2})\gamma + \lambda_{R_2}d_1 + (\lambda_{R_2} - \lambda_{R_1})\delta \\
&+ \sigma_6 q_{12},
\end{aligned}
$$

with transversality conditions $\lambda_{S_i}(t_f) = 0$,$\lambda_{V_i}(t_f) = 0$,$\lambda_{E_i}(t_f) = 0$,$\lambda_{I_i}(t_f) = 0$, $\lambda_{H_i}(t_f) = 0$, $\lambda_{R_i}(t_f) = 0$, where $\chi_{[0,T-\tau_i]}$ are indicator functions on $[0, T - \tau_i]$ $(i = 1, 2)$satisfying

$$
\chi_{[0,T-\tau_i)} = \begin{cases} 1 & t \in [0, T - \tau_i) \\ 0 & t \notin [0, T - \tau_i). \end{cases} \tag{6}
$$

Considering the optimality conditions, the Hamiltonian function is differentiated with respect to the control variables resulting in

$$\begin{aligned}\frac{\partial H}{\partial u_1} &= B_1u_1(t) - (\lambda_{S_1} - \lambda_{E_1})e^{-m_1I_1(t-\tau_1)}\frac{S_1I_1}{N_1}\\ &- (\lambda_{S_2} - \lambda_{E_2})e^{-m_1I_2(t-\tau_2)}\frac{S_2I_2}{N_2} = 0,\\ \frac{\partial H}{\partial u_2} &= B_2u_2(t) - (\lambda_{S_1} - \lambda_{E_1})e^{-m_2I_1(t-\tau_1)}\frac{\delta S_2I_1}{N_2}\\ &- (\lambda_{S_2} - \lambda_{E_2})e^{-m_2I_2(t-\tau_2)}\frac{\delta S_1I_2}{N_1} = 0,\\ \frac{\partial H}{\partial u_3} &= B_3u_3(t) - (\lambda_{H_1} - \lambda_{R_1})H_1 - (\lambda_{H_2} - \lambda_{R_2})H_2\\ &= 0,\\ \frac{\partial H}{\partial u_4} &= B_4u_4(t) - (\lambda_{S_1} - \lambda_{V_1})S_1 - (\lambda_{S_2} - \lambda_{V_2})S_2\\ &= 0,\end{aligned}$$

on the interior of the control set U. So we can get an unique solution $\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t), \bar{u}_3(t), \bar{u}_4(t))$ in terms of state variables and adjoint variable, as following

$$\begin{aligned}\bar{u_1}(t) &= \frac{1}{B_1}[(\lambda_{S_1} - \lambda_{E_1})e^{-m_1I_1(t-\tau_1)}\frac{S_1I_1}{N_1}\\ &+ (\lambda_{S_2} - \lambda_{E_2})e^{-m_1I_2(t-\tau_2)}\frac{S_2I_2}{N_2}],\\ \bar{u_2}(t) &= \frac{1}{B_2}[(\lambda_{S_1} - \lambda_{E_1})e^{-m_2I_1(t-\tau_1)}\frac{\delta S_2I_1}{N_2}\\ &+ (\lambda_{S_2} - \lambda_{E_2})e^{-m_2I_2(t-\tau_2)}\frac{\delta S_1I_2}{N_1}],\\ \bar{u_3}(t) &= \frac{1}{B_3}[(\lambda_{H_1} - \lambda_{R_1})H_1 + (\lambda_{H_2} - \lambda_{R_2})H_2],\\ \bar{u_4}(t) &= \frac{1}{B_4}[(\lambda_{S_1} - \lambda_{V_1})S_1 - (\lambda_{S_2} - \lambda_{V_2})S_2],\end{aligned}$$

Finally, we get the solution shown below

$$\begin{aligned}u_1^* &= \min\{b, \max[a, \bar{u}_1(t)]\},\\ u_2^* &= \min\{b, \max[a, \bar{u}_2(t)]\},\\ u_3^* &= \min\{b, \max[a, \bar{u}_3(t)]\},\\ u_4^* &= \min\{b, \max[a, \bar{u}_4(t)]\}.\end{aligned}$$

5 Numerical Simulations

In this section, we use Milsteins Higher Order Method for discretization and the RK4 techniques for iteration to perform numerical simulations. The following three examples are numerically simulated for extinction, persistence, and optimal control of diseases.

Example 1. According to Theorem 2.2, we obtain a sufficient condition for disease extinction, which is $R_0^{jp} < 1$. As shown in Fig.1 & Fig.2, it can be intuitively seen that when $R_0^{jp} < 1$ is satisfied, the number of patients increases to a certain number and then gradually decreases until it approaches 0, which represents the gradual extinction of the disease.

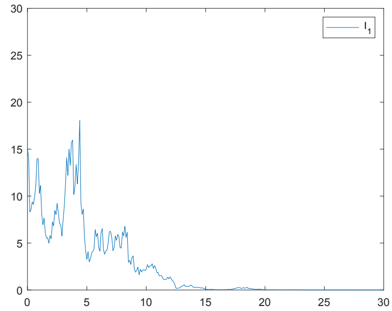

Fig. 1: $I_1(t)$-infected individuals in the A place

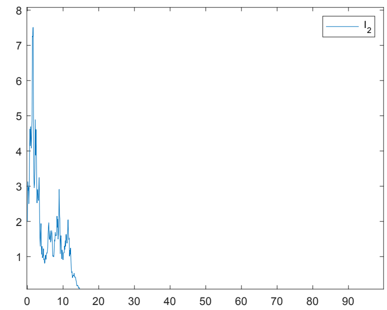

Fig. 2: $I_2(t)$-infected individuals in the B place

Example 2. According to Theorem 3.1, we obtain a sufficient condition for the persistence of disease, which is $R_0^p > 1$. As shown in Fig.3 & Fig.4, it can be intuitively seen that under the premise of meeting $R_0^p > 1$, although the number of patients in the image is decreasing, it does not approach 0, which indicates that the disease will always exist for a long time.

Example 3. As can be seen from Fig.5 & Fig.6, implementing appropriate control strategies after a disease outbreak can better control the spread of the disease than not implementing measures.

6 Conclusion

By incorporating urban transport correlation, media coverage, and time delays into the SVEIHR epidemic model, we can gain a more realistic understanding of disease transmission. In this paper, we use suitable Lyapunov functions to study the existence of global positive solutions and establish the conditions for disease extinction or persistence. At the same time, we use the stochastic optimal theory to establish the optimal control strategy. In addition, we

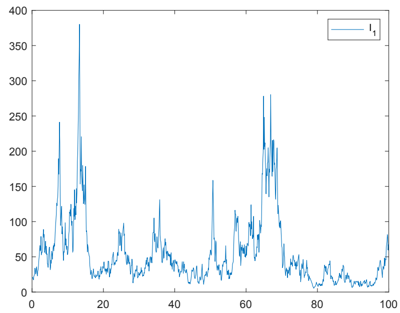

Fig. 3: $I_1(t)$-infected individuals in the A place

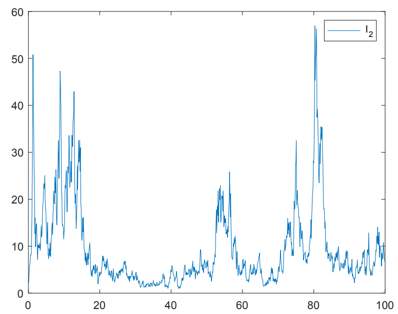

Fig. 4: $I_2(t)$-infected individuals in the B place

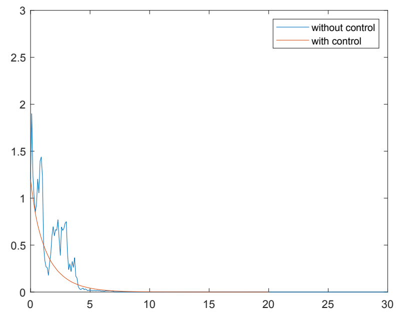

Fig. 5: $I_1(t)$-infected individuals in the A place

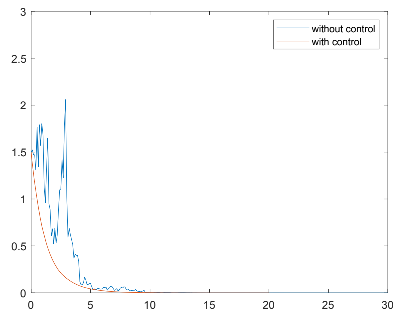

Fig. 6: $I_2(t)$-infected individuals in the B place

construct an efficient numerical format based on the Milstein method to support our results. Our analysis also highlights the impact of time delay on model dynamics, which provides an aspect of disease control that needs attention. According to our research, the movement of people between two cities can increase the spread of disease. Therefore, in the outbreak of infectious diseases, we should pay attention to the control of the flow of people, timely and rational use of the media, and do our best to quickly control the spread of the disease and protect people's physical and mental health.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Hong Qiu, instructed and checked the reasonableness and correctness of the article. Rujie Yang is responsible for the derivation of calculations, simulation design and the writing of articles.

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Conflict of Interest

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