Discontinuous Solutions for a non-local Regularization of the Short Pulse Equation

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Abstract: The propagation of ultra-short light pulses in silica optical fibers is modeled by the short pulse equation. We introduce a nonlocal regularization of the problem and study the existence of solutions in a class of possibly discontinuous functions.

Key-Words: Existence. Short pulse equation. Non-local formulation. Hyperbolc-elliptic system. Discontinuous solutions. Cauchy problem.

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1 Introduction

The short pulse equation reads

$$\partial_x \left(\partial_t u + q\partial_x u^3\right) = bu, \qquad q,\, b \in \mathbf{R}, \tag{1}$$

and has beed deduced in several context

- in [1] for the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media,
- in [2] for the propagation of ultra-short light pulses in silica optical fibers,
- in [3], [4], [5], [6], [7], [8] as non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons,
- in [9], [10], [11] for pseudospherical surfaces,
- in [12] for the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response,
- in [13], [14] in the context of plasma physics,
- in [15], [16] for the dynamics of radiating gases,
- in [17], [18], [19] for ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime.

The mathematical features of (1) have been widely studied

- the wellposedness of the Cauchy problem in the context of energy spaces can be found in [20], [21], [22],
- the wellposedness of the Cauchy problem in the context of entropy solution can be found in [23], [24], [25],
- the wellposedness of the homogeneous initial boundary value problem is in [26],
- the convergence of a finite difference numerical scheme is studied in [27].

Here we regularize (1) with the following nonlocal problem

$$\begin{cases} \partial_t u + q\partial_x v = bP, & t > 0,\, x \in \mathbf{R}, \\ \partial_x P = u, & t > 0,\, x \in \mathbf{R}, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^3, & t > 0,\, x \in \mathbf{R}, \\ P(t,0) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{2}$$

where $q,\, b,\, \alpha,\, \beta,\, \gamma,\, \kappa \in \mathbf{R}$.

Nonlocal regularizations are widely used for conservation laws

- in the context of traffic flow [28], [29], [30], [31],

- in the context of sedimentation [32],
- in the context of slow erosion [33], [34],
- in the context of the linearly polarized continuum spectrum pulses in optical waveguides [35], [36].

Coherently with [23], [24], [37],[38].

- on the initial datum we assume

$$u_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}), \quad \int_{\mathbf{R}} u_0(x)dx = 0; \quad (3)$$

- on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \quad (4)$$

we assume that

$$\begin{aligned} &\int_{\mathbf{R}} P_0(x)dx \\ &\quad = \int_{\mathbf{R}} \left(\int_{-\infty}^{x} u_0(y)dy\right) dx = 0, \\ &\|P_0\|^2_{L^2(\mathbf{R})} \\ &\quad = \int_{\mathbf{R}} \left(\int_{-\infty}^{x} u_0(y)dy\right)^2 dx < \infty; \end{aligned} \quad (5)$$

- on the constants q, b, α, β, γ, κ, we assume that

$$\frac{q\beta}{\kappa} \geq 0, \quad b = \frac{2q\kappa}{\gamma}, \quad \alpha,\, \beta,\, \kappa,\, \gamma \neq 0, \quad (6)$$

or

$$b = \frac{2q\kappa}{\gamma}, \quad \alpha = -\gamma, \quad \beta = 0, \quad \gamma \neq 0. \quad (7)$$

Since in (6) and (6) we assume $\alpha \neq 0$, it is not restrictive to set it equal to 1. The assumptions (6) and (7) are necessary to keep the solutions of (2) in the energy space.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume (3), (4), (5), and (6) or (7). There exists a distributional solution* $(u,\, v,\, P)$ *of (2) such that*

$$\begin{aligned} &u \in L^\infty((0,T)\times\mathbf{R}) \cap L^\infty(0,T;L^2(\mathbf{R})), \\ &v \in H^2((0,T)\times\mathbf{R}) \cap L^\infty(0,T;H^2(\mathbf{R}))\cap \\ &\qquad \cap W^{1,\infty}((0,T)\times\mathbf{R})\cap \\ &\qquad \cap L^\infty(0,T;W^{2,\infty}(\mathbf{R})), \\ &\partial_t u \in L^\infty(0,T;W^{1,\infty}(\mathbf{R}))\cap \\ &\qquad \cap L^\infty(0,T;H^1(\mathbf{R})), \\ &\partial_t\partial_x v \in L^\infty(0,T;L^\infty(\mathbf{R}))\cap \\ &\qquad \cap L^\infty(0,T;L^2(\mathbf{R})), \\ &P \in L^\infty((0,T)\times\mathbf{R}) \cap L^\infty(0,T;L^2(\mathbf{R})). \end{aligned} \quad (8)$$

for every $T > 0$.

The well-posedness of (2) was proved in [39] and [35] under the assumption:

$$u_0 \in L^1(\mathbf{R}) \cap H^2(\mathbf{R}), \quad (9)$$

and

$$u_0 \in L^1(\mathbf{R}) \cap H^1(\mathbf{R}), \quad (10)$$

respectively. Finally, we observe that the assumptions (3), (4) and (5) are the ones used in [23], [24] in order to prove the well-posedness of entropy solutions of (1).

The remaining part of the manuscript is organized as follows. Section 2 is dedicated to several a priori estimates on a vanishing viscosity approximation of (2). Those play a key role in the proof of our main result, that is given in Section 3.

2 Vanishing Viscosity Approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (2).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem, [40]:

$$\begin{cases} \partial_t u_\varepsilon + q\partial_x v_\varepsilon = bP_\varepsilon + \varepsilon\partial_x^2 u_\varepsilon, \\ \partial_x P_\varepsilon = u_\varepsilon, \\ \alpha\partial_x^2 v_\varepsilon + \beta\partial_x v_\varepsilon + \gamma v_\varepsilon = \kappa u_\varepsilon^3, \\ P_\varepsilon(t,0) = 0, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), \end{cases} \quad (11)$$

where $t > 0$, $x \in \mathbf{R}$ and $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &\|u_{\varepsilon,0}\|_{L^\infty(\mathbf{R})} \leq \|u_0\|_{L^\infty(\mathbf{R})}, \\ &\|u_{\varepsilon,0}\|_{L^2(\mathbf{R})} \leq \|u_0\|_{L^2(\mathbf{R})}, \\ &\int_{\mathbf{R}} u_{\varepsilon,0}(x)dx = 0, \\ &\|P_{\varepsilon,0}\|_{L^2(\mathbf{R})} \leq \|P_0\|_{L^2(\mathbf{R})}, \\ &\int_{\mathbf{R}} P_{\varepsilon,0}(x)dx = 0, \\ &\sqrt{\varepsilon}\,\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbf{R})} \\ &\qquad + \sqrt{\varepsilon}\,\|\partial_x^2 u_{\varepsilon,0}\|^2_{L^2(\mathbf{R})} \leq C_0 \\ &\varepsilon\,\|\partial_x^3 u_{\varepsilon,0}\|^2_{L^2(\mathbf{R})} \\ &\qquad + \varepsilon\sqrt{\varepsilon}\,\|\partial_x^4 u_{\varepsilon,0}\|^2_{L^2(\mathbf{R})} \leq C_0, \\ &\sqrt{\varepsilon}\,\|\partial_t\partial_x u_{\varepsilon,0}\|_{L^2(\mathbf{R})} \leq C_0, \\ &u_{\varepsilon,0} \to u_0(x) \text{ in } L^p_{loc}(\mathbf{R}) \text{ and a.e. in } \mathbf{R}, \end{aligned} \quad (12)$$

and C_0 is a constant independent on ε.

Let us prove some a priori estimates on u_ε, P_ε and v_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

Following [5, Lemma 2.1], we prove the following result.

Lemma 2.1 *For each $t > 0$, we have that*

$$\begin{aligned} &P_\varepsilon(t,-\infty) = P_\varepsilon(t,\infty) = 0,\\ &\int_{-\infty}^{0} u_\varepsilon(t,x)dx = \int_0^\infty u_\varepsilon(t,x)dx = 0,\\ &\int_{\mathbf{R}} u_\varepsilon(t,x)dx = 0. \end{aligned} \tag{13}$$

Remark 2.1 *In light of (13), we have that*

$$P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y)dy = \int_{-\infty}^x u_\varepsilon(t,y)dy. \tag{14}$$

Proof of Lemma 2.1. We begin by proving

$$P_\varepsilon(t,-\infty) = 0. \tag{15}$$

Thanks to the smoothness of u_ε, from the first equation of (11), we have

$$\lim_{x\to-\infty} (\partial_t u_\varepsilon + q\partial_x v_\varepsilon) = bP_\varepsilon(t,-\infty) = 0, \tag{16}$$

that is (15).

In a similar way, we can prove that

$$P_\varepsilon(t,\infty) = 0. \tag{17}$$

(15) and (17) give (13).

We prove (13). Integrating the second equation of (11) $(0,x)$, again by (11), we have

$$P_\varepsilon(t,x) = \int_0^x u_\varepsilon(t,y)dy. \tag{18}$$

(13) follows from (13) and (18).

Finally, we prove (13). We begin by observing that, by (13),

$$\int_{-\infty}^0 u_\varepsilon(t,x)dx = 0. \tag{19}$$

Therefore, by (13) and (19),

$$\begin{aligned} \int_{-\infty}^0 u_\varepsilon(t,x)dx \quad &+ \int_0^\infty u_\varepsilon(t,x)dx\\ &= \int_{\mathbf{R}} u_\varepsilon(t,x)dx = 0, \end{aligned} \tag{20}$$

that is (13). ♠

Following [35, Lemma 2.5], we have the following result.

Lemma 2.2 *For each $t \geq 0$, we have that*

$$\begin{aligned} &\int_0^{-\infty} P_\varepsilon(t,x)dx = -\frac{1}{b}\partial_t P_\varepsilon(t,0)\\ &\quad -\frac{q}{b}v_\varepsilon(t,0) + \frac{\varepsilon}{b}\partial_x u_\varepsilon(t,0),\\ &\int_0^\infty P_\varepsilon(t,x)d = -\frac{1}{b}\partial_t P_\varepsilon(t,0)x\\ &\quad -\frac{q}{b}v_\varepsilon(t,0) + \frac{\varepsilon}{b}\partial_x u_\varepsilon(t,0),\\ &\int_{\mathbf{R}} P_\varepsilon(t,x)dx = 0. \end{aligned} \tag{21}$$

Proof. We begin by proving (21). Integrating the first equation on $(0,x)$, we have that

$$\begin{aligned} &\int_0^x \partial_t u_\varepsilon(t,y)dy + qv_\varepsilon(t,x) - qv_\varepsilon(t,0)\\ &\quad -\varepsilon\partial_x u_\varepsilon(t,x) + \varepsilon\partial_x u_\varepsilon(t,0)\\ &= b\int_0^x P_\varepsilon(t,y)dy. \end{aligned} \tag{22}$$

By (13), we obtain that

$$\begin{aligned} &\frac{d}{dt}\int_0^{-\infty} u_\varepsilon(t,x)dx\\ &= \int_0^{-\infty} \partial_t u_\varepsilon(t,x)dx = 0. \end{aligned} \tag{23}$$

Moreover, the regularity of u_ε and v_ε give

$$\lim_{x\to-\infty} (qv_\varepsilon(t,x) - \varepsilon\partial_x u_\varepsilon(t,x)) = 0. \tag{24}$$

Therefore, by (22), (23) and (24),

$$b\int_0^x P_\varepsilon(t,y)dy = -qv_\varepsilon(t,0) + \varepsilon\partial_x u_\varepsilon(t,0), \tag{25}$$

which gives (21).

We prove (21). Observe that by (13),

$$\frac{d}{dt}\int_0^\infty u_\varepsilon(t,x)dx = \int_0^\infty \partial_t u_\varepsilon(t,x)dx = 0, \tag{26}$$

while, thanks to the regularity of u_ε, v_ε,

$$\lim_{x\to\infty} (qv_\varepsilon(t,x) - \varepsilon\partial_x u_\varepsilon(t,x)) = 0. \tag{27}$$

(21) follows from (22), (26) and (27).

Finally, (21) and (21) give (21). ♠

Arguing as in [39, Lemma 2.2], we have the following result.

Lemma 2.3 *We have that*

$$\begin{aligned} &\int_{\mathbf{R}} u_\varepsilon^3 \partial_x v_\varepsilon dx\\ &= \begin{cases} \dfrac{\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}, & \text{if (6) holds,}\\ 0, & \text{if (7) holds.} \end{cases} \end{aligned} \tag{28}$$

We continue with some L^2 type estimates of the solution.

Lemma 2.4 *Let $T > 0$. If (6) or (7) hold*

$$\begin{aligned} &\|u_\varepsilon(t,\cdot)\|_{L^4(\mathbf{R})} \leq C(T),\\ &\|P_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})} \leq C(T),\\ &\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})} \leq C(T),\\ &\varepsilon\int_0^t \|(u_\varepsilon\partial_x u_\varepsilon)(s,\cdot)\|^2_{L^2(\mathbf{R})} ds \leq C(T),\\ &\varepsilon\int_0^t \|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbf{R})} ds \leq C(T),\\ &\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbf{R})} ds \leq C(T), \end{aligned} \tag{29}$$

for every $0 \le t \le T$. *In particular, if (6) holds, we have*

$$\begin{aligned}&\int_0^t \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbf{R})}\, ds \le C(T),\\&\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})} \le C(T),\end{aligned} \tag{30}$$

for every $0 \le t \le T$.

Proof. We begin by observing that, thanks to (21), we can consider the following function:

$$F_\varepsilon(t,x) = \int_{-\infty}^x P_\varepsilon(t,y)dy. \tag{31}$$

Integrating the second equation of (11) on $(-\infty, x)$, thanks (31) and Remark 2.1, we have the following equation:

$$\begin{aligned}&\partial_t P_\varepsilon(t,x) + qv_\varepsilon(t,x)\\&\quad = bF_\varepsilon(t,x) + \varepsilon\partial_x u_\varepsilon(t,x).\end{aligned} \tag{32}$$

Therefore, arguing as in [39, Lemma 2.3], we have (29), (30) and (30). ♠

A key role in our compactness argument is played by the following a priori estimates.

Lemma 2.5 *Assume (6) or (7). Let* $T > 0$. *We have that*

$$\begin{aligned}&\|\partial_x v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})} \le C(T),\\&, \|\partial_x v_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})} \le C(T),\\&\|v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})}, \|v_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})} \le C(T),\end{aligned} \tag{33}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (29) and the Young inequality, we have that

$$\kappa u^3_\varepsilon(t,\cdot) \in L^1(\mathbf{R}), \quad 0 \le t \le T. \tag{34}$$

Therefore, by [35, Lemma 2.1], (33) holds. ♠

Arguing as in [5, Lemma 2.6] and [35, Lemma 2.8], we have the following result.

Lemma 2.6 *Assume (6) or (7).We have that*

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})} \le C(T),\\&\left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \le C(T),\\&\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})} \le C(T),\end{aligned} \tag{35}$$

for every $0 \le t \le T$.

In the next lemma we prove an H^1 energy estimate.

Lemma 2.7 *Assume (6) or (7). Fix* $T > 0$. *We have that*

$$\begin{aligned}&\sqrt{\varepsilon}\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad +2\varepsilon\sqrt{\varepsilon}e^t\int_0^t e^{-s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})} ds\\&\quad \le C(T),\end{aligned} \tag{36}$$

for every $0 \le t \le T$.

Proof. Multiplying the first equation of (11) by $-2\sqrt{\varepsilon}\partial_x^2 u_\varepsilon$, an integration on $\mathbf{R}$ gives

$$\begin{aligned}&\sqrt{\varepsilon}\tfrac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad +2\varepsilon\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&= 2\sqrt{\varepsilon}b\int_{\mathbf{R}} P_\varepsilon\partial_x^2 u_\varepsilon dx\\&\quad -2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x v_\varepsilon dx.\end{aligned} \tag{37}$$

Observe that, by (11) and (13),

$$\begin{aligned}&2b\int_{\mathbf{R}} P_\varepsilon\partial_x^2 u_\varepsilon d\\&= -2b\int_{\mathbf{R}}\partial_x P_\varepsilon\partial_x u_\varepsilon dx\\&= -2b\int_{\mathbf{R}} u_\varepsilon\partial_x u_\varepsilon dx = 0.\end{aligned} \tag{38}$$

Moreover,

$$\begin{aligned}&2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x v_\varepsilon dx\\&= -2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x u_\varepsilon\partial_x^2 v_\varepsilon dx.\end{aligned} \tag{39}$$

Consequently, by (37),

$$\begin{aligned}&\sqrt{\varepsilon}\tfrac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad +2\varepsilon\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&= -2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x u_\varepsilon\partial_x^2 v_\varepsilon dx.\end{aligned} \tag{40}$$

Since $0 < \varepsilon < 1$, thanks to (35) and the Young inequality,

$$\begin{aligned}&2\sqrt{\varepsilon}|q|\int_{\mathbf{R}}|\partial_x u_\varepsilon||\partial_x v_\varepsilon|dx\\&= 2\sqrt{\varepsilon}\int_{\mathbf{R}}|\partial_x u_\varepsilon||q\partial_x^2 v_\varepsilon|dx\\&\le \sqrt{\varepsilon}\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad +\sqrt{\varepsilon}q^2\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le \sqrt{\varepsilon}\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad +q^2\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le \sqrt{\varepsilon}\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})} + C(T).\end{aligned} \tag{41}$$

Therefore, by (40),

$$\begin{aligned}&\sqrt{\varepsilon}\tfrac{d}{dt}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\varepsilon\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le\sqrt{\varepsilon}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{42}$$

The Gronwall Lemma and (12) give

$$\begin{aligned}&\sqrt{\varepsilon}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\sqrt{\varepsilon}e^t\int_0^t e^{-s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\&\le C_0e^t+C(T)e^t\int_0^t e^{-s}ds\le C(T),\end{aligned}\tag{43}$$

which gives (36). ♠

The following lemma gives an estimate on the blow-up of the H^3 norm of the solution.

Lemma 2.8 *Assume (6) or (7). We have that*

$$\sqrt{\varepsilon}\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C(T),\tag{44}$$

for every $0\le t\le T$.

Proof. Differentiating the third equation of (11) with respect to x, we have

$$\alpha\partial_x^3 v_\varepsilon+\beta\partial_x^2 v_\varepsilon+\gamma\partial_x v_\varepsilon=3\kappa u_\varepsilon^2\partial_x u_\varepsilon.\tag{45}$$

Since

$$\begin{aligned}&u_\varepsilon(t,\pm\infty)=\partial_x u_\varepsilon(t,\pm\infty)\\&=v_\varepsilon(t,\pm\infty)=\partial_x v_\varepsilon(t,\pm\infty)\\&=\partial_x^2 v_\varepsilon(t,\pm\infty)=0,\end{aligned}\tag{46}$$

then

$$\partial_x^3 v_\varepsilon(t,\pm\infty)=0.\tag{47}$$

Multiplying (46) by $2\alpha\varepsilon\partial_x^3 v_\varepsilon$ an integration on **R** of (45 gives

$$\begin{aligned}&2\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=6\sqrt{\varepsilon}\alpha\kappa\int_{\mathbf{R}}u_\varepsilon^2\partial_x u_\varepsilon\partial_x^3 v_\varepsilon dx\\&\quad-2\sqrt{\varepsilon}\alpha\beta\int_{\mathbf{R}}\partial_x^2 v_\varepsilon\partial_x^3 v_\varepsilon dx\\&\quad-2\sqrt{\varepsilon}\alpha\gamma\int_{\mathbf{R}}\partial_x v_\varepsilon\partial_x^3 v_\varepsilon dx.\end{aligned}\tag{48}$$

Observe that, by (47),

$$\begin{aligned}&-2\sqrt{\varepsilon}\alpha\beta\int_{\mathbf{R}}\partial_x^2 v_\varepsilon\partial_x^3 v_\varepsilon dx\\&=\sqrt{\varepsilon}\alpha\beta\int_{\mathbf{R}}\partial_x(\partial_x^2 v_\varepsilon)^2=0,\end{aligned}\tag{49}$$

and

$$\begin{aligned}&-2\sqrt{\varepsilon}\alpha\gamma\int_{\mathbf{R}}\partial_x v_\varepsilon\partial_x^3 v_\varepsilon dx\\&=2\sqrt{\varepsilon}\alpha\gamma\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.\end{aligned}\tag{50}$$

Consequently, since $0<\varepsilon<1$, by (35) and (48),

$$\begin{aligned}&2\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 6\sqrt{\varepsilon}|\alpha||\kappa|\int_{\mathbf{R}}u_\varepsilon^2|\partial_x u_\varepsilon||\partial_x^3 v_\varepsilon|dx\\&\quad+2\sqrt{\varepsilon}|\gamma\alpha|\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 6\sqrt{\varepsilon}|\alpha||\kappa|\int_{\mathbf{R}}u_\varepsilon^2|\partial_x u_\varepsilon||\partial_x^3 v_\varepsilon|dx\\&\quad+C(T).\end{aligned}\tag{51}$$

Due to (35), (36) and the Young inequality,

$$\begin{aligned}&6\sqrt{\varepsilon}|\alpha||\kappa|\int_{\mathbf{R}}u_\varepsilon^2|\partial_x u_\varepsilon||\partial_x^3 v_\varepsilon|dx\\&=2\sqrt{\varepsilon}\int_{\mathbf{R}}|3\kappa u_\varepsilon^2\partial_x u_\varepsilon||\alpha\partial_x^3 v_\varepsilon|dx\\&\le 9\sqrt{\varepsilon}\kappa^2\int_{\mathbf{R}}u_\varepsilon^4(\partial_x u_\varepsilon)^2dx\\&\quad+\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 9\sqrt{\varepsilon}\kappa^2\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbf{R})}\times\\&\quad\times\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le C(T)+\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.\end{aligned}\tag{52}$$

It follows from (51) that

$$\sqrt{\varepsilon}\alpha^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C(T),\tag{53}$$

which gives (44). ♠

In the next lemma we prove an H^2 energy estimate.

Lemma 2.9 *Assume (6) or (7). Fix* $T>0$. *We have that*

$$\begin{aligned}&\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\sqrt{\varepsilon}e^t\int_0^t e^{-s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\&\le C(T),\end{aligned}\tag{54}$$

and

$$\sqrt[4]{\varepsilon}\left\|\partial_x u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\le C(T),\tag{55}$$

for every $0\le t\le T$.

Proof. Multiplying the first equation of (11) by

$2\sqrt{\varepsilon}\partial_x^4 u_\varepsilon$, it follows from integration on **R** that

$$\begin{aligned}&\sqrt{\varepsilon}\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\sqrt{\varepsilon}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=2\sqrt{\varepsilon}b\int_{\mathbf{R}}P_\varepsilon\partial_x^4 u_\varepsilon dx\\&\quad-2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_x v_\varepsilon dx.\end{aligned}\tag{56}$$

Observe that, by (11) and (13),

$$\begin{aligned}&2b\sqrt{\varepsilon}\int_{\mathbf{R}}P_\varepsilon\partial_x^4 u_\varepsilon dx\\&=-2b\int_{\mathbf{R}}\partial_x P_\varepsilon\partial_x^3 u_\varepsilon dx\\&=-2\sqrt{\varepsilon}\int_{\mathbf{R}}u_\varepsilon\partial_x^3 u_\varepsilon dx\\&=2b\int_{\mathbf{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx=0,\end{aligned}\tag{57}$$

and

$$\begin{aligned}&-2q\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_x v_\varepsilon dx\\&=2\sqrt{\varepsilon}q\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_x^2 v_\varepsilon dx\\&=-2q\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^3 v_\varepsilon dx.\end{aligned}\tag{58}$$

It follows from (44), (56) and the Young inequality that

$$\begin{aligned}&\sqrt{\varepsilon}\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\sqrt{\varepsilon}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=-2\sqrt{\varepsilon}q\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^3 v_\varepsilon dx\\&\le 2\sqrt{\varepsilon}|q|\int_{\mathbf{R}}|\partial_x^2 u_\varepsilon||\partial_x^3 v_\varepsilon|dx\\&\le\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+q^2\sqrt{\varepsilon}\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{59}$$

The Gronwall Lemma and (12) give

$$\begin{aligned}&\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\sqrt{\varepsilon}e^t\int_0^t e^{-s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\&\le C_0+C(T)e^t\int_0^t e^{-s}ds\le C(T),\end{aligned}\tag{60}$$

which gives (54).

Finally, we prove (55). Thanks to (36), (54) and the Hölder inequality,

$$\begin{aligned}&(\partial_x u_\varepsilon(t,x))^2\\&=2\int_{-\infty}^x\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dy\\&\le 2\int_{\mathbf{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|dx\\&\le 2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\\&\le\frac{C(T)}{\sqrt{\varepsilon}}.\end{aligned}\tag{61}$$

Hence,

$$\sqrt{\varepsilon}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\le C(T),\tag{62}$$

which gives (55). ♠

The following lemma gives a bound on the time derivative of the solution.

Lemma 2.10 *Assume (6) or (7). We have that*

$$\left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C(T),\tag{63}$$

for every $0\le t\le T$.

Proof. Multiplying the first equation of (11) by $2\partial_t u_\varepsilon$, an integration on **R** gives

$$\begin{aligned}&2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=2b\int_{\mathbf{R}}\partial_t u_\varepsilon P_\varepsilon dx\\&\quad+2\varepsilon\int_{\mathbf{R}}\partial_t u_\varepsilon\partial_x^2 u_\varepsilon dx\\&\quad-2q\int_{\mathbf{R}}\partial_t u_\varepsilon\partial_x v_\varepsilon dx.\end{aligned}\tag{64}$$

Since $0<\varepsilon<1$, thanks to (29), (33), (36) and the Young inequality,

$$\begin{aligned}&2|b|\int_{\mathbf{R}}|\partial_t u_\varepsilon||P_\varepsilon|dx\\&=\int_{\mathbf{R}}|\partial_t u_\varepsilon||2bP_\varepsilon|dx\\&\le\frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2b^2\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le\frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T),\end{aligned}\tag{65}$$

and

$$\begin{aligned}&2\varepsilon\int_{\mathbf{R}}|\partial_t u_\varepsilon||\partial_x^2 u_\varepsilon|dx\\&=\int_{\mathbf{R}}|\partial_t u_\varepsilon||2\varepsilon\partial_x^2 u_\varepsilon|dx\\&\le\frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\frac{\varepsilon^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le\frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\frac{\sqrt{\varepsilon}}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le\frac{1}{2}\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T),\end{aligned}\tag{66}$$

and

$$\begin{aligned}&2|q|\int_{\mathbf{R}}|\partial_t u_\varepsilon||\partial_x v_\varepsilon|dx\\&=\int_{\mathbf{R}}|\partial_t u_\varepsilon||2q\partial_x v_\varepsilon|dx\\&\le\tfrac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\quad+2q^2\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\\&\le\tfrac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{67}$$

Therefore, by (64), we have that

$$\frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\le C(T),\tag{68}$$

which gives (63). ♠

We continue with some estimates of the high order derivatives of mixed type.

Lemma 2.11 *Assume (6) or (7). We have that*

$$\begin{aligned}&\|\partial_t\partial_x v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})}\le C(T),\\&\|\partial_t\partial_x v_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})}\le C(T),\\&\|\partial_t v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})}\le C(T),\\&\|\partial_t v_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})}\le C(T),\end{aligned}\tag{69}$$

for every $0\le t\le T$.

Proof. Differentiating the third equation of (11) with respect to t, we have that

$$\alpha\partial_t\partial_x^2 v_\varepsilon+\beta\partial_t\partial_x v_\varepsilon+\gamma\partial_t v_\varepsilon=3\kappa u_\varepsilon^2\partial_t u_\varepsilon.\tag{70}$$

We begin by observing that, thanks to (29), (63) and the Young inequality, we have that

$$\left\|3\kappa u_\varepsilon^2(t,\cdot)\partial_t u_\varepsilon(t,\cdot)\right\|_{L^1(\mathbf{R})}\le C(T),\tag{71}$$

for every $0\le t\le T$. Therefore, by [35, Lemma 2.1], (69) holds. ♠

We continue with blow-up rate of the H^4 norm.

Lemma 2.12 *Assume (6) or (7). We have that*

$$\varepsilon\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\le C(T),\tag{72}$$

for every $0\le t\le T$.

Proof. Differentiating (45) with respect to x, we have that

$$\begin{aligned}&\alpha\partial_x^4 v_\varepsilon+\beta\partial_x^3 v_\varepsilon+\gamma\partial_x^2 v_\varepsilon\\&=6\kappa u_\varepsilon(\partial_x u_\varepsilon)^2+3\kappa u_\varepsilon^2\partial_x^2 u_\varepsilon.\end{aligned}\tag{73}$$

Observe that, since

$$\partial_x^2 u_\varepsilon(t,\pm\infty)=0,\tag{74}$$

by (46) and (47),

$$\partial_x^4 v_\varepsilon(t,\pm\infty)=0.\tag{75}$$

Multiplying (73 by $2\varepsilon\alpha\partial_x^4 v_\varepsilon$, an integration on $\mathbf{R}$ gives

$$\begin{aligned}&2\alpha^2\varepsilon\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=12\alpha\kappa\varepsilon\int_{\mathbf{R}}u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 v_\varepsilon dx\\&\quad+6\alpha\kappa\varepsilon\int_{\mathbf{R}}u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 v_\varepsilon dx\\&\quad-2\alpha\beta\varepsilon\int_{\mathbf{R}}\partial_x^3 v_\varepsilon\partial_x^4 v_\varepsilon dx\\&\quad-2\alpha\gamma\varepsilon\int_{\mathbf{R}}\partial_x^2 v_\varepsilon\partial_x^4 v_\varepsilon dx.\end{aligned}\tag{76}$$

Observe that, by (47) and (75),

$$\begin{aligned}&-2\alpha\beta\varepsilon\int_{\mathbf{R}}\partial_x^3 v_\varepsilon\partial_x^4 v_\varepsilon dx\\&=-\alpha\beta\varepsilon\int_{\mathbf{R}}\partial_x((\partial_x^3 v_\varepsilon)^2dx=0,\end{aligned}\tag{77}$$

and

$$\begin{aligned}&-2\alpha\gamma\varepsilon\int_{\mathbf{R}}\partial_x^2 v_\varepsilon\partial_x^4 v_\varepsilon dx\\&=2\alpha\gamma\varepsilon\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.\end{aligned}\tag{78}$$

Consequently, since $0<\varepsilon<1$, by (44) and (76),

$$\begin{aligned}&2\alpha^2\varepsilon\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 12|\alpha\kappa|\varepsilon\int_{\mathbf{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|dx\\&\quad+6|\alpha\kappa|\varepsilon\int_{\mathbf{R}}u_\varepsilon^2|\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx\\&\quad+2|\alpha\gamma|\varepsilon\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 12|\alpha\kappa|\varepsilon\int_{\mathbf{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|dx\\&\quad+6|\alpha\kappa|\varepsilon\int_{\mathbf{R}}u_\varepsilon^2|\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx\\&\quad+2|\alpha\gamma|\sqrt{\varepsilon}\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\le 12|\alpha\kappa|\varepsilon\int_{\mathbf{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|dx\\&\quad+6|\alpha\kappa|\varepsilon\int_{\mathbf{R}}u_\varepsilon^2|\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx\\&\quad+C(T).\end{aligned}\tag{79}$$

Since $0<\varepsilon<1$, by (35), (36), (54), (55) and the

Young inequality,

$$\begin{aligned}
&12|\alpha\kappa|\varepsilon\int_{\mathbf{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 v_\varepsilon|\\
&=\varepsilon\int_{\mathbf{R}}|12\kappa u_\varepsilon(\partial_x u_\varepsilon)^2||\alpha\partial_x^4 v_\varepsilon|dx\\
&\le 72\kappa^2\varepsilon\int_{\mathbf{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^4dx\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le 72\kappa^2\varepsilon\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\int_{\mathbf{R}}(\partial_x u_\varepsilon)^4dx\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)\varepsilon\int_{\mathbf{R}}(\partial_x u_\varepsilon)^4dx\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)\varepsilon\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\times\\
&\quad\times\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{80}$$

and

$$\begin{aligned}
&6|\alpha\kappa|\varepsilon\int_{\mathbf{R}}u_\varepsilon^2|\partial_x^2 u_\varepsilon||\partial_x^4 v_\varepsilon|dx\\
&=\varepsilon\int_{\mathbf{R}}|6\kappa u_\varepsilon^2\partial_x^2 u_\varepsilon||\alpha\partial_x^4 v_\varepsilon|dx\\
&\le 18\kappa^2\varepsilon\int_{\mathbf{R}}u_\varepsilon^4(\partial_x^2 u_\varepsilon)^2dx\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le 18\kappa^2\sqrt{\varepsilon}\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbf{R})}\times\\
&\quad\times\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)+\tfrac{\alpha^2\varepsilon}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{81}$$

It follows from (79) that

$$\alpha^2\varepsilon\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C(T),\tag{82}$$

which gives (72). ♠

In the next lemma we prove an H^3 energy estimate.

Lemma 2.13 *Assume (6) or (7). We have that*

$$\begin{aligned}
&\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+2\varepsilon^2 e^t\int_{\mathbf{R}}e^{-s}\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\le C(T),
\end{aligned}\tag{83}$$

and

$$\sqrt[8]{\varepsilon^3}\left\|\partial_x^3 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\le C(T),\tag{84}$$

for every $0\le t\le T$.

Proof. Multiplying the first equation of (11) by $-2\varepsilon\partial_x^6 u_\varepsilon$, we have that

$$\begin{aligned}
&-2\varepsilon\partial_x^6 u_\varepsilon\partial_t u_\varepsilon\\
&=-2b\varepsilon P_\varepsilon\partial_x^6 u_\varepsilon-2\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon\\
&\quad-2q\varepsilon\partial_x^6 u_\varepsilon\partial_x v_\varepsilon.
\end{aligned}\tag{85}$$

Observe that, thanks the second equation of (11) and (13),

$$\begin{aligned}
&-2b\varepsilon\int_{\mathbf{R}}P_\varepsilon\partial_x^6 u_\varepsilon dx\\
&=2b\varepsilon\int_{\mathbf{R}}\partial_x P_\varepsilon\partial_x^5 u_\varepsilon dx\\
&=2b\varepsilon\int_{\mathbf{R}}u_\varepsilon\partial_x^5 u_\varepsilon dx\\
&=-2b\varepsilon\int_{\mathbf{R}}\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx\\
&=2b\varepsilon\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon=0.
\end{aligned}\tag{86}$$

Moveover,

$$\begin{aligned}
&-2\varepsilon\int_{\mathbf{R}}\partial_x^6 u_\varepsilon\partial_t u_\varepsilon dx\\
&=2\varepsilon\int_{\mathbf{R}}\partial_x^5 u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&=-2\varepsilon\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\
&=2\varepsilon\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_t\partial_x^3 u_\varepsilon dx\\
&=\varepsilon\tfrac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{87}$$

and

$$\begin{aligned}
&-2\varepsilon^2\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&=2\varepsilon^2\int_{\mathbf{R}}\partial_x^5 u_\varepsilon\partial_x^3 u_\varepsilon\\
&=-2\varepsilon^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{88}$$

and

$$\begin{aligned}
&-2q\varepsilon\int_{\mathbf{R}}\partial_x^6 u_\varepsilon\partial_x v_\varepsilon dx\\
&=2q\varepsilon\int_{\mathbf{R}}\partial_x^5 u_\varepsilon\partial_x^2 v_\varepsilon dx\\
&=-2q\varepsilon\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_x^3 v_\varepsilon dx\\
&=2q\varepsilon\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_x^4 v_\varepsilon dx.
\end{aligned}\tag{89}$$

It follows from (86), (88) and an integration of (85) on **R** that

$$\begin{aligned}
&\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+2\varepsilon^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&=2q\varepsilon\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_x^4 v_\varepsilon dx.
\end{aligned}\tag{90}$$

Due to (72) and the Young inequality,

$$\begin{aligned} &2|q|\varepsilon \int_{\mathbf{R}} |\partial_x^3 u_\varepsilon||\partial_x^4 v_\varepsilon| dx \\ &\le \varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\quad + q^2\varepsilon \left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\le \varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + C(T). \end{aligned} \tag{91}$$

Therefore, by (90),

$$\begin{aligned} &\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\quad + 2\varepsilon^2 \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\le \varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} + C(T). \end{aligned} \tag{92}$$

The Gronwall Lemma and (12) give (83).

Finally, we prove (84). Thanks to (54), (83) and the Hölder inequality,

$$\begin{aligned} &(\partial_x^2 u_\varepsilon(t,x))^2 \\ &= 2\int_{-\infty}^{x} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dy \\ &\le 2\int_{\mathbf{R}} |\partial_x^2 u_\varepsilon||\partial_x^3| dx \\ &\le \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\le \frac{C(T)}{\sqrt[4]{\varepsilon^3}}. \end{aligned} \tag{93}$$

Hence,

$$\sqrt[4]{\varepsilon^3} \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbf{R})} \le C(T), \tag{94}$$

which gives (84). ♠

We prove an uniform L^∞ bound on the time derivative.

Lemma 2.14 *Assume (6) or (7). We have that*

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})} \le C(T). \tag{95}$$

Proof. By the first equation of (11), (30) and (33), we have

$$\begin{aligned} &|\partial_t u_\varepsilon| \\ &= |bP_\varepsilon - q\partial_x v_\varepsilon + \varepsilon\partial_x^2 u_\varepsilon| \\ &\le |b||P_\varepsilon| + |q| + \varepsilon|\partial_x^2 u_\varepsilon| \\ &\le |b| \left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \\ &\quad + |q| \left\|\partial_x v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \\ &\quad + \varepsilon \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \\ &\le C(T) + \varepsilon \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}. \end{aligned} \tag{96}$$

Since $0 < \varepsilon < 1$, thanks to (84),

$$\begin{aligned} &\varepsilon \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \\ &= \sqrt[8]{\varepsilon^5}\sqrt[8]{\varepsilon^3} \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \\ &\le \sqrt[8]{\varepsilon^3} \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})} \le C(T). \end{aligned} \tag{97}$$

It follows from (96) and (97) that

$$|\partial_t u_\varepsilon| \le C(T), \tag{98}$$

which gives (95). ♠

We prove an uniform L^2 bound on the mixed time-space second derivative.

Lemma 2.15 *Assume (6) or (7). We have that*

$$\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbf{R})} \le C(T), \tag{99}$$

for every $0 \le t \le T$.

Proof. Multiplying the first equation of (11) by $-2\partial_t\partial_x^2 u_\varepsilon$, we have that

$$\begin{aligned} &-2\partial_t\partial_x^2 u_\varepsilon\partial_t u_\varepsilon \\ &= -2b\partial_t\partial_x^2 u_\varepsilon P_\varepsilon + 2q\partial_t\partial_x^2 u_\varepsilon\partial_x v_\varepsilon \\ &\quad -2\varepsilon\partial_t\partial_x^2\partial_x^2 u_\varepsilon. \end{aligned} \tag{100}$$

Observe that by the second equation of (11),

$$\begin{aligned} &-2b\int_{\mathbf{R}} \partial_t\partial_x^2 u_\varepsilon P_\varepsilon dx \\ &= 2b\int_{\mathbf{R}} \partial_x P_\varepsilon \partial_t\partial_x u_\varepsilon dx \\ &= 2b\int_{\mathbf{R}} u_\varepsilon\partial_t\partial_x u_\varepsilon dx. \end{aligned} \tag{101}$$

Moreover,

$$\begin{aligned} &-2\int_{\mathbf{R}} \partial_t\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx \\ &= 2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}, \\ &2q\int_{\mathbf{R}} \partial_t\partial_x^2 u_\varepsilon\partial_x v_\varepsilon dx \\ &= -2q\int_{\mathbf{R}} \partial_t\partial_x u_\varepsilon\partial_x^2 v_\varepsilon dx, \\ &-2\varepsilon\int_{\mathbf{R}} \partial_t\partial_x^2 u_\varepsilon\partial_x^2 u_\varepsilon dx \\ &= 2\varepsilon\int_{\mathbf{R}} \partial_t\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx. \end{aligned} \tag{102}$$

An integration of (100) on **R**, (101) and (102) give

$$\begin{aligned} &2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &= 2b\int_{\mathbf{R}} u_\varepsilon\partial_t\partial_x u_\varepsilon dx \\ &\quad -2q\int_{\mathbf{R}} \partial_t\partial_x u_\varepsilon\partial_x^2 v_\varepsilon dx \\ &\quad +2\varepsilon\int_{\mathbf{R}} \partial_t\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx. \end{aligned} \tag{103}$$

Since $0 < \varepsilon < 1$, due to (29), (35), (83) and the Young inequality,

$$\begin{aligned} &2|b|\int_{\mathbf{R}} |u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx \\ &\le b^2 \left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\quad + \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})} \\ &\le C(T) + \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}, \end{aligned} \tag{104}$$

and

$$\begin{aligned}&2\varepsilon\int_{\mathbf{R}}\left|\partial_t\partial_x u_\varepsilon\right|\left|\partial_x^3 u_\varepsilon\right|dx\\&=\int_{\mathbf{R}}\left|\partial_t\partial_x u_\varepsilon\right|\left|2\varepsilon\partial_x^3 u_\varepsilon\right|dx\\&\leq\tfrac{1}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\tfrac{1}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\tfrac{1}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T),\end{aligned}\tag{105}$$

and

$$\begin{aligned}&2|q|\int_{\mathbf{R}}\left|\partial_t\partial_x u_\varepsilon\right|\left|\partial_x^2 v_\varepsilon\right|dx\\&=\int_{\mathbf{R}}\left|\frac{\sqrt{2}\partial_t\partial_x^2 u_\varepsilon}{\sqrt{3}}\right|\left|\sqrt{6}q\partial_x^2 v_\varepsilon dx\right|dx\\&\leq\tfrac{1}{3}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+3q^2\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\tfrac{1}{3}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{106}$$

Therefore, by (103),

$$\frac{1}{6}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\leq C(T),\tag{107}$$

which gives (99). ♠

We continue with the blow-up rate of the H^5 norm of the solution.

Lemma 2.16 *Assume (6) or (7). We have that*

$$\varepsilon\sqrt{\varepsilon}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\leq C(T),\tag{108}$$

for every $0\leq t\leq T$.

Proof. Differentiating (73) with respect to x, we have

$$\begin{aligned}&\alpha\partial_x^5 v_\varepsilon+\beta\partial_x^4 v_\varepsilon+\gamma\partial_x^3 v_\varepsilon\\&=6\kappa(\partial_x u_\varepsilon)^3+18\kappa u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\\&\quad+3\kappa u_\varepsilon^2\partial_x^3 u_\varepsilon.\end{aligned}\tag{109}$$

Observe that, since $\partial_x^3 u_\varepsilon(t,\pm)=0$, by (46), (47), (74) and (75),

$$\partial_x^5 v_\varepsilon(t,\pm\infty)=0.\tag{110}$$

Multiplying (109) by $2\varepsilon\sqrt{\varepsilon}\alpha\partial_x^5 v_\varepsilon$, an integration on $\mathbf{R}$ gives

$$\begin{aligned}&2\varepsilon\sqrt{\varepsilon}\alpha^2\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=12\varepsilon\sqrt{\varepsilon}\alpha\kappa\int_{\mathbf{R}}(\partial_x u_\varepsilon)^3\partial_x^5 v_\varepsilon dx\\&\quad+32\varepsilon\sqrt{\varepsilon}\alpha\kappa\int_{\mathbf{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 v_\varepsilon dx\\&\quad+6\varepsilon\sqrt{\varepsilon}\alpha\kappa\int_{\mathbf{R}}u_\varepsilon^2\partial_x^3 u_\varepsilon\partial_x^5 v_\varepsilon dx\\&\quad-2\varepsilon\sqrt{\varepsilon}\alpha\beta\int_{\mathbf{R}}\partial_x^4 v_\varepsilon\partial_x^5 v_\varepsilon dx\\&\quad-2\varepsilon\sqrt{\varepsilon}\alpha\gamma\int_{\mathbf{R}}\partial_x^3 v_\varepsilon\partial_x^5 v_\varepsilon dx.\end{aligned}\tag{111}$$

Since $0<\varepsilon<1$, thanks to (47), (75) and (110),

$$\begin{aligned}&-2\varepsilon\sqrt{\varepsilon}\alpha\beta\int_{\mathbf{R}}\partial_x^4 v_\varepsilon\partial_x^5 v_\varepsilon dx\\&=-\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x((\partial_x^4 v_\varepsilon))^2dx=0,\end{aligned}\tag{112}$$

and

$$\begin{aligned}&-2\varepsilon\sqrt{\varepsilon}\alpha\gamma\int_{\mathbf{R}}\partial_x^3 v_\varepsilon\partial_x^5 v_\varepsilon dx\\&=2\varepsilon\sqrt{\varepsilon}\alpha\gamma\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq 2\varepsilon|\alpha\gamma|\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.\end{aligned}\tag{113}$$

Therefore, by (111),

$$\begin{aligned}&2\varepsilon\sqrt{\varepsilon}\alpha^2\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq 12\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|\partial_x u_\varepsilon|^3|\partial_x^5 v_\varepsilon|dx\\&\quad+32\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon|\times\\&\quad|\partial_x^5 v_\varepsilon|dx\\&\quad+6\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|u_\varepsilon^2\partial_x^3 u_\varepsilon||\partial_x^5 v_\varepsilon|dx\\&\quad+C(T).\end{aligned}\tag{114}$$

Since $0<\varepsilon<1$, due to (35), (36), (54), (55), (83) and the Young inequality,

$$\begin{aligned}&12\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|\partial_x u_\varepsilon|^3|\partial_x^5 v_\varepsilon|dx\\&=\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}|12\kappa(\partial_x u_\varepsilon)^3||\alpha\partial_x^5 v_\varepsilon|dx\\&\leq 72\kappa^2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}(\partial_x u_\varepsilon)^6dx\\&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq 72\kappa^2\varepsilon\sqrt{\varepsilon}\left\|\partial_x u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbf{R})}\times\\&\quad\times\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\sqrt{\varepsilon}C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq C(T)+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},\end{aligned}\tag{115}$$

and

$$\begin{aligned}
&32\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^5 v_\varepsilon|dx\\
&=\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}|32\kappa u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon||\alpha\partial_x^5 v_\varepsilon|dx\\
&\le 512\kappa^2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2dx\\
&+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le 512\kappa^2\varepsilon\sqrt{\varepsilon}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\times\\
&\quad\times\int_{\mathbf{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2dx\\
&+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)\varepsilon\sqrt{\varepsilon}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\times\\
&\quad\times\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{116}$$

and

$$\begin{aligned}
&6\varepsilon\sqrt{\varepsilon}|\alpha\kappa|\int_{\mathbf{R}}|u_\varepsilon^2\partial_x^3 u_\varepsilon||\partial_x^5 v_\varepsilon|dx\\
&=\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}|6\kappa u_\varepsilon^2\partial_x^3 u_\varepsilon||\alpha\partial_x^5 v_\varepsilon|dx\\
&\le 18\kappa^2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}u_\varepsilon^4(\partial_x^3 u_\varepsilon)^2dx\\
&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le 18\kappa^2\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbf{R})}\times\\
&\quad\times\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)\varepsilon\sqrt{\varepsilon}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\le C(T)+\tfrac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}.
\end{aligned}\tag{117}$$

Consequently, by (114),

$$\frac{\varepsilon\sqrt{\varepsilon}\alpha^2}{2}\left\|\partial_x^5 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\le C(T),\tag{118}$$

which gives (108). ♠

We continue by proving an H^4 energy type estimate.

Lemma 2.17 *Assume (6) or (7). We have that*

$$\begin{aligned}
&\varepsilon\sqrt{\varepsilon}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\
&\quad+\tfrac{2\varepsilon^2\sqrt{\varepsilon}e^t}{2}\int_0^t e^{-s}\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbf{R})}ds\\
&\le C(T),
\end{aligned}\tag{119}$$

and

$$\sqrt[8]{\varepsilon^5}\left\|\partial_x^3 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\le C(T),\tag{120}$$

for every $0\le t\le T$.

Proof. Multiplying the first equation of (11) by $2\varepsilon\sqrt{\varepsilon}\partial_x^8 u_\varepsilon$, we get

$$\begin{aligned}
&2\varepsilon\sqrt{\varepsilon}\partial_x^8 u_\varepsilon\partial_t u_\varepsilon\\
&=2b\varepsilon\sqrt{\varepsilon}P_\varepsilon\partial_x^8 u_\varepsilon\\
&\quad+2\varepsilon^2\sqrt{\varepsilon}\partial_x^2 u_\varepsilon\partial_x^8 u_\varepsilon\\
&\quad-2q\varepsilon\sqrt{\varepsilon}\partial_x^8 u_\varepsilon\partial_x v_\varepsilon.
\end{aligned}\tag{121}$$

Observe that by (13) and the second equation of (11),

$$\begin{aligned}
&2b\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}P_\varepsilon\partial_x^8 u_\varepsilon dx\\
&=2b\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x P_\varepsilon\partial_x^7 u_\varepsilon dx\\
&=2b\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}u_\varepsilon\partial_x^7 u_\varepsilon dx\\
&=2b\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&=2b\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon dx\\
&=2b\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx=0.
\end{aligned}\tag{122}$$

Moreover,

$$\begin{aligned}
&2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^8 u_\varepsilon\partial_t u_\varepsilon dx\\
&=-2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^7 u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&=2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^6 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\
&=-2\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^5 u_\varepsilon\partial_t\partial_x^3 u_\varepsilon dx\\
&=\varepsilon\sqrt{\varepsilon}\tfrac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{123}$$

and

$$\begin{aligned}
&2\varepsilon^2\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^2 u_\varepsilon\partial_x^8 u_\varepsilon dx\\
&=-2\varepsilon^2\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^3 u_\varepsilon\partial_x^7 u_\varepsilon dx\\
&=2\varepsilon^2\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&=-2\varepsilon^2\sqrt{\varepsilon}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})},
\end{aligned}\tag{124}$$

and

$$\begin{aligned}
&-2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^8 u_\varepsilon\partial_x v_\varepsilon dx\\
&=2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^7 u_\varepsilon\partial_x^2 v_\varepsilon dx\\
&=-2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^6 u_\varepsilon\partial_x^3 v_\varepsilon dx\\
&=2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^5 u_\varepsilon\partial_x^4 v_\varepsilon dx\\
&=-2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^4 u_\varepsilon\partial_x^5 v_\varepsilon dx.
\end{aligned}\tag{125}$$

Integrating (121), by (122) and (124), we have that

$$\begin{aligned}&\varepsilon\sqrt{\varepsilon}\tfrac{d}{dt}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+2\varepsilon^2\sqrt{\varepsilon}\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&=-2q\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}\partial_x^4u_\varepsilon\partial_x^5v_\varepsilon dx.\end{aligned}\tag{126}$$

Due to (108) and the Young inequality,

$$\begin{aligned}&2|q|\varepsilon\sqrt{\varepsilon}\int_{\mathbf{R}}|\partial_x^4u_\varepsilon||\partial_x^5v_\varepsilon|dx\\&\leq\varepsilon\sqrt{\varepsilon}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+q^2\varepsilon\sqrt{\varepsilon}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\varepsilon\sqrt{\varepsilon}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{127}$$

It follows from (126) that

$$\begin{aligned}&\varepsilon\sqrt{\varepsilon}\tfrac{d}{dt}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&+2\varepsilon^2\sqrt{\varepsilon}\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\leq\varepsilon\sqrt{\varepsilon}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}+C(T).\end{aligned}\tag{128}$$

The Gronwall Lemma and (12) gives (119).

Finally, we prove (120). Thanks to (83), (119) and the Hölder inequality,

$$\begin{aligned}&(\partial_x^3u_\varepsilon(t,x))^2\\&=2\int_{-\infty}^{x}\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon dx\\&\leq2\int_{\mathbf{R}}|\partial_x^3u_\varepsilon||\partial_x^4u_\varepsilon|dx\\&\leq\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbf{R})}\\&\leq\tfrac{C(T)}{\sqrt[4]{\varepsilon^5}}.\end{aligned}\tag{129}$$

Hence,

$$\sqrt[4]{\varepsilon^5}\left\|\partial_x^3u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbf{R})}\leq C(T),\tag{130}$$

which gives (120). ♠

We show an uniform L^∞ bound on the second order mixed derivative.

Lemma 2.18 *Assume (6) or (7). We have that*

$$\|\partial_t\partial_xu_\varepsilon\|_{L^\infty((0,T)\times\mathbf{R})}\leq C(T).\tag{131}$$

Proof. Differentiating the first equation of (11) with respect to, thanks to the second one of (11), we have that

$$\partial_t\partial_xu_\varepsilon=bu_\varepsilon+\varepsilon\partial_x^3u_\varepsilon-q\partial_x^3v_\varepsilon.\tag{132}$$

Due to (35) and (35),

$$\begin{aligned}&|\partial_t\partial_xu_\varepsilon|\\&=|bu_\varepsilon-q\partial_x^3v_\varepsilon+\varepsilon\partial_x^3u_\varepsilon|\\&\leq|b||u_\varepsilon|+|q||\partial_x^2v_\varepsilon|+\varepsilon|\partial_x^3u_\varepsilon|\\&\leq|b|\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\\&\quad+|q|\left\|\partial_x^2v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\\&\quad+\varepsilon\left\|\partial_x^3u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\\&\leq C(T)+\varepsilon\left\|\partial_x^3u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}.\end{aligned}\tag{133}$$

Observe that, since $0<\varepsilon<1$, thanks to (120),

$$\begin{aligned}&\varepsilon\left\|\partial_x^3u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\\&=\sqrt[8]{\varepsilon^3}\sqrt[8]{\varepsilon^5}\left\|\partial_x^3u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\\&\leq\sqrt[8]{\varepsilon^5}\left\|\partial_x^3u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbf{R})}\leq C(T).\end{aligned}\tag{134}$$

(131) follows from (133) and (134). ♠

Consider the fast decaying function

$$\chi(x)=e^{-|x|},\qquad x\in\mathbf{R},\tag{135}$$

that satisfies

$$0\leq\chi\leq1,\qquad|\chi'|=\chi.\tag{136}$$

We prove the following result

Lemma 2.19 *Assume (6) or (7). We have that*

$$\begin{aligned}&\varepsilon\int_{\mathbf{R}}(\partial_t\partial_xu_\varepsilon)^2\chi dx\\&\quad+\int_0^t\int_{\mathbf{R}}(\partial_t^2u_\varepsilon)^2\chi dtdx\leq C(T),\end{aligned}\tag{137}$$

for every $0\leq t\leq T$.

Proof. Differentiating the first equation of (11) with respect to t, we have

$$\partial_t^2u_\varepsilon=b\partial_tP_\varepsilon+\varepsilon\partial_t\partial_x^2u_\varepsilon-q\partial_t\partial_xv_\varepsilon.\tag{138}$$

Multiplying (138) by $2\partial_t^2u_\varepsilon\chi$, and integration on $\mathbf{R}$ give,

$$\begin{aligned}&2\int_{\mathbf{R}}(\partial_t^2u_\varepsilon)^2\chi dx\\&=2b\int_{\mathbf{R}}\partial_tP_\varepsilon\partial_t^2u_\varepsilon\chi dx\\&\quad+2\varepsilon\int_{\mathbf{R}}\partial_t\partial_x^2u_\varepsilon\partial_t^2u_\varepsilon\chi dx\\&\quad-2q\int_{\mathbf{R}}\partial_t\partial_xv_\varepsilon\partial_t^2u_\varepsilon\chi dx.\end{aligned}\tag{139}$$

Observe that

$$\begin{aligned}&2\varepsilon\int_{\mathbf{R}}\partial_t\partial_x^2u_\varepsilon\partial_t^2u_\varepsilon\chi dx\\&=-\varepsilon\tfrac{d}{dt}\int_{\mathbf{R}}\chi(\partial_tu_\varepsilon)^2dx\\&\quad-2\varepsilon\int_{\mathbf{R}}\partial_t\partial_xu_\varepsilon\partial_t^2u_\varepsilon\chi'dx.\end{aligned}\tag{140}$$

Consequently, by (139),

$$\begin{aligned}&\varepsilon\tfrac{d}{dt}\int_{\mathbf{R}}\chi(\partial_t u_\varepsilon)^2dx\\&\quad+2\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&=2b\int_{\mathbf{R}}\partial_t P_\varepsilon\partial_t^2 u_\varepsilon\chi dx\\&\quad-2\varepsilon\int_{\mathbf{R}}\partial_t\partial_x u_\varepsilon\partial_t^2 u_\varepsilon\chi' dx\\&-2q\int_{\mathbf{R}}\partial_t\partial_x v_\varepsilon\partial_t^2 u_\varepsilon\chi dx.\end{aligned}\tag{141}$$

Since $0<\varepsilon<1$, thanks to (69), (99) and the Young inequality,

$$\begin{aligned}&2\varepsilon\int_{\mathbf{R}}|\partial_t\partial_x u_\varepsilon||\partial_t^2 u_\varepsilon||\chi'|dx\\&\le 2C_0\int_{\mathbf{R}}|\partial_t\partial_x u_\varepsilon||\partial_t^2 u_\varepsilon|\chi dx\\&\le C_0\int_{\mathbf{R}}\chi(\partial_t\partial_x u_\varepsilon)^2dx\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C_0\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C(T)+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx,\\&2|b|\int_{\mathbf{R}}|\partial_t P_\varepsilon||\partial_t^2 u_\varepsilon|\chi dx\\&\le 2b^2\int_{\mathbf{R}}(\partial_t P_\varepsilon)^2\chi dx\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx,\\&2|q|\int_{\mathbf{R}}|\partial_t\partial_x v_\varepsilon||\partial_t^2 u_\varepsilon|\chi dx\\&\le 2q^2\int_{\mathbf{R}}\chi(\partial_t\partial_x v_\varepsilon)^2dx\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C_0\left\|\partial_t\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbf{R})}\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C(T)+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx.\end{aligned}\tag{142}$$

It follows from (141) that

$$\begin{aligned}&\varepsilon\tfrac{d}{dt}\int_{\mathbf{R}}\chi(\partial_t u_\varepsilon)^2dx\\&\quad+\tfrac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C(T)+2b^2\int_{\mathbf{R}}(\partial_t P_\varepsilon)^2\chi dx.\end{aligned}\tag{143}$$

Observe that, by the second equation of (11),

$$\partial_t P_\varepsilon=\int_0^x\partial_t u_\varepsilon(t,y)dy.\tag{144}$$

Therefore, by (144), (63) and the Jensen inequality

$$\begin{aligned}&2b^2\int_{\mathbf{R}}(\partial_t P_\varepsilon)^2\chi dx\\&=\int_{\mathbf{R}}\chi\left(\int_0^x\partial_t u_\varepsilon(t,y)dy\right)^2dx\\&\le\int_{\mathbf{R}}\chi|x|\left|\int_0^x(\partial_t u_\varepsilon)^2dy\right|dx\\&\le\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbf{R})}\int_{\mathbf{R}}\chi|x|\le C(T).\end{aligned}\tag{145}$$

Thus, by (143), we have

$$\begin{aligned}&\varepsilon\tfrac{d}{dt}\int_{\mathbf{R}}\chi(\partial_t u_\varepsilon)^2dx+\frac12\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dx\\&\le C(T).\end{aligned}\tag{146}$$

Integrating on $(0,t)$, by (12), we get

$$\begin{aligned}&\varepsilon\int_{\mathbf{R}}\chi(\partial_t u_\varepsilon)^2dx\\&\quad+\tfrac12\int_0^t\int_{\mathbf{R}}(\partial_t^2 u_\varepsilon)^2\chi dsdx\\&\le C_0+C(T)t\le C(T)\end{aligned}\tag{147}$$

that is (137). ♠

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.
Proof of Theorem 1.1. Thanks to Lemmas 2.4, 2.5, 2.6, 2.10, 2.11, 2.15, (95), (131), and (2.19),

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0}\text{ is bounded in }H^1_{loc}((0,\infty)\times\mathbf{R})\tag{148}$$

Consequentially, there exists $w\in H^1_{loc}((0,\infty)\times\mathbf{R})$ such that

$$\begin{aligned}&\partial_t u_\varepsilon\rightharpoonup w\text{ in }H^1_{loc}((0,\infty)\times\mathbf{R}),\\&\partial_t u_\varepsilon\to w\text{ in }L^p_{Loc}((0,\infty)\times\mathbf{R}),\\&\quad 1\le p<\infty\text{ and a.e. in }(0,\infty)\times\mathbf{R}.\end{aligned}\tag{149}$$

We define the following functon:

$$u(t,x)=\int_0^t w(s,x)ds+u_0(x).\tag{150}$$

We prove that

$$\begin{aligned}&u_\varepsilon\to u\text{ in }L^p_{Loc}((0,\infty)\times\mathbf{R}),\\&\quad 1\le p<\infty\text{ and a.e. in }(0,\infty)\times\mathbf{R}.\end{aligned}\tag{151}$$

Observe that

$$u_\varepsilon(t,x)=\int_0^t u_\varepsilon(s,x)ds+u_{\varepsilon,0}(x).\tag{152}$$

consequentially, we have that

$$\begin{aligned}&u_\varepsilon(t,x)-u(t,x)\\&=\int_0^t(\partial_t u_\varepsilon(s,x)-w(s,x))ds\\&+u_{\varepsilon,0}(x)-u_0(x).\end{aligned}\tag{153}$$

Therefore, by (149),

$$\begin{aligned}&\int_0^T\int_{-R}^R |u_\varepsilon(t,x)-u(t,x)|dtdx\\ &\le \int_0^T\int_{-R}^R\int_0^t |\partial_t u_\varepsilon(s,x)-w(t,x)|dsdtdx\\ &\quad +T\int_{-R}^R |u_{\varepsilon,0}(x)-u_0(x)|dx \to 0,\end{aligned} \tag{154}$$

which gives (151).

By (151), we have that

$$\begin{aligned}&P_{\varepsilon_\kappa}\to P \text{ in } L^p_{loc}((0,T);W^{1,p}(\mathbf{R}))\\ &\quad 1\le p<\infty, \text{ and a.e. in } (0,\infty)\times\mathbf{R},\end{aligned} \tag{155}$$

where

$$P(t,x)=\int_0^x u(t,y)dy, \quad t>0, \quad x\in\mathbf{R}. \tag{156}$$

Moreover, thanks to Lemmas 2.4, 2.5, 2.6, 2.10, 2.11, 2.15 , (95), (131), and (2.19),

$$\{v_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } H^1_{loc}((0,\infty)\times\mathbf{R}) \tag{157}$$

Consequentially, there exists $v\in H^1_{loc}((0,\infty)\times\mathbf{R})$ such that

$$\begin{aligned}&v_\varepsilon \rightharpoonup v \text{ in } H^1_{loc}((0,\infty)\times\mathbf{R}),\\ &v_\varepsilon \to v \text{ in } L^p_{Loc}((0,\infty)\times\mathbf{R}),\\ &\quad 1\le p<\infty \text{ and a.e. in } (0,\infty)\times\mathbf{R}.\end{aligned} \tag{158}$$

Therefore, the triple (u, v, P) is a distributional solution of (2) and (8) hold. ♠

4 Conclusion

We consider the short pulse equations that is a second order evolutive PDE that appear in the modeling of several physical and mathematical phenomena. Moreover, if can rewritten in the form of an hyperbolic equation of the first order with a nonlocal source term. Here we consider a nonlocal regularization fo the flux and studied the existence of possibly discontinuous solutions using a vanishing viscosity type argument and energy type estimates.

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