Application of Odd Chen-Log-Logistic Distribution to Covid-19 Data

KLODIANA BANI, ARBËR QOSHJA, AURORA SIMONI, MARSELINA UZUNI
Department of Applied Mathematics,
Tirana University,
Sheshi "Nënë Tereza", Tirana,
ALBANIA

Abstract: - This article created the Odd Chen-Log-Logistic distribution from Odd Chen-G family distributions. We derive various statistical features. The parameter estimation theory focuses on selecting the best estimators. We estimate distribution parameters using maximum likelihood, moment, least squares, weighted least, L-moment, maximum product spacing, and minimal distance methods. We will examine Kolmogorov-Smirnov simulation studies that compare estimator efficiency. Finally, we analyze a genuine COVID-19 data set to demonstrate the flexibility of our model and its accuracy compared to other distributions.

Key-Words: - Log-Logistic distribution, Hazard Function, Maximum Likelihood, Moment Estimation, Simulations, COVID-19.

Received: May 9, 2024. Revised: September 23, 2024. Accepted: October 15, 2024. Published: November 8, 2024.

1 Introduction

The log-logistic distribution, also known as the Fisk distribution, is a massive continuous probability distribution with a huge tail. Indeed, it has a singular form parameter and a single scale or rate. This distribution uses a non-negative random variable whose logarithm has the common logistic distribution, [1]. The log-logistic distribution enables the closed representation of the cumulative distribution and helps one to estimate incomplete (or censored) data. Indeed, in the domains of business, medicine, economics, income, wealth, and the social sciences, the log-logistic distribution models do find their use. They help to depict data with a substantial degree of fluctuation.

In several areas, the log-logistic distribution differs from many parametric distributions used in survival and reliability research.

Many disciplines, including demography [2], economics [3], engineering [4], and hydrology [5], use the log-logistic distribution as a basic yet effective parametric model.

Definition 1.1. A random variable X has a Log-Logistic distribution with shape parameter $\beta > 0$ and scale parameter $\alpha > 0$, based on the probability density function:

$$g_X(x;\theta,\mu) = \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} \quad (1)$$

and cumulative distribution function

$$G_X(x;\theta,\mu) = \frac{x^{\mu}}{\theta^{\mu}+x^{\mu}} \quad (2)$$

According to [6], the Odd Chen-G Family of distributions generator provides the pdf and cdf of a continuous distribution:

$$f_X\left(x;\lambda,\beta,\underline{\xi}\right) = \lambda\beta g\left(x;\underline{\xi}\right) G\left(x;\underline{\xi}\right)^{\beta-1} \times$$
$$\times \left[1-G\left(x;\underline{\xi}\right)\right]^{-(\beta+1)} e^{\left(\frac{G\left(x;\underline{\xi}\right)}{1-G\left(x;\underline{\xi}\right)}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{G\left(x;\underline{\xi}\right)}{1-G\left(x;\underline{\xi}\right)}\right)^{\beta}}\right)} \quad (3)$$

and

$$F_X\left(x;\lambda,\beta,\underline{\xi}\right) = 1-e^{\lambda\left(1-e^{\left(\frac{G\left(x;\underline{\xi}\right)}{1-G\left(x;\underline{\xi}\right)}\right)^{\beta}}\right)} \quad (4)$$

respectively, for $x > 0, \lambda > 0, \beta > 0$ and parameter vector $\underline{\xi}$.

This study examines Log-Logistic (3) and (4), often known as the Odd Chen distribution, as baseline functions for (5) and (6).

Definition 1.2. The probability density function of a random variable X with a vector parameter is called an Odd Chen Log-Logistic distribution $(\lambda, \beta, \theta, \mu)$.

$$f_X(x;\lambda,\beta,\theta,\mu) = \lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} \frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}^{\beta-1} \times$$

$$\times\left[1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}\right]^{-(\beta+1)} e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right)} \quad (5)$$

and cumulative distribution function

$$F_X(x;\lambda,\beta,\theta,\mu)=1-e^{\lambda\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right)} \quad (6)$$

respectively, for $x>0,\lambda>0,\beta>0,\theta>0,\mu>0$.

Some alternative shapes of the Odd Chen Log-Logistic distribution (OC-LL) distribution for given values of λ,β,θ, and μ are shown in Figure 1 and Figure 2.

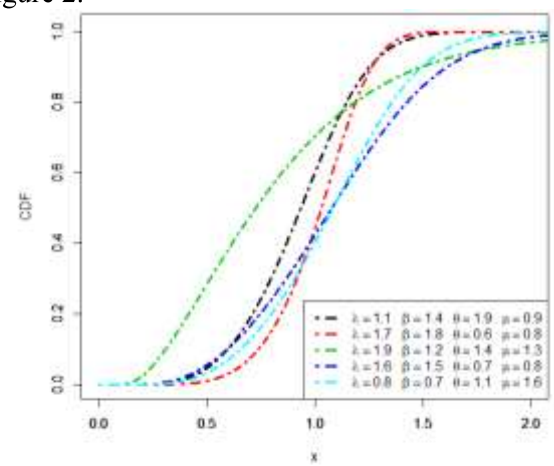

Fig. 1: CDF

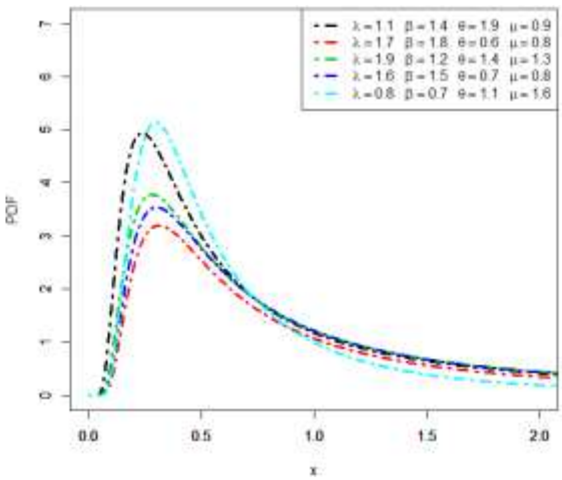

Fig. 2: PDF

2 Some Properties

2.1 Survival Function

Survival function or reliability function of the Odd Chen Log-Logistic distribution:

$$R(x;\alpha,\beta,\theta)=1-F_X(x;\alpha,\beta,\theta)$$

$$1-1+e^{\lambda\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right)}=e^{\lambda\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right)} \quad (7)$$

2.2 Hazard Function

Hazard rate function, or failure rate, of the Odd Chen Log-Logistic distribution is:

$$h(x;\alpha,\beta,\theta)=\lambda\beta\frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^{2}}\times\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}^{\beta-1}\times$$

$$\times\left[1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}\right]^{-(\beta+1)} e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}} \quad (8)$$

Figure 3 and Figure 4 display the potential Reliability and Hazard functions of the Odd Chen Log-Logistic (OC-LL) distribution for specific values λ,β,θ and μ.

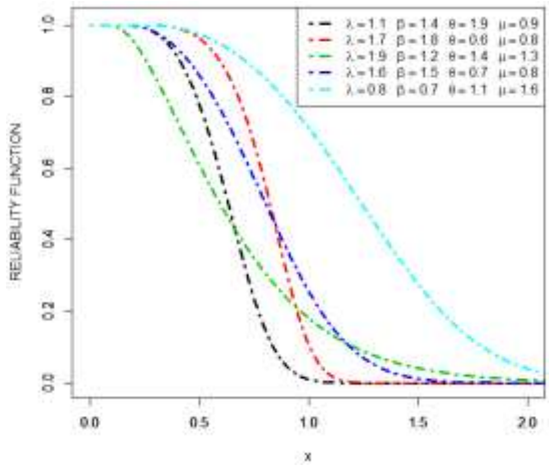

Fig. 3: Reliability Function

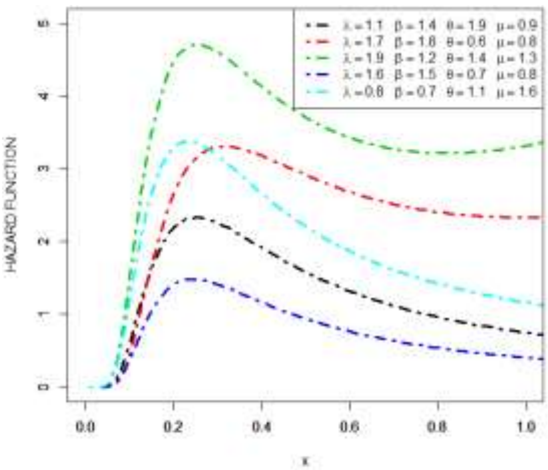

Fig. 4: Hazard Function

2.3 Quantiles

Solving equation $G(x_p) = p$, we get the quantile of some distribution, for $0 < p < 1$. Odd Chen Log-Logistic distribution quantile function is:

$$x(p) = G^{-1}\left[\frac{\left(log\left(1-\frac{log(1-p)}{\lambda}\right)\right)^{1/\beta}}{1+\left(log\left(1-\frac{log(1-p)}{\lambda}\right)\right)^{1/\beta}}\right] \tag{9}$$

2.4 Some Useful Expression

Extension of Taylor series, [7], the pdf (7) of X becomes:

$$f_X(x;) = \lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} e^{\lambda} \times$$
$$\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i\lambda^i}{i!}\frac{(i+1)^j}{j!}\left(\frac{x^\mu}{\theta^\mu+x^\mu}\right)^{\beta(j+1)-1} \times$$
$$\times \left[1-\left(\frac{x^\mu}{\theta^\mu+x^\mu}\right)\right]^{-(\beta(j+1)-1)}$$

Generalized binomial expansion:

$$f_X(x;\lambda,\beta,\theta,\mu) = \lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} e^{\lambda} \times$$
$$\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i\lambda^i}{i!}\frac{(i+1)^j}{j!}\binom{\beta(j+1)+k}{k} \times$$
$$\times \left[\frac{x^\mu}{\theta^\mu+x^\mu}\right]^{\beta(j+1)+k-1}$$

2.5 Order Statistics

The maximum order density for i.i.d. continuous random variables $X_1, X_2, \ldots, X_n$ with pdf (7) and cdf (8) is:

$$f_{(n)}(x) = nf(x)F(x)^{n-1} = n\lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} {\frac{x^\mu}{\theta^\mu+x^\mu}}^{\beta-1} \times$$
$$\left[1-\frac{x^\mu}{\theta^\mu+x^\mu}\right]^{-(\beta+1)} e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)} \times$$
$$\left[1-exp\left(\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)\right)\right]^{n-1} \tag{10}$$

The minimum order density for i.i.d. continuous random variables $X_1, X_2, \ldots, X_n$ with pdf (7) and cdf (8) is:

$$f_{(1)}(x) = nf(x)\left(1-F(x)\right)^{n-1}$$
$$= n\lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} {\frac{x^\mu}{\theta^\mu+x^\mu}}^{\beta-1} \times$$
$$\times \left[1-\frac{x^\mu}{\theta^\mu+x^\mu}\right]^{-(\beta+1)} e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)} \times$$
$$\left[exp\left(\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)\right)\right]^{n-1} \tag{11}$$

The minimum order density for i.i.d. continuous random variables $X_1, X_2, \ldots, X_n$ with pdf (7) and cdf (8) is:

$$f_{(k)}(x) = nf(x)\binom{n-1}{k-1}F(x)^{k-1}\left(1-F(x)\right)^{n-k} =$$
$$= n\lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} {\frac{x^\mu}{\theta^\mu+x^\mu}}^{\beta-1} \times \left[1-\frac{x^\mu}{\theta^\mu+x^\mu}\right]^{-(\beta+1)}$$
$$\times e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)} \times \binom{n-1}{k-1}$$
$$\times \left[1-exp\left(\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)\right)\right]^{k-1} \times$$
$$\times \left[exp\left(\lambda\left(1-e^{\left(\frac{\frac{x^\mu}{\theta^\mu+x^\mu}}{1-\frac{x^\mu}{\theta^\mu+x^\mu}}\right)^{\beta}}\right)\right)\right]^{n-k} \tag{12}$$

2.6 Rényi Entropy

Information theory's Rényi entropy generalises collision, min-entropy, Shannon, and Hartley entropies. The Rényi entropy studies the largest method for information quantification that maintains additivity for separate events, [8]. In fractal dimension estimation, generalized dimensions are based on Rényi entropy. Rényi entropy is a measure of diversity in statistics and ecology. The Rényi entropy is an important entanglement gauge in quantum information. It is possible to exactly determine the Rényi entropy as a function of α in the Heisenberg XY spin chain model, as it is an automorphic function for a specific subgroup of the modular. Rényi entropy $I_R(\nu)$ for the Odd Chen Log-Logistic distribution is as follows:

$$I_R(\nu) = (1-\nu)^{-1} log\left[\int_{-\infty}^{\infty} f^{\nu}(x)dx\right] = (1-\nu)^{-1} log$$

$$\left[\int_{-\infty}^{\infty}\left(\begin{array}{c}\lambda\beta \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x}{\theta}\right)^{\mu}\right)^2} \frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}^{\beta-1}\left[1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}\right]^{-(\beta+1)} \times \\ \times e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}} e^{\lambda\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right)}\end{array}\right)^{\nu}\right],$$

$\nu \neq 1, \nu > 0$.

3 Approaches to Parameter Estimation

3.1 Maximum Likelihood

The most prevalent ML method is full information maximum likelihood (ML) because it gives estimates with desirable large sample quality. In finite samples, these traits hold roughly. For independent $x_1, x_2, \dots, x_n$ consider a parametric model with probability density or frequency distribution functions $f_i(x_i; \boldsymbol{\xi})$. We know that:

$$L(\boldsymbol{\xi}) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} f_i(x_i; \boldsymbol{\xi}) \tag{13}$$

$L(\boldsymbol{\xi})$ represents the probability of the sample being observed for $\boldsymbol{\xi}$. So, the value of $\tilde{\boldsymbol{\xi}}$ that maximises $L(\boldsymbol{\xi})$ defines the MLE.

So, $x_1, x_2, \dots, x_n$ be i.i.d. random variables with pdf (7). The likelihood function of parameters λ, β, θ and μ is:

$$\ell = nlog\lambda\beta + \sum_{i=1}^{n} log \frac{\left(\frac{\mu}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\mu-1}}{\left(1+\left(\frac{x_i}{\theta}\right)^{\mu}\right)^2} +$$

$$(\beta-1)\sum_{i=1}^{n} log \frac{x^{\mu}}{\theta^{\mu}+x^{\mu}} + \lambda \sum_{i=1}^{n}\left(1-e^{\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta}}\right) -$$

$$(\beta+1)\sum_{i=1}^{n} log\left[1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}\right] \sum_{i=1}^{n}\left(\frac{\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}{1-\frac{x^{\mu}}{\theta^{\mu}+x^{\mu}}}\right)^{\beta} \tag{14}$$

Solving nonlinear equations simultaneously estimates unknown parameters, which cannot be solved analytically. Iterative methods like the Newton-Raphson approach simplify nonlinear situations. Newton Raphson estimated parameters using these beginning values. The parameter estimates for the $100(1-\alpha)$ two-sided confidence range are asymptotically close to standard normal, as indicated by the z-score.

3.2 Moment Estimation

Since sample moments are estimates of population moments, the method of moments is one of the oldest point estimator methods, [9]. Equalising the first three theoretical moments with the three sample moments yields the Odd Chen Log-Logistic distribution's moment estimators. These four moments are examples:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i, m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{14}$$

$$m_3 = \frac{1}{n}\sum_{i=1}^{n} x_i^3, m_4 = \frac{1}{n}\sum_{i=1}^{n} x_i^4 \tag{15}$$

and the first four theoretical moments were characterised as:

$$\mu_1' = E(X^1) = \int_{-\infty}^{+\infty} x f(x)dx$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$

$$\mu_3' = E(X^3) = \int_{-\infty}^{+\infty} x^3 f(x)dx$$

$$\mu_4' = E(X^4) = \int_{-\infty}^{+\infty} x^4 f(x)dx$$

The moment's estimators $\hat{\lambda}_{ME}, \hat{\beta}_{ME}, \hat{\theta}_{ME}, \hat{\mu}_{ME}$ of the parameters $\lambda, \beta, \theta, \mu$ can be obtained by solving numerically the following system of equations:

$$m_1 = \mu_1'(\lambda, \beta, \theta, \mu)$$
$$m_2 = \mu_2'(\lambda, \beta, \theta, \mu)$$
$$m_3 = \mu_3'(\lambda, \beta, \theta, \mu)$$
$$m_4 = \mu_4'(\lambda, \beta, \theta, \mu)$$

Modified moment estimate is a good alternative to moment estimation. This first-order statistics approach can be adjusted, as mentioned by [10].

Let $X_1, X_2, \dots, X_n$ be a sample from Odd Chen Log-Logistic distribution, with observed values $x_1, x_2, \dots, x_n$. Solving the following equations yields Odd Chen Log-Logistic distribution modified moment estimators:

$$E(X) = \bar{x}$$
$$V(X) = s^2$$
$$E\left(F\left(X_{(1)}\right)\right) = F(x_1)$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and s^2 is the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

3.3 Least Square Estimation

Beta distribution parameters can be found using least square estimators and weighted LSEs, [11].

The Odd Chen Log-Logistic distribution's unknown parameters' LSEs can be found by minimizing:

$$\sum_{j=1}^{n}\left(F\left(x_{(j)}\right)-\frac{j}{n+1}\right)^2 \qquad (16)$$

regarding unknown parameters λ,β,θ,μ.

To calculate the least squares estimate (LSE) of λ,β,θ,μ, can be derived by minimizing the respective values:

$$LS(x_j)=\sum_{j=1}^{n}\left(1-exp\left(\lambda\left(1-e^{\left(\frac{\frac{x_j{}^{\mu}}{\theta^{\mu}+x_j{}^{\mu}}}{1-\frac{x_j{}^{\mu}}{\theta^{\mu}+x_j{}^{\mu}}}\right)^{\beta}}\right)\right)-\frac{j}{n+1}\right)^2$$

Hence, $\hat{\lambda}_{LSE},\hat{\beta}_{LSE},\hat{\theta}_{LSE},\hat{\mu}_{LSE}$ of λ,β,θ,μ can be found by solving the following system of equations:

$$\frac{\partial LS(x_j;\lambda,\beta,\theta,\mu)}{\partial\lambda}=0,\frac{\partial LS(x_j;\lambda,\beta,\theta,\mu)}{\partial\beta}=0$$
$$\frac{\partial LS(x_j;\lambda,\beta,\theta,\mu)}{\partial\theta}=0,\frac{\partial LS(x_j;\lambda,\beta,\theta,\mu)}{\partial\mu}=0$$

We can calculate estimates by solving these equations numerically $\hat{\lambda}_{LSE},\hat{\beta}_{LSE},\hat{\theta}_{LSE},\hat{\mu}_{LSE}$.

3.4 The Weighted Least Square Estimation

The unknown parameters' weighted least squares estimators (WLSEs) can be calculated by minimizing:

$$\sum_{j=1}^{n}\omega_j\left(F\left(x_{(j)}\right)-\frac{j}{n+1}\right)^2 \qquad (17)$$

consider α,β,θ, and ω_j denote the weight function:

$$\omega_j=\frac{1}{V(F(X_{(j)})}\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

The WLSEs say $\hat{\lambda}_{LSE},\hat{\beta}_{LSE},\hat{\theta}_{LSE},\hat{\mu}_{WLSE}$ by minimizing:

$$WLSE(x_j|\lambda,\beta,\theta,\mu)=\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\times$$
$$\times\left(1-exp\left(\lambda\left(1-e^{\left(\frac{\frac{x_j{}^{\mu}}{\theta^{\mu}+x_j{}^{\mu}}}{1-\frac{x_j{}^{\mu}}{\theta^{\mu}+x_j{}^{\mu}}}\right)^{\beta}}\right)\right)-\frac{j}{n+1}\right)^2$$

We may calculate the estimators $\hat{\lambda}_{WLSE}$, $\hat{\beta}_{WLSE}$, $\hat{\theta}_{WLSE}$, $\hat{\mu}_{WLSE}$ by taking the first partial derivative of λ,β,θ,μ and setting the result to zero:

$$\frac{\partial WLS(x_j;\lambda,\beta,\theta,\mu)}{\partial\lambda}=0,\frac{\partial WLS(x_j;\lambda,\beta,\theta,\mu)}{\partial\beta}=0$$
$$\frac{\partial WLS(x_j;\lambda,\beta,\theta,\mu)}{\partial\theta}=0,\frac{\partial WLS(x_j;\lambda,\beta,\theta,\mu)}{\partial\mu}=0$$

We can calculate estimates by solving these equations numerically $\hat{\lambda}_{WLSE}$, $\hat{\beta}_{WLSE}$, $\hat{\theta}_{WLSE}$, $\hat{\mu}_{WLSE}$.

3.5 L-Moments Estimators

Equating sample and population L-moments yields L-moment estimators, [12]. Equating sample and population L-moments yields these estimators. The L-moment estimators are more reliable than the moment estimators, more immune to outliers, and more efficient than the maximum likelihood estimators for specific distributions, [13].

Equating the first three sample L-moments with the population L-moments yields the Odd Chen Log-Logistic distribution L-moments estimators. Example's first three L-moments:

$$l_1=\frac{1}{n}\sum_{j=1}^{n}x_{(j)},$$
$$l_2=\frac{2}{n(n-1)}\sum_{j=2}^{n}(j-1)x_{(j)}-l_1$$
$$l_3=\frac{6}{n(n-1)(n-2)}\sum_{j=3}^{n}(j-1)(j-2)x_{(j)}-6l_2+l_1$$

a first three population L-moments of:

$$\lambda_1=E(X_{1:1})=\int_{-\infty}^{+\infty}x\,f(x)dx=E(X),$$
$$\lambda_2=\frac{1}{2}[E(X_{2:2})-E(X_{2:1})]=\int_{-\infty}^{+\infty}x[2F(x)-1]f(x)dx,$$
$$\lambda_3=\frac{1}{3}[E(X_{3:3})-2E(X_{2:3})+E(X_{1:3})]=\int_{-\infty}^{+\infty}x[6\left(F(x)\right)^2-6F(x)+1]f(x)dx$$

$X_{j:n}$ is the j^{th} order statistic of an n-sample. To calculate the L-moments estimators for the parameters λ,β,θ,μ solve the following equations numerically:

$$\lambda_1(\hat{\lambda}_{LME},\hat{\beta}_{LME},\hat{\theta}_{LME},\hat{\mu}_{LME})=l_1$$
$$\lambda_2(\hat{\lambda}_{LME},\hat{\beta}_{LME},\hat{\theta}_{LME},\hat{\mu}_{LME})=l_2$$
$$\lambda_3(\hat{\lambda}_{LME},\hat{\beta}_{LME},\hat{\theta}_{LME},\hat{\mu}_{LME})=l_3$$

3.6 Maximum Product Spacing Estimators

For continuous univariate distributions, [13], [14] developed the maximum product of spacings (MPS) approach to estimate parameters and approximate the Kullback-Leibler measure of information. This approach assumes an equal distribution of consecutive point discrepancies.

Consider $X_1, \ldots, X_n$ as a random sample from the Odd Chen Log-Logistic distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ as an ordered sample. We refer to $X_0 = -\infty$ and $X_n = +\infty$. I The method of maximum product of spacings estimates distribution parameters $\lambda, \beta, \theta, \mu$ by maximising the geometric mean of distances D_i, denoted as:

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x;\theta)\,dx = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)$$

for $i = 1, 2, \ldots, n+1$ (18)

where $F\left(x_{(0)}\right) = 0$, $F\left(x_{(n+1)}\right) = 1$ and

$$\sum_{i=1}^{n+1} D_i = 1.$$

The geometric mean of distances is expressed as:

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \quad (19)$$

The MPS estimators $\hat{\lambda}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}, \hat{\mu}_{MPS}$ are calculated by maximising the geometric mean (GM) of spacings concerning $\lambda, \beta, \theta, \mu$ or the logarithm of the geometric mean of sample spacings:

$$\log(GM) = \log\left(\sqrt[n+1]{\prod_{i=1}^{n+1} D_i}\right) = \frac{1}{n+1}\sum_{i=1}^{n+1} logD_i$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)\right] =$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left(1 - exp\left(\lambda\left(1 - e^{\left(\frac{\frac{x_i{}^{\mu}}{\theta^{\mu}+x_i{}^{\mu}}}{1-\frac{x_i{}^{\mu}}{\theta^{\mu}+x_i{}^{\mu}}}\right)^{\beta}}\right)\right) - \left(1 - exp\left(\lambda\left(1 - e^{\left(\frac{\frac{x_{i-1}{}^{\mu}}{\theta^{\mu}+x_{i-1}{}^{\mu}}}{1-\frac{x_{i-1}{}^{\mu}}{\theta^{\mu}+x_{i-1}}}\right)^{\beta}}\right)\right)\right)\right) \quad (20)$$

The MPS estimators $\hat{\lambda}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}, \hat{\mu}_{MPS}$ can be derived by solving the following equations simultaneously:

$$\frac{\partial logGM}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\lambda}\left(x_{(i)}\right) - F'_{\lambda}\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0$$

$$\frac{\partial logGM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\beta}\left(x_{(i)}\right) - F'_{\beta}\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0$$

$$\frac{\partial logGM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}\left(x_{(i)}\right) - F'_{\theta}\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0$$

$$\frac{\partial logGM}{\partial \mu} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\mu}\left(x_{(i)}\right) - F'_{\mu}\left(x_{(i-1)}\right)}{F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right)}\right] = 0$$

3.7 Methods of Minimum Distances

The goodness-of-fit statistics method minimizes empirical distribution function statistics to estimate distribution parameters, [15]. The generic minimal distance method assumes establishing a distribution function that closely matches the empirical distribution of the observed data. The minimal distance approach has estimators based on the empirical distribution function statistic. This section presents three Odd Chen Log-Logistic distribution estimate methods based on goodness-of-fit statistics minimization for $\lambda, \beta, \theta, \mu$. This statistical class is based on the difference between the empirical distribution function and the cumulative distribution function estimate, [16], [17].

3.7.1 Method of Cramér-von-Mises

The minimal distance estimator (CME) is a type of estimator based on the Cramér-von-Mises statistic [18], [19]. The real-world data in [20] shows that Cramér-von-Mises-type minimal distance estimators are less biased than other minimum distance estimators, which explains why they are used. The Cramér-von-Mises estimates

$\hat{\lambda}_{CME}, \hat{\beta}_{CME}, \hat{\theta}_{CME}, \hat{\mu}_{CME}$ of parameters $\lambda, \beta, \theta, \mu$ of Odd Chen Log-Logistic distribution are obtained by minimizing, concerning $\lambda, \beta, \theta, \mu$ the function:

$$C(\lambda, \beta, \theta, \mu) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)}\right) - \frac{2i-1}{n}\right)^2$$

$$C(\lambda, \beta, \theta, \mu) = \frac{1}{12n} + \sum_{i=1}^{n}\left(1 - exp\left(\lambda\left(1 - e^{\left(\frac{\frac{x_i{}^{\mu}}{\theta^{\mu}+x_i{}^{\mu}}}{1-\frac{x_i{}^{\mu}}{\theta^{\mu}+x_i{}^{\mu}}}\right)^{\beta}}\right)\right) - \frac{2i-1}{n}\right)^2 \quad (21)$$

To obtain these estimates, you can solve the following nonlinear equations:

$$\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{n}\right)^2\frac{\partial F(x_{(i)})}{\partial\lambda}=0$$
$$\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{n}\right)^2\frac{\partial F(x_{(i)})}{\partial\beta}=0$$
$$\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{n}\right)^2\frac{\partial F(x_{(i)})}{\partial\theta}=0$$
$$\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{n}\right)^2\frac{\partial F(x_{(i)})}{\partial\mu}=0$$

3.7.2 Anderson-Darling and Right-tail Methods

The Anderson-Darling estimator (ADE) is derived from the statistic, another minimum distance estimator. Besides using a weighted squared difference, the Anderson-Darling test is similar to the Cramér-von-Mises criterion. These weights are determined by the deviation from the empirical distribution function. The Anderson-Darling test is an alternative to traditional statistical procedures used to identify deviations from normality in sample distributions, [21], [22]. We minimize a function to estimate the Anderson-Darling parameter λ,β,θ,μ:

$$A(\lambda,\beta,\theta,\mu)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF(x_{(i)})+\log\bar{F}\left(x_{(n+1-i)}\right)\right] \quad (22)$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\lambda(x_{(i)})}{F(x_{(i)})}-\frac{\bar{F}'_\lambda(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}\right]=0$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\beta(x_{(i)})}{F(x_{(i)})}-\frac{\bar{F}'_\beta(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}\right]=0$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\theta(x_{(i)})}{F(x_{(i)})}-\frac{\bar{F}'_\theta(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}\right]=0$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_\mu(x_{(i)})}{F(x_{(i)})}-\frac{\bar{F}'_\mu(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}\right]=0$$

To obtain right-tail Anderson-Darling estimations of parameters, minimise function about λ,β,θ,μ.

$$R(\lambda,\beta,\theta,\mu)=\frac{n}{2}-2\sum_{i=1}^{n}F(x_{(i)})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{F}\left(x_{(n+1-i)}\right) \quad (23)$$

The following non-linear equations may be solved for these estimates:

$$-2\sum_{i=1}^{n}\frac{F'_\lambda(x_{(i)})}{F(x_{(i)})}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_\lambda(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}=0$$
$$-2\sum_{i=1}^{n}\frac{F'_\beta(x_{(i)})}{F(x_{(i)})}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_\beta(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}=0$$
$$-2\sum_{i=1}^{n}\frac{F'_\theta(x_{(i)})}{F(x_{(i)})}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_\theta(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}=0$$
$$-2\sum_{i=1}^{n}\frac{F'_\mu(x_{(i)})}{F(x_{(i)})}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_\mu(x_{(n+1-i)})}{\bar{F}(x_{(n+1-i)})}=0$$

4 Computer Applications

4.1 Simulation Study

Using Monte Carlo simulation, this section tests multiple estimation methods for forecasting Odd Chen Log-Logistic distribution parameters. The Kolmogorov-Smirnov test compares the recommended estimators. This approach uses KS statistics.

$$D_n=\max_x|F_n(x)-F(x|\lambda,\beta,\theta,\mu)|$$

where $\max_x$ denotes the maximum of the set of distances, $F_n(x)$ is the empirical distribution function, and $F(x|\lambda,\beta,\theta,\mu)$ is the cumulative distribution function.

We proposed a method to randomly sample the Odd Chen Log-Logistic distribution given parameter values and sample size n.

We take $\lambda=0.5,\beta=0.9,\theta=1.2,\mu=1.7$ arbitrarily and $n=10,20,\ldots,50$.

We implemented all techniques in the statistical computing environment R.

Simulations were done using the approach.

Table 1. Estimation techniques and Kolmogorov-Smirnov values

i	Methods of Estimations	Kolmogorov-Smirnov test	Ranking
1	Maximum Product Spacing Estimating	0.068542	5
2	Moment Estimation	0.066021	3
3	Least Square Estimation	0.066254	4
4	Weighted Least Square Estimation	0.062741	2
5	L-Moment Estimation	0.070517	6
6	**Maximum Likelihood Estimation**	0.061254	1
7	Maximum Product Spacing Estimating	0.072749	9
8	Anderson-Darling Estimation	0.070654	7
9	Right-tail Anderson-Darling	0.071154	8

The simulation study shows that the Maximum Likelihood Estimation (MLE) technique estimates Odd Chen Log-Logistic distribution parameters more efficiently than other approaches. Table 1 shows that MLE produces the lowest Kolmogorov-Smirnov test result. Additionally, maximal likelihood estimators (MLE) have excellent theoretical properties.. These are consistency, asymptotic efficiency, normalcy, and invariance. These data suggest that MLE estimators are best for calculating Odd Chen Log-Logistic distribution parameters.

4.2 Actual Data

Now, we'll assess the enlarged distribution's effectiveness. In this investigation, our model outperforms other models on a real data set (Table 2). The data shows the case fatality ratio of COVID-19 in China from March 8th to April 1st, 2022, relative to a new strain. Data is obtained from the WHO website (https://covid19.who.int/).
The data are as follows: 1.09, 1.00, 1.08, 1.12, 1.50, 1.60, 1.77, 1.81, 2.07, 1.75, 2.58, 2.59,
2.65, 3.09, 3.20, 3.47, 3.21, 2.77, 3.17, 2.65, 3.00, 3.61, 3.08, 2.70, 2.41.

Table 2. Covid-19 case fatality ratio in China: MLEs and comparability criteria

Distribution	Parameter Estimate	$-\ell$	AIC	BIC	CAIC
Chen Log-Logistic	λ=0.5 β=0.9 θ=1.2 μ=1.7	89.2419	157.326	151.322	148.231
HLOPGW-ILD	α=1.25458712 β=0.36458756 θ=4.25145235	91.1254	165.236	160.631	159.374
EPL	α=2.67052921 β=0.66547111 λ=1.56820413	132.2541	198.254	191.365	196.784
L	α=0.65354891	195.3512	290.354	289.951	290.457
E	θ=0.26732123	201.3264	340.587	342.614	341.753

For the dataset, AIC, CAIC, and BIC are used to evaluate distribution models. Lower criteria values indicate a better dispersion.

$$AIC = -2log\ell(\tilde{x}, \alpha, \beta, \theta) + 2p$$
$$CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$
$$BIC = -2log\ell(\tilde{x}, \alpha, \beta, \theta) + plog(n)$$

The sample size is n, and the p-value reflects the number of parameters calculated from the data.

According to our preview work [23], we can say that our new modified distribution, Odd Chen Log-Logistic, fits the data better than other models.

5 Conclusion

This paper derives the Odd Chen Log-Logistic distribution from the Odd Chen Log-G family distributions. We analyzed various statistical aspects of the distribution and tried to design a parameter estimation model. We used Kolmogorov-Smirnov simulations to compare multiple estimators. To compare our model's adaptability to other distributions' correctness, this research analyses an actual COVID-19 data set. This broader spread may be useful in different study areas.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflict of Interest
The authors have no conflicts of interest to declare.