Bifurcation on 4-dimensional Canards with Hyper Catastrophe

KIYOYUKI TCHIZAWA[1], SHUYA KANAGAWA[2]

[1]Institute of Administration Engineering, Ltd.
Sotokanda 2-2-2, Chiyoda-ku, Tokyo 101-0021
JAPAN

[2]Department of Mathematics
Tokyo City University
1-28-1 Tamazutsumi, Setagaya-ku,Tokyo 158-8557
JAPAN

Abstract: In 4-dimensional slow-fast system, under the condition of ”symmetry”, there exists ”structural stability”. It is, however, of one parameter for the slow vector. On other parameters for all slow/fast vectors, it is not yet discussed still now as it is very complicated geometrical structure. In the ”Hyper catastrophe on 4-dimensional canards”, it is confirmed due to the existence of ”bifurcation”, because ”catastrophe” is a bifurcation problem itself. In the beginning of catastrophe theory, the word ”structural stability” is used for the original differential equations and not be used for the multi variable functions. In the slow-fast system having canards, what kinds of structure are there? It is used for the parameter, which depends on the existence of canards. Through this paper, it will become clear.

Key-Words: canards flying, 4-dimensional slow-fast system, hyper catastrophe

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1 Introduction

Since 4-dimensional slow-fast system is analyzed on ”hyper finite time line” in [1] and [2] or done by using ”non-standard analysis”, it is called ”Hyper Catastrophe”. In the slow-fast system which includes a very small parameter ϵ, it is difficult to do precise analysis. Thus, it is useful to get the orbits as a singular limit. When trying to do simulations, it is also faced with difficulty due to singularity. Using very small time intervals corresponding small ϵ, we shall overcome the difficulty, because the ”difference equation” on the small time interval adopts the standard ”differential equation”. These small intervals are defined on hyper finite number N, which is nonstandard. As ϵ and the intervals are linked to use $1/N$, the simulation should be done exactly. In our previous paper ”Hyper catastrophe on 4-dimensional canards” ([3]), a neuron system induced from the FitzHugh-Nagumo equation is taken up as a concrete system, but there is no simulations. In this paper, bifurcation structure having catastrophe developed by [4], [5], [6] will be described through the neuron system with simulations. As a result, a new structure on the bifurcation parameters b_3, b_4 will be provided. For more details of the catastrophe, see e.g. [7], [8], [9], [10], [11] and [12].

2 Bifurcation on Slow/Fast Vectors

Let us consider the following system extended having parameters for slow/fast vectors:

$$\begin{cases} \varepsilon \frac{dx}{dt} = h\left(\alpha x, \beta y, \epsilon\right) \\ \dfrac{dy}{dt} = g\left(\alpha x, \beta y, \epsilon\right) \end{cases}, \tag{1}$$

where $\alpha = (b_1, b_2)$, $\beta = (b_3, b_4)$.

The following is established, see [3],

$$det\left[\frac{\partial h}{\partial x}\right] = b_1 b_2 \left\{\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right\}. \tag{2}$$

In fact,

$$\frac{\partial h_1\left(b_1 x_1, b_2 x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_1} = \tag{3}$$

$$b_1 \frac{\partial h_1\left(x_1, x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_1},$$

$$\frac{\partial h_1\left(b_1 x_1, b_2 x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2} =$$

$$b_2 \frac{\partial h_1\left(x_1, x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2},$$

$$\frac{\partial h_2\left(b_1 x_1, b_2 x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2} =$$

$$b_1 \frac{\partial h_2\left(x_1, x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2},$$

$$\frac{\partial h_2\left(b_1 x_1, b_2 x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2} =$$

$$b_2 \frac{\partial h_2\left(x_1, x_2, b_3 y_1, b_4 y_2, 0\right)}{\partial x_2}.$$

Lemma 1.

$$det\left[\frac{\partial h}{\partial x}\right] = \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} = 0. \tag{4}$$

It does not depend on bifurcation parameter b_1, b_2.

Theorem 1. *If the system having parameters for slow/fast vectors is "symmetric", then it has a potential classified by [4].*

Proof. Under the rank condition $(A4)$ in [3], proceeding a projection (changing the co-ordinates):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = P \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{5}$$

like as $y_1 + y_2 = Y, x_1 + x_2 = X$, then, the potentials are obtained as "elementary catastrophe" under the following conditions.
At around $(x_0, y_0) \in PS$,

(i) $\dfrac{\partial^3 h_1}{\partial x_1^3} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0.$

(ii) $\dfrac{\partial^4 h_1}{\partial x_1^4} \neq 0,\ \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0.$

(iii) $\dfrac{\partial^5 h_1}{\partial x_1^5} \neq 0,\ \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0,\ \dfrac{\partial^4 h_1}{\partial x_1^4} = 0$.

(iv) $\dfrac{\partial^6 h_1}{\partial x_1^6} \neq 0,\ \dfrac{\partial^4 h_1}{\partial x_1^4} \neq 0,\ \dfrac{\partial^3 h_1}{\partial x_1^3} \neq 0,$
$\dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0.$

(v) $\dfrac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0,\ \dfrac{\partial^3 h_2}{\partial x_2^3} \neq 0,\ \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0,$
$\dfrac{\partial h_2}{\partial x_2} \neq 0,\ \dfrac{\partial^3 h_1}{\partial x_2^2 \partial x_1} = 0,\ \dfrac{\partial^2 h_1}{\partial x_1 \partial x_2} = 0.$

(vi) $\dfrac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0,\ \dfrac{\partial^3 h_2}{\partial x_2^3} \neq 0,\ \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0,$
$\dfrac{\partial h_1}{\partial x_2} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0.$

(vii) $\dfrac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0,\ \dfrac{\partial^4 h_2}{\partial x_2^4} \neq 0,\ \dfrac{\partial^2 h_2}{\partial x_2^2} \neq 0,$
$\dfrac{\partial^2 h_1}{\partial x_2^1} \neq 0,\ \dfrac{\partial h_2}{\partial x_2} \neq 0,\ \dfrac{\partial h_1}{\partial x_1} \neq 0.$ □

Remark 1. *As the system is "symmetric", the conditions are described exclusively, for example, the condition (1) is as the following,*

$$\frac{\partial^3 h_2}{\partial x_2^3} \neq 0, \quad \frac{\partial h_2}{\partial x_2} \neq 0.$$

Remark 2. *It is called "Hyper Catastrophe", which is composed of the potential reduced from the slow manifold ($\epsilon = 0$). Although it is using non-standard analysis, for example ϵ is infinitesimal, "Transfer Principle" ensures that it is established in standard analysis. Then, the slow manifold is obtained as the singular limit (ϵ tends to zero). They are "dynamical catastrophe" but not "statical one".*

Theorem 2. *The capital $Y = y_1 + y_2$ in the equation (5) includes bifurcation parameters α and β implicitly. Because the implicit function theorem ensures that there exists a function $y = \phi(x, \alpha, \beta)$ where $y = (y_i = \phi_i(x, b_1, b_2, b_3, b_4))(i = 1, 2)$ by taking $\epsilon = 0$. Then, the corresponding capital X axis, which satisfies $Y = 0$, sometimes changes the sign, i.e., switching the direction. It causes another bifurcation.*

3 Concrete Example

Consider the equation, for $0 \leqslant t \leqslant T$

$$\begin{cases} \varepsilon \dfrac{dx_1}{dt} = b_2 x_2 + b_3 y_1 - \dfrac{b_1^3 x_1^3}{3} \\ \varepsilon \dfrac{dx_2}{dt} = b_1 x_1 + b_4 y_2 - \dfrac{b_2^3 x_2^3}{3} \\ \dfrac{dy_1}{dt} = -\dfrac{1}{c}\left(b_1 x_1 + b_3 y_1\right) \\ \dfrac{dy_2}{dt} = -\dfrac{1}{c}\left(b_2 x_2 + b_4 y_2\right) \end{cases}, \tag{6}$$

where $b_i (i = 1, ..., 4)$ are bifurcation parameters and c is a positive constant. Changing the coordinates by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{3b_3}{b_1^3} & \frac{3b_4}{b_2^3} \end{pmatrix}, \tag{7}$$

like as

$$\begin{cases} x_1 + x_2 = X \\ \dfrac{3b_3}{b_1^3} y_1 + \dfrac{3b_4}{b_2^3} y_2 = Y \end{cases}, \tag{8}$$

then on the axis X, getting the function

$$Y = x_1^3 + x_2^3 - \frac{3b_2}{b_1^3}x_2 - \frac{3b_1}{b_2^3}x_1. \tag{9}$$

This potential function is "hyperbolic umbilic" classified by [4].

Using variables X, Y, when satisfying $b_1 = b_2$,

$$Y = X^3 - kX + O(X^3) \quad \text{and} \quad k = \frac{3}{b_1^2}. \tag{10}$$

Remark 3. *One of the pseudo singular point $(x_0, y_0) = (1, -1)$ is structurally stable, and the other one is $(1, 1)$ is unstable. At around the pseudo singular point, the above function keeps approximation. In [3], p201, the equations (17), (18), (19) are mistyped, and the above form is correct one.*

Theorem 3. *The system induced from FitzHugh-Nagumo equation has "Hyper catastrophe" at around the pseudo singular point. The multi variable function does not depend on the parameter b_4 but depends on b_1, and b_2. If $b_4 < -3/4$ the pseudo singular point is saddle, and it is structurally stable. If $-3/4 < b_4 < 0$ it is node, which is unstable. In case $b_4 > 0$, they are unstable.*

Theorem 4. *The changing coordinates in (8) includes another bifurcation structure with respect to the parameters b_3 and b_4.*

Proof. On the capital X axis, that is $Y = 0$, there exist two cases $\frac{3b_3}{b_1^3}\frac{3b_4}{b_2^3} > 0$ $(or < 0)$. $\square$

Remark 4. *Take a notice that $y_1 = \frac{b_1^3}{3b_3}x_1^3 - \frac{b_2}{b_3}x_2$, and $y_2 = \frac{b_2^3}{3b_4}x_2^3 - \frac{b_1}{b_4}x_1$ are satisfied.*

Remark 5. *The potential function depends on the parameters b_1, and b_2. It implies that the value of b_4 may take some constant on the function. The condition on b_4 is ensured in [3].*

4 Simulation results induced from FitzHugh-Nagumo equations

In this section, let us provide computer simulations for the equation (6) using the following difference equation. For $t_k = k\Delta t, \quad k = 1, 2, \cdots, NT$,

$$\begin{cases} \varepsilon\{x_1(t_k) - x_1(t_{k-1})\} \\ = \left\{ b_2x_2(t_{k-1}) + b_3y_1(t_{k-1}) - \frac{b_1x_1(t_{k-1})^3}{3} \right\}\Delta t \\ \varepsilon\{x_2(t_k) - x_2(t_{k-1})\} \\ = \left\{ b_1x_1(t_{k-1}) + b_4y_2(t_{k-1}) - \frac{b_2x_2(t_{k-1})^3}{3} \right\}\Delta t \\ y_1(t_k) - y_1(t_{k-1}) \\ = -\frac{1}{c}\{b_1x_1(t_{k-1}) + b_3y_1(t_{k-1})\}\Delta t \\ y_2(t_k) - y_2(t_{k-1}) \\ = -\frac{1}{c}\{b_2x_2(t_{k-1}) + b_4y_2(t_{k-1})\}\Delta t \end{cases}, \tag{11}$$

where $\Delta t = 1/N$ and N is a hyper number in the sense of nonstandard. When doing simulations N takes standard number in the equation (11).

In Figure 1, Figure 2, Figure 3, Figure 4 and Figure 5 in the Appendix, the line $x_1 = x_2$ is an invariant manifold and two red points are pseudo singular points. Furthermore, $\varepsilon = 0.01$, $c = 1$ and $\Delta t = 0.0001$ in (11). The curves, which satisfy $x_1x_2 = 1$ and $x_1x_2 = -1$, respectively, are Pli set.

Figure 1. Figure 1 in Appendix shows an orbit of $\{(x_1(t), x_2(t)), \ 0 \le t \le T = 5\}$ satisfying the equation (11) with $b_1 = b_2 = b_3 = b_4 = 1$ and starting from $(x_1(t_0), x_2(t_0)) = (0.5, 1.5)$ near the pseudo singular point $\left(\sqrt{\frac{1}{2}(3-\sqrt{5})}, \sqrt{\frac{1}{2}(3+\sqrt{5})}\right)$. The orbit converges to the invariant manifold $x_1 = x_2$. Then, corresponding potential is $Y = x_1^3 + x_2^3 - 3x_1 - 3x_2$.

Figure 2. Figure 2 in Appendix shows an orbit of $\{(x_1(t), x_2(t)), \ 0 \le t \le T = 5\}$ satisfying the equation (11) with $b_1 = 1.4, b_2 = 0.6, b_3 = 0.6, b_4 = 1.4$ and starting from $(0.5, 1.5)$ near the pseudo singular point $\left(\sqrt{\frac{1}{2}(3-\sqrt{5})}, \sqrt{\frac{1}{2}(3+\sqrt{5})}\right)$. From Figure 2 (Appendix) we observe that the orbit converges to the invariant manifold $x_1 = x_2$.

Figure 3. Figure 3 in Appendix shows an orbit of $\{(x_1(t), x_2(t)), \ 0 \le t \le T = 5\}$ satisfying the equation (11) with $b_1 = 0.6, b_2 = 0.8, b_3 = -0.6, b_4 = -0.8$ and starting from $(-1.2, 1.2)$ near the pseudo singular point.

Figure 4. Figure 4 in Appendix shows an orbit

of $\{(x_1(t), x_2(t)),\ 0 \leq t \leq T = 5\}$ satisfying the equation (11) with $b_1 = 1.4, b_2 = 0.8, b_3 = 0.1, b_4 = 0.8$ and starting from $(0.8, 1.5)$ near the pseudo singular point. From Figure 4 (Appendix) we observe that the orbit converges to the invariant manifold $x_1 = x_2$.

Figure 5. Figure 5 in Appendix shows an orbit of $\{(x_1(t), x_2(t)),\ 0 \leq t \leq T = 5\}$ satisfying the equation (11) with $b_1 = 0.6, b_2 = 0.8, b_3 = -0.1, b_4 = -0.8$ and starting from $(1.2, -1.2)$ near the pseudo singular point.

5 Conclusion

In the system induced from the FitzHugh-Nagumo equation, when $b_1 = b_2 = b_3 = 1$ it is composed of only one parameter b_4. Then, this state is quite the same as the system in [3]. Bifurcation problem on 4-dimensional canards makes its appearance through constructing "Hyper catastrophe", which is a dynamical model, not a statical one. Notice that there is no parameter b_4 in the multi variable function but it is fixed. Notice that Figure 3 and Figure 5 (Appendix), which satisfy $b_3 < 0$, $b_4 < 0$, provide a new jumping direction along $x_2 = -x_1$. The parameters b_3, b_4 give a new bifurcation along the orthogonal complement of the invariant set different from our previous paper. When satisfying $b_3 = 1$, b_4 changes the positive sign to negative one, corresponding canards are flying on the function.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

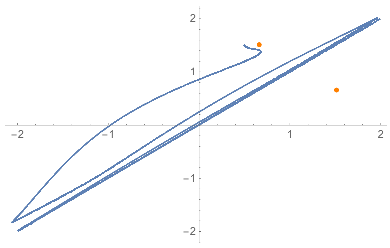

Figure 1: $(x_1(0), x_2(0)) = (0.5, 1.5)$, $b_1 = b_2 = b_3 = b_4 = 1$

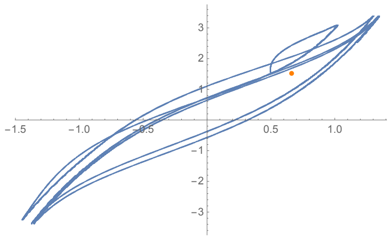

Figure 2: $(x_1(0), x_2(0)) = (0.5, 1.5), b_1 = 1.4, b_2 = 0.6, b_3 = 0.6, b_4 = 1.4$

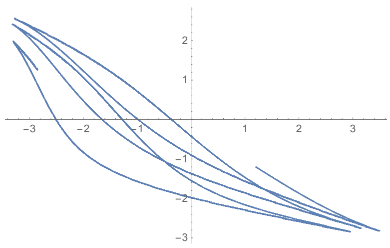

Figure 3: $(x_1(0), x_2(0)) = (1.2, -1.2)$, $b_1 = 0.6, b_2 = 0.8, b_3 = -0.6, b_4 = -0.8$

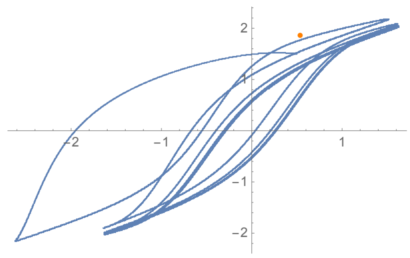

Figure 4: $(x_1(0), x_2(0)) = (0.5, 1.5), b_1 = 1.4, b_2 = 0.8, b_3 = 0.1, b_4 = 0.8$

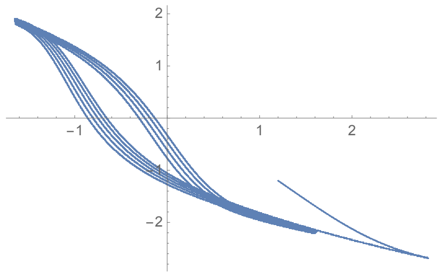

Figure 5: $(x_1(0), x_2(0)) = (1.2, -1.2)$, $b_1 = 0.6, b_2 = 0.8, b_3 = -0.1, b_4 = -0.8$