Certain Subordination and Superordination Properties of Analytic Functions

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Abstract: - In this paper, we extract some subordination and Superordination properties using the characteristics of the generalized byproduct operator. The article aims to demonstrate some applications of the differential subordination concept to univalent function subclasses that contain specific convolutions as operators. During this time, several highly complex mathematical detectives have emerged, including Riemann, Cauchy, Gauss, Euler, and several others. Geometric function theory combines or involves geometry and analysis. The main objectives of the paper above are to investigate the dependence principle and to introduce an extra subset over polyvalent functions through a further operator related to higher-order derivative products. The results were important when taking into account the numerous geometric characteristics, including radii over stiffness, close-to-convexity, and convexity; value estimation; deformation and expansion bounds; and so on.

Key-Words: - Derivative Operator, Convex function, Subordination, Superordination, Univalent Function, Analytic Functions.

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1 Introduction

A lot of investigators are interested in this concept because it is one of among the most fascinating subjects in geometric function theory. Analytical univalent and multivalent functions are holomorphic and meromorphic functions. This paper is dedicated solely to the study of meromorphic functions, [1], [2], [3], [4]. The theory of univalent as well as multivalent functions has emerged as one of the most important areas to research in the area that links geometry alongside analysis. The aim of the research is to become capable to look into a new class of multivalent functions called an unusual linear operator and begin studying the new linear operator with the aid of the product, which stands for the basic higher-order products resulting from subordination for various functions, [4], [5]. Using the universal the characteristics of the broader lead operator as well as the hypergeometric function, various solutions are going to be obtained as a more complex product of deviations and subordination inside the open unit disk. Very intriguing findings are found regarding harmonious multivalent functions that are determined by operators with differential properties. The idea of shift subordination is connected to the study of the univalent function subclass, [6], [7]. We examined some consequences of both subordination and Superordination, including a class devoted to the field of univalent meromorphic functions on a large unit disc.

The goal of the paper is to show some applications of the notion of differential subordination to subclasses of univalent functions that have certain convolutions as operators. The aforementioned paper's primary goals are to explore the dependence principle and introduce an additional subset over polyvalent functions via an additional operator associated with higher-order derivative products.

The following are some of the most significant scientific applications that can be researched: communications (hide function), electrostatic fields, magnetism, heat flow, fluid mechanics, and gravity.

If the functions k and h are analytic functions in $\Lambda = \{t \in \mathbb{C} : |t| < 1\}$ be an open unit disc in $\mathbb{C}$ as well as $\overline{\Lambda} = \{\mathrm{t} \in \mathbb{C} : |\mathrm{t}| \leq 1\}$. The class of analytic functions in Λ is denoted by $\mathcal{H}(\Lambda)$, as well as the subclass of $\mathcal{H}(\Lambda)$ of the kind represented by $\mathcal{H}[e, l]$ $k(t) = e + e_l t^l + e_{l+1} t^{l+1} + \cdots$ where $e \in \mathbb{C}$ and $l \in \mathbb{N}$.

A function k is subordinate to h and h is called superordinate to k if a function is available $j \in \mho = \{j(0) = 0 \text{ and } |j(t)| < 1, s.t. j \in \mathcal{A}\}$, the Schwarz function, so $k(t) = h(j(t))$.

In case, we compose a letter $k(t) \prec h(t)$ where $t \in \Lambda$. There would then be equivalency if h is univalent in Λ.

$k(0) = h(0) \Leftrightarrow k(t) \prec h(t)$ with $k(\Lambda) \subset h(\Lambda)$. If $k(t)$ is one of $\mathcal{A}$, then

$$k(t) = t + \sum_{l=2}^{\infty} e_l t^l \quad \text{where } (t \in \Lambda) \qquad (1)$$

The differential subordinations approach was proposed by Miller and Mocanu in 1978, along with the theory started to take shape in 1981.

Allow $\mho, \Omega \in \mathbb{C}$, with $\Delta: \mathbb{C}^3 \times \Lambda \to \mathbb{C}$ and g is univalent. If function d is analytic with $d(0) = e$ and broad generalizations of inference:

$$\mho \supset [\Delta\{d(t), td'(t), td''(t); t\}] \Longrightarrow d(\Lambda) \subset \Omega,$$

In addition to satisfies the differential subordination:

$$\Delta\{d(t), td'(t), td''(t); t\} \prec g(t) \qquad (2)$$

If d and $\Upsilon(d(t), td'(t), td''(t); t)$ are univalent in Λ in addition to satisfies the differential subordination:

$$g(t) \prec \Delta\{d(t), td'(t), td''(t); t\} \qquad (3)$$

by Eq. (3), $\mho \subset [\Delta\{d(t), td'(t), td''(t); t\}]$.

An explanation of the paper's structure is provided below. We cover some fundamental definitions in complex analysis in section two, along with some examples and fundamental ideas in geometric function theory. In the third section, we discuss results concerning the linear operator.

2 Definitions and Lemmas

In this section, the definitions you need in the theorems are mentioned, a new operator is mentioned, and the special cases obtained are mentioned.

Definition 2.1: [8] Allow W the set of analytic as well as injective functions q on $\overline{\Lambda}/E(w)$, so that

$$\overline{\Lambda} = \Lambda \cup \{t \in \partial\Lambda\} \qquad (4)$$

In addition to $E(w) = \left\{x \in \partial\Lambda\, ; \lim_{t \to x} w(t) = \infty\right\}$ such that $w'(x) \neq 0$ for $x \in \partial\Lambda/E(w)$. $W(e)$ is subclass of W for which $w(0) = e$.

Definition 2.2: [9] Allow $\mho$ be a set in $\mathbb{C}$ and this functions $\Delta: \mathbb{C}^3 \times \Lambda \to \mathbb{C}$ that satisfy the admissibility condition $\Delta(r, p, q; x) \in \mho$, in addition to $p = \frac{tw'(t)}{s}$, $r = w(t)$, $w \in \mathcal{H}[e, l]$, $t \in \Lambda$ with $w'(t) \neq 0$. The class of admissible functions $\Upsilon'[\mho, w]$ and $\Re e\left\{1 + \frac{q}{p}\right\} \leq \frac{1}{s} \Re e\left\{1 + \frac{tw''(t)}{w'(t)}\right\}$, for $t \in \Lambda, x \in \partial\Lambda$ and $s \geq l \geq 1$.

Definition 2.3: [10] The class of admissible functions $\Upsilon_l[\mho, w]$ consist of functions $\Delta: \mathbb{C}^3 \times \Lambda \to$

$\mathbb{C}$ that satisfy the admissibility condition $\Delta(r,p,q;\,t) \notin \mho$, in addition to $r = w(x)$, $p = yxw(x)$, $\Re e\left\{1+\frac{q}{p}\right\} \geq y\,\Re e\left\{1+\frac{xw''(x)}{w'(x)}\right\}$, so that $t \in \Lambda$, $x \in \partial\Lambda / E(w), y \geq l$. Let $\mho$ be a set in $\mathbb{C}$, $w(t) \in W$ and l be a positive integer. In particular $w(t) = Y\frac{Yt+e}{Y+\bar{e}t}, Y > 0$ as well as $|e| < Y$, then $w(\Lambda) = \Lambda Y = \{j : |j| < Y\}, w(0) = e,\ E(w) = \emptyset$ and $w \in W$.
In this case, we set $\Upsilon_l[\mho, Y, e] = \Upsilon[\mho, w]$, and in the special case when $\mho = \Lambda Y$, the class is simply denoted by $\Upsilon_l[Y, \alpha]$.

Definition 2.4: [11] A derivative operator k, which we apply to the function $k \in \mathcal{A}$ in cases where $0 \leq \beta_1 \leq \beta_2$ and v is a fixed positive natural number

$$Y^{v}_{\beta_1,\beta_2}k(\mathrm{t}) = t + \sum_{l=2}^{\infty}\left(\frac{1+\beta_2(l-1)}{1+(\beta_1+\beta_2)(l-1)}\right)^{v+1} e_l t^l \qquad (5)$$

By Eq. (5), we obtain

$$t\left(Y^{v}_{\beta_1,\beta_2}k(\mathrm{t})\right)' = (\beta_1+\beta_2)Y^{v}_{\beta_1,\beta_2}k(\mathrm{t}) - \left((\beta_1+\beta_2)+1\right)Y^{v+1}_{\beta_1,\beta_2}k(\mathrm{t}) \qquad (6)$$

Definition 2.5: [12] The generalized derivative operator $\mathrm{D}^{v,s}_{\lambda,\eta}: \mathcal{A} \to \mathcal{A}$ for $k \in \mathcal{A}$ is described by

$$D^{v,s}_{\lambda,\eta}k(t) = t + \sum_{l=2}^{\infty}\frac{\left(1+\eta(l-1)\right)^{v-1}}{\left(1+\lambda(l-1)\right)^{v}}\binom{s+l-1}{s} e_l t^l \qquad (7)$$

Where $, v \in \mathbb{N}_0, \eta \geq \lambda \geq 0$.
Clearly visible is the fact that

$$\mathrm{D}^{0,0}_{\lambda,0}k(t) = \mathrm{D}^{1,0}_{0,\eta}k(t) = k(t), \text{ and}$$

$$\mathrm{D}^{1,0}_{\lambda,0}k(t) = \mathrm{D}^{1,1}_{0,\eta}k(t) = tk'(t).$$

and, $\mathrm{D}^{b-1,0}_{\lambda,0}k(t) = \mathrm{D}^{1,b-1}_{0,\eta}k(t),\ (b = 1,2,3,\ldots)$.
We are able to confirm that:

$$t\left(\mathrm{D}^{v,s}_{\lambda,\eta}k(t)\right)' = (1+\mathrm{s})\mathrm{D}^{v,s}_{\lambda,\eta}k(t) + s\left(\mathrm{D}^{v,s+1}_{\lambda,\eta}k(t)\right) \qquad (8)$$

Definition 2.6: By use the Hadamard product of the derivative operator $Y^{v}_{\beta_1,\beta_2}$ and the generalized operator $\mathrm{D}^{v,s}_{\lambda,\eta}$ and for $k \in \mathcal{A}$ the operator $DY^{v,\lambda}_{\beta,\eta}: \mathcal{A} \to \mathcal{A}$ is defined
$DY^{v,\lambda}_{\beta,\eta}k(t) = \left(Y^{v}_{\beta_1,\beta_2} * \mathrm{D}^{v,s}_{\lambda,\eta}\right)k(t)$ where $z \in U$, and

$$DY^{v,\lambda}_{\beta,\eta}k(t) = t + \sum_{l=2}^{\infty}\left(\frac{1}{1+\eta(l-1)}\right)^{n}\left(\frac{1+\beta_2(l-1)}{1+\lambda(\beta_1+\beta_2)(l-1)}\right)^{v}\binom{s+l-1}{s} e_l t^l \quad (9)$$

so that $n = v-1$ and $v, s \in \mathbb{N}_0, 0 \leq \beta_1 \leq \beta_2, \eta \geq \lambda \geq 0$.
By Eq. (9),

$$t\left(Y^{v,\lambda}_{\beta,\eta}k(\mathrm{t})\right)' = s\beta Y^{v+1,\lambda}_{\beta,\eta}k(\mathrm{t}) - (s\beta+1)Y^{v,\lambda}_{\beta,\eta}k(\mathrm{t}) \qquad (10)$$

As an alternative, consider a few special cases involving operator $DY^{v}_{\beta_1,\beta_2}$.

1. The generalized derived operator $\mathrm{D}^{v,s}_{\lambda,\eta}$ is present if $Y^{v}_{1,0} = 1$ by [13], [14], [15].
   - **a.** The Srivastava-Attiya derivative operator [16] introduces $\mathrm{D}^{v,s}_{0,0}$, which is what happens if $\lambda = 0, \eta = 0$ and $\mathrm{D}^{v,s}_{\lambda,\eta}$ reduces to $\mathrm{D}^{v,s}_{0,0}$.
   - **b.** The Salagean derivative operator [17], [18], [19] introduces $\mathrm{D}^{v,0}_{1,0}$, which is what happens if $\lambda = 1, \eta = 0, s = 0$ and $\mathrm{D}^{v,s}_{\lambda,\eta}$ reduces to $\mathrm{D}^{v,0}_{1,0}$.
   - **c.** The Generalized Salagean derivative operator introduced by [20] introduces $\mathrm{D}^{v,0}_{\lambda,0}$, which is what happens if $\eta = 0, s = 0$ and $\mathrm{D}^{v,s}_{\lambda,\eta}$ reduces to $\mathrm{D}^{v,0}_{\lambda,0}$.
2. The derivative operator $Y^{v}_{\beta_1,\beta_2}$ is present if $\mathrm{D}^{v,0}_{1,0} = 1$ by [21].
   - **a.** [22] introduces $Y^{v}_{\beta_1}$, which is what happens if $\beta_2 = 0$ and $Y^{v}_{\beta_1,\beta_2}$ reduces to $Y^{v}_{\beta_1}$.
   - **b.** [23], [24], [25] introduces Y^{v}, which is what happens if $\beta_1 = 1, \beta_2 = 0$ and $Y^{v}_{\beta_1,\beta_2}$ reduces to Y^{v}.

Lemma 2.1: [26] If $d \in W(e)$ with $\Delta \in \Upsilon_l[\mho, w]$ and $\Delta\{d(t), td'(t), td''(t); t\}$ is univalent in Λ, then

$$\mho\ \Delta(d(t), td'(t), td''(t); t) \qquad (11)$$

Implies $w(t) \prec d(t)$.

Lemma 2.2: [26] Let $\Delta \in \Upsilon_l[\mho, w]$ wtih $w(0) = e$. If the analytic function

$$d(t) = e + d_l t^l + d_{l+1}t^{l+1} + \ldots \quad (12)$$

$\mathrm{t} \in \Lambda$, satisfies the following inclusion relationship

$$\Delta(d(t), td'(t), td''(t); t) \in \mho \qquad (13)$$

Then $d(t) \prec w(t), t \in \Lambda$.

3 Findings Related to the Linear Operator

In the field of computational science, an operator which satisfies specific requirements, like having a closed a subspace and adhering to particular equations, and keeps linear combinations is called a linear operator $DY^{v,\lambda}_{\beta,\eta}$. Results related to the linear operator were obtained and are discussed in this section.

Definition 3.1: Allow $\mho \in \mathbb{C}, w \in W_0 \cap \mathcal{H}[0,1]$ and class of admissible functions $\Phi_l[\mho, w]$ consist of this functions $\phi : \mathbb{C}^3 \times \Lambda \to \mathbb{C}$ in addition to satisfy the admissibility condition $\phi(u,v,j;t;x) \notin \mho$, so that:

$$u = w(x),$$
$$v = \frac{yxw'(x) + 2(s\beta-2)w(x)}{(s\beta)^2},$$
$$j = \frac{y^2x^2w''(x) + 4yx(s\beta-3)w'(x) + (s\beta-2)^2w(x)}{(s^2\beta^2)^3}$$

and

$$\Re e\left\{\frac{s^2\beta^2 j + 4(s\beta-3)s\beta v + 6(s\beta-2)^2u}{s\beta v + (s\beta-2)u}\right\} \geq y\Re e\left\{1 + \frac{xw''(x)}{w'(x)}\right\},$$

Where $x \epsilon \partial\Lambda/E(w)$, and $\beta, s \geq 0$.

Theorem 3.1: If $k \in \mathcal{A}$ and $\phi \in \Phi_l[\mho, w]$ satisfies

$$\left\{\phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) : t \in \Lambda\right\} \subset \mho \quad (14)$$

hence,

$$DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t).$$

Proof. By Eq. (5) in addition to Eq. (6),

$$DY_{\beta,\eta}^{v+1,\lambda}k(t) = \frac{t\left(DY_{\beta,\eta}^{v,\lambda}k(t)\right)' + 2(s\beta-2)DY_{\beta,\eta}^{v,\lambda}k(t)}{(s\beta)^2} \quad (15)$$

and $h(t) \in \Lambda$ define

$$h(t) = DY_{\beta,\eta}^{v,\lambda}k(t) \quad (16)$$

Hence,

$$DY_{\beta,\eta}^{v+1,\lambda}k(t) = \frac{th'(t) + 2(s\beta-2)h(t)}{(s\beta)^2} \quad (17)$$

In light of that

$$Y_{\beta_1,\beta_2}^{c+2}k(t) = \frac{t^2h''(t) + (4s\beta-3)th'(t) + (s\beta-2)^2h(t)}{(s^2\beta^2)^3} \quad (18)$$

Describe the conversion of $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,p,q) = r,$$
$$v(r,p,q) = \frac{p + 2(s\beta-2)r}{(s\beta)^2},$$
$$j(r,p,q) = \frac{q + (4s\beta-6)p + (s\beta-2)^2r}{(s^2\beta^2)^3} \quad (19)$$

Let

$$\Delta(r,p,q;t) = \phi\left(r, \frac{p + 2(s\beta-2)r}{(s\beta)^2}, \frac{q + (4s\beta-6)p + (s\beta-2)^2r}{(s^2\beta^2)^3}; t\right) = \phi(u,v,j;t) \quad (20)$$

By use of Lemma 2.1 and Eqs. (16), (17) and (18), from Eq. (20), we get:

$$\Delta(d(t), td'(t), td''(t); t) = \phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) \quad (21)$$

and by Eq. (13), obtaining

$$\Delta(d(t), td'(t), td''(t); t) \in \mho \quad (22)$$

See that

$$1 + \frac{q}{p} = \frac{s^2\beta^2 j + 4(s\beta-3)s\beta v - 6(s\beta-2)^2u}{s\beta v + (s\beta-2)u},$$

since Lemma 2.2. and $\Delta \in \Upsilon_l[\mho, w]$

$$DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t).$$

Theorem 3.2: If $k \in \mathcal{A}$ and $\phi \in \Phi_l[g, w]$ satisfies

$$\phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) \prec g(t) \quad (23)$$

Therefore, $DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t)$.

The following is an expansion of Theorem 3.2 to include the situation where $w(t)$ is behavior on $\partial\Lambda$ is unknown. For the following result, the best dominant of the differential subordination is required Eq. (23).

Theorem 3.3: If $k(t) \in \mathcal{A}$, $\mho \in \mathbb{C}$ and allow $w(t)$ be univalent in Λ in addition to $w_\rho(t) = w(\rho t)$ with $w(0) = 0$ and let $\phi \in \Phi_l[\mho, w_\rho]$ where $\rho \in (0,1)$ satisfies

$$\phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) \in \mho,$$

then $DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t)$.

Proof: By use Theorem 3.1., $DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(\rho t)$ the conclusion that can now be drawn from the subsequent subordination relationship $w_\rho(t) \prec w(t)$.

Definition 3.2. If class of admissible functions $\Phi_l[\mho, Y]$ of functions $\phi : \mathbb{C}^3 \times \Lambda \to \mathbb{C}$ in addition to satisfy the admissibility condition:

$$\phi\left(Ye^{i\theta}, \frac{k + (s\beta+2)Ye^{i\theta}}{(s\beta)^2}, \frac{L + [(4s\beta+3)k + (s\beta+2)^2]Ye^{i\theta}}{(s^2\beta^2)^2}; t\right) \notin \mho \quad (24)$$

so that $\mho$ be a set in $\mathbb{C}$, $\Re\{c\} > 0$, $\beta \geq 1$ and $Y > 0$, $\theta \in R$, $R\left(Le^{i\theta}\right) \geq k(k-1)Y$ for all real θ, $k \geq 1$ and $t \in \Lambda$.

Corollary 3.1. If $k(t) \in \mathcal{A}$ and $\phi \in \Phi_l[Y]$ satisfies

$$\phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) \in \mho,$$

so that $Y > 0$ and $t \in \Lambda$. Then $\left|DY_{\beta,\eta}^{v,\lambda}k(t)\right| < Y$.

Proof: By using Theorem 3.1., obtaining

$$DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t) = Yt, \text{and}$$
$$DY_{\beta,\eta}^{v,\lambda}k(t) \prec w(t) = Yj(t).$$

Hence,

$$\left|DY_{\beta,\eta}^{v,\lambda}k(t)\right| < Y, \text{ so that } |j(t)| < 1.$$

Corollary 3.2. Let $k(t) \in \mathcal{A}$ and $\phi \in \Phi_l[\mho, Y]$ satisfy inclusion relationship

$$\phi\left(DY_{\beta,\eta}^{v,\lambda}k(t), DY_{\beta,\eta}^{v+1,\lambda}k(t), DY_{\beta,\eta}^{v+2,\lambda}k(t); t\right) \in \mho,$$

so that $t \in \Lambda$ and $Y > 0$. Then $DY_{\beta,\eta}^{v,\lambda}k(t) \prec Yt$.

4 Conclusions

In the present work, some applications of the notion of different subordination to subclasses of univalent functions which employ specific transformations as operators are found. The geometric properties of these kinds of functions were studied including, among other things, convexity radii, starlikeness, distortion theorem, as well as coefficient bounds. We investigated both the essential operator as well as extreme points. Here, we investigated some characteristics related to the differences in subordination of analytical univalent functions in Λ. We also used the characteristics of the more general result operator to determine some facets of subordination as well as Superordination. Utilizing a broader hypergeometric function and the convexity operator, several results on subordination in Λ. Applying the principle of dependence and introducing a new subset of polyvalent functions via an additional operator associated with higher order derivative products. The applications of science that are most relevant to study are fluid mechanics, communications (hide function), magnetism, gravity, electrostatic fields, and heat flow.

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