Stability Analysis of Matrices and Single-Parameter Matrix Families on Special Regions

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Abstract: This paper proposes an efficient method for determining the $\mathcal{D}$-stability of matrices using Kelley's cutting-plane method. The study examines the $\mathcal{D}$-stability of matrices in symmetric regions of the complex plane, defined by quadratic matrix inequalities (QMI) and polynomial functions. Using Kelley's cutting-plane method, we present an iterative algorithm for determining the $\mathcal{D}$-stability of a given matrix. Further, the robustness of single-parameter matrix families is analyzed, and a method is proposed for considering their $\mathcal{D}$-stability. Through examples, we indicate the possible applicability of the proposed approach in addressing stability problems in linear systems.

Key-Words: $\mathcal{D}$-Stability, Single-Parameter Matrix Families, Robust Stability, QMI region, Linear Matrix Inequality.

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1 Introduction

In this study, we have developed an algorithm based on Kelley's cutting plane method to determine the $\mathcal{D}$-stability of a matrix. This algorithm aims to present a different approach to stability analysis techniques. Our method solves a matrix inequality using simple iterative operations without requiring direct eigenvalue calculations and the cases of eigenvalue inclusion in set $\mathcal{D}$. In recent studies, various methods have been proposed to address stability problems in linear systems using advanced optimization techniques and Linear Matrix Inequalities (LMIs), [1], [2], [3], [4], [5], [6]. The techniques discussed can also be used to solve robust stability problems for a family of matrices. This study focuses on the stability region defined by second-order matrix inequalities (QMIs) and polynomial functions. Our method addresses the problem of determining the stability of a matrix within these regions and the robust stability of a matrix segment. We tested our method on matrices and tried to illustrate the effectiveness of the proposed algorithm using examples.

Determining whether a matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{D}$-stable is a critical problem in control theory, where $\mathcal{D}$ is a symmetric region of the complex plane $\mathbb{C}$.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be $\mathcal{D}$-stable if all its eigenvalues lie within a specified region $\mathcal{D} \subset \mathbb{C}$, [7]. For instance, a matrix is called Hurwitz stable if all its eigenvalues are in the open left half-plane, i.e., $\mathcal{D} = \{z \in \mathbb{C} : \text{Re}(z) < 0\}$. Hurwitz stability of a matrix A provides that the continuous-time linear system described by $\dot{x} = Ax$ is asymptotically stable. Similarly, a matrix is called Schur stable if all its eigenvalues lie inside the open unit disk, i.e., $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$. This stability guarantees that the discrete-time linear system described by $x_{k+1} = Ax_k$ is asymptotically stable. Likewise, a matrix is called sector stable if all its eigenvalues lie within a specified sector in the complex plane. Particularly, for an angle θ with $0 \leq \theta \leq \pi$, the sector is defined as $\mathcal{D} = \{z \in \mathbb{C} : \pi - \theta < \arg(z) < \pi + \theta\}$. When the arguments of the eigenvalues of the system matrix are restricted, it gives rise to the sector stability problem of that matrix.

In [8], the stability region

$$\mathcal{D} = \{z \in \mathbb{C} : a + b(z + \bar{z}) + cz\bar{z} < 0\}$$

is considered, where a, b, and c are real numbers, $b \geq 0$, $c \geq 0$. The authors propose that the problem can be reduced to the positivity of two specially constructed determinants in the continuous case. In contrast, in the discrete-time case, three determinants are required.

Results on the $\mathcal{D}$-stability of matrices for regions defined by linear matrix inequalities (LMIs) are provided in the study, [3]. Specifically, a region $\mathcal{D}$ of the complex plane is defined by

$$\mathcal{D} = \left\{z \in \mathbb{C} : \; Q + Sz + S^T\bar{z} + Rz\bar{z} < 0\right\} \quad (1)$$

for some symmetric matrix $Q \in \mathbb{R}^{m\times m}$, a general matrix $S \in \mathbb{R}^{m\times m}$, and a symmetric positive semi-definite $R \geq 0 \in \mathbb{R}^{m\times m}$. This is called a quadratic matrix inequality region (QMI) of order m.

QMI regions possess crucial characteristics such as being open, convex, and symmetrical concerning the real axis. Moreover, since the intersection of any QMI region results in another QMI region, it is possible to approximate practically any convex region in the complex plane using a QMI region. The following theorem ascertains a criterion for the $\mathcal{D}$-stability of a matrix $A \in \mathbb{R}^{n\times n}$.

Theorem 1 ([3]). *A matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{D}$-stable (1) if and only if there exists a symmetric positive definite matrix $P > 0$ of dimension $n \times n$ such that*

$$Q\otimes P+S\otimes(AP)+S^T\otimes(PA^T)+R\otimes(APA^T) < 0. \quad (2)$$

Here, the symbol '$\otimes$' stands for the Kronecker product of matrices.

Another region of stability that draws attention is the region defined by polynomials with real coefficients in complex variables. In [9], the stability of a matrix for such defined regions has been characterized. In [9], the region

$$\mathcal{D} := \{z \in \mathbb{C} \mid \mathrm{Re} f_i(z) < 0,\ i = 1, \ldots, N\} \quad (3)$$

is considered, where the coefficients of the polynomial $f_i(z)$ are all real. This region is open and symmetrical about the real axis. The following theorem suggests a necessary and sufficient condition for a common positive definite solution to a set of Lyapunov inequalities for the eigenvalues of a matrix to lie in $\mathcal{D}$.

Theorem 2 ([9]). *A matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{D}$-stable (3) if and only if there exists a matrix $P = P^T > 0$ satisfying*

$$\{f_i(A)\}^{\mathrm{T}} P + P f_i(A) < 0, \quad i = 1, \ldots, N. \quad (4)$$

Consider the switched system

$$\dot{x}(t) = Ax(t), \quad A \in \{A_1, A_2, \ldots, A_N\} \quad (5)$$

where $x(t) \in \mathbb{R}^n$ and $t \geq 0$. In (5), the matrix A switches among N Hurwitz stable matrices $A_1, A_2, \ldots, A_N$.

A key issue is to determine the existence of a quadratic Lyapunov function $V(x) = x^T P x$, where $P = P^T > 0$, such that:

$$A_i^T P + PA_i < 0 \text{ for all } i \in \{1, 2, \ldots, N\}. \quad (6)$$

This function $V(x)$, known as a common quadratic Lyapunov function (CQLF), guarantees the uniform asymptotic stability of the switched system.

In [10], the common $P > 0$ solution of Lyapunov inequalities given by equation (6) is investigated by Kelley's cutting-plane method.

In our research, we discuss the solution of the inequality system (4) in Theorem 2 using the method presented in [10]. We will use this method to determine the $\mathcal{D}$-stability of a matrix A, as given by equation (3). The $\mathcal{D}$-stability of matrix A for the QMI region defined by equation (1) is equivalent to the LMI problem (2) formulated for this matrix. In our study, we propose an algorithm to solve this problem.

2 Kelley's Cutting-Plane Method and LMI's

Kelley's cutting-plane method is an iterative algorithm for convex optimization problems. The method works by iteratively refining a feasible region, using linear hyperplanes to exclude regions that do not contain the optimal solution. At each iteration, a subgradient of the objective function is computed, and the algorithm checks whether the current solution encounters the stopping criterion. If not, a new hyperplane is added, and the process continues until convergence. This method is particularly useful for solving linear matrix inequalities (LMIs) in $\mathcal{D}$-stability problems.

Let $x \in \mathbb{R}^r$ be $x^T = (x_1, x_2, \ldots, x_r)$ and P be an $n \times n$ symmetric matrix defined as

$$P = P(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{pmatrix}, \quad (7)$$

with $r = n(n + 1)/2$.

For the inequalities involving the matrix P linearly, as in (2) or (4), let us define the symmetric matrices

$$M(x) := Q \otimes P + S \otimes (AP) + S^T \otimes (PA^T) + R \otimes (APA^T), \quad (8)$$

and

$$L(x, i) := \{f_i(A)\}^{\mathrm{T}} P + P f_i(A), \quad i = 1, 2, \ldots, N. \quad (9)$$

The function $\lambda_{\max}(A)$ for a symmetric matrix A represents the largest eigenvalue of the matrix A. Define

$$\phi_M(x) = \lambda_{\max}(M(x)) = \max_{\|u\|=1} u^T M(x) u. \quad (10)$$

and

$$\begin{aligned}\phi_L(x) &= \max_{1\le i\le N} \lambda_{\max}(L(x,i)) \\ &= \max_{1\le i\le N, \|u\|=1} u^T L(x,i)u.\end{aligned} \quad (11)$$

We consider two problems:

1) Is there $P > 0$ such that $M(x) < 0$ (referred to as (2))? If there exists a point $\tilde{x}$ such that $P(\tilde{x}) > 0$ and $\phi_M(\tilde{x}) < 0$, then the matrix $P(\tilde{x})$ is the solution we are pursuing.
2) Is there a common matrix $P > 0$ such that $L(x,i) < 0$ for all $i = 1, 2, \ldots, m$ (referred to as (4))? If there exists a point $\hat{x}$ such that $P(\hat{x}) > 0$ and $\phi_L(\hat{x}) < 0$, then $P(\hat{x})$ satisfies the necessary conditions.

These problems can be reduced to minimizing a convex function under convex constraints.

Consider the following convex minimization problem:

$$\begin{array}{c}\phi(x) \to \text{minimize} \\ \min_{\|v\|=1} v^T P(x) v > 0\end{array} \cdot \quad (12)$$

Assume $X \subset \mathbb{R}^n$ is a convex set and $F : X \to \mathbb{R}$ is a convex function. We say that a vector $g \in \mathbb{R}^n$ is a subgradient of $F(x)$ at $x_* \in X$ if the following inequality holds for every $x \in X$:

$$F(x) \ge F(x_*) + g^T(x - x_*).$$

The collection of all subgradients of $F(x)$ at x_* is represented by $\partial F(x_*)$. If x_* is an interior point of X, then the set $\partial F(x_*)$ is guaranteed to be nonempty and convex. The subsequent proposition follows from the theory of nondifferentiable optimization.

Proposition 3 ([10]). *Define $\phi(x)$ as follows:*

$$\phi(x) = \max_{y\in Y} f(x,y) \quad (13)$$

where Y is a compact set, and $f(x,y)$ is continuous and differentiable with respect to x. Then,

$$\partial\phi(x) = \text{conv}\left\{\frac{\partial f(x,y)}{\partial x} : y \in Y(x)\right\}$$

where $Y(x)$ is the set of all maximizing elements y in (13), that is,

$$Y(x) = \{y \in Y : f(x,y) = \phi(x)\}.$$

For a given x, if the the maximizing element is unique, i.e. $Y(x) = \{y(x)\}$ then $\phi(x)$ is differentiable at x and its gradient is

$$\nabla\phi(x) = \frac{\partial f(x, y(x))}{\partial x}$$

For the function $\phi_M(x)$ as defined in (10):

$$\partial\phi_M(x) = \text{conv}\left\{\frac{\partial}{\partial x}\left(u^T M(x) u\right) : u \text{ is a unit eigenvector corresponding to } \lambda_{\max}(M(x))\right\}. \quad (14)$$

Similarly, for the function $\phi_L(x)$ as defined in (11):

$$\partial\phi_L(x) = \text{conv}\left\{\frac{\partial}{\partial x}\left(u^T L(x,i) u\right) : i \text{ maximizes } \lambda_{\max}(L(x,i)), u \text{ is a corresponding unit eigenvector}\right\}. \quad (15)$$

If the maximizing value of i for the given x is unique and $\lambda_{\max}(L(x,i))$ is a simple eigenvalue, then the differentiability of ϕ_L at the point x is guaranteed, [11].

We examine problem (12) using Kelley's cutting-plane method

Kelley's method reformulates problem (12) as follows:

$$\begin{array}{l} c^T z \to \min \\ c_1(z) \ge 0, \\ c_2(z) \ge 0, \\ -1 \le x_i \le 1 \quad (i = 1, 2, \ldots, r) \end{array} \quad (16)$$

where $z = (x_1, x_2, \ldots, x_r, L)^T$, $c = (0, \ldots, 0, 1)^T$, $c_1(z) = L - \phi(x)$, and $c_2(z) = \min_{\|v\|=1} v^T P v$.

Let z^0 be a starting point and $z^0, z^1, \ldots, z^k$ be $k+1$ distinct points. (Here z^k denotes the point at the k-th iteration.)

At the $(k+1)$th iteration, the cutting-plane algorithm solves the following LP problem

$$\begin{array}{ll} \text{minimize} & L \\ \text{subject to} & -h_1^T(z^0)z \ge -h_1^T(z^0)z^0 - c_1(z^0) \\ & -h_2^T(z^0)z \ge -h_2^T(z^0)z^0 - c_2(z^0) \\ & \vdots \\ & -h_1^T(z^k)z \ge -h_1^T(z^k)z^k - c_1(z^k) \\ & -h_2^T(z^k)z \ge -h_2^T(z^k)z^k - c_2(z^k) \\ & -1 \le x_i \le 1 \end{array} \quad (17)$$

where $h_j(z^i)$ denotes a subgradient of $-c_j(z)$ at z^i $(j = 1, 2)$.

Let z_*^k be the minimizer of the problem (17).

If z_*^k satisfies the inequality

$$\min\{c_1(z_*^k), c_2(z_*^k)\} \ge -\varepsilon,$$

where ε is a tolerance, then z_*^k is considered an approximate solution to the problem in (16).

If this condition is not satisfied, we define j^* as the index of the most negative $c_j(z_*^k)$. Update the constraints in (17) by adding the linear constraint:

$$c_{j^*}(z^{k+1}) - h_{j^*}^T(z^{k+1})(z - z^{k+1}) \ge 0, \quad (18)$$

then repeat the process.

Remember, our objective is to find x_* such that $P(x_*) > 0$ and $\phi(x_*) < 0$, rather than solving the minimization problem (12), (16).

Theorem 4 ([10]). *If there exists a k such that:*

$$c_1(z_*^k) > L^k \quad and \quad c_2(z_*^k) > 0,$$

where $z_*^k = (x_*^k, L^k)$ is the minimizer of the problem (17), then the matrix $P = P(x_*^k)$ is a solution to (12).

This algorithm can be adapted to our problem as follows:

Algorithm 1.

(1) Select an initial point $z^0 = (x^0, L^0)^T$.
Compute $\phi(x^0)$ and $c_2(z^0)$.
If $\phi(x^0) < 0$ and $c_2(z^0) > 0$, terminate;
Otherwise, continue to the next step.

(2) Find z_*^k by solving the LP problem in (17).
If $c_1(z_*^k) > L^k$ and $c_2(z_*^k) > 0$, terminate;
Otherwise, proceed.
Set $z^{k+1} = z_*^k$, update the constraints in (17) and repeat the procedure.

We will present applications of Kelley's method to $\mathcal{D}$-stability problems.

3 Examples and Applications

We demonstrate the $\mathcal{D}$-stability of a matrix A using Algorithm 1, where the set $\mathcal{D}$ is defined as a QMI region given by equation (1) and a region defined by equation (3), respectively. In the proposed method, we calculate the subgradient of the convex function by using the maximal eigenvalue and its corresponding eigenvector. We then update the set of constraints (17) for the linear programming problem. This approach requires fewer operations than other methods, reducing computational complexity.

Example 1. Let us consider the set $\mathcal{D}$ defined by equation (2) with the following matrices (see, [3]):

$$Q = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

This $\mathcal{D} \subset \mathbb{C}$ region is as shown in Fig. 1. Given the

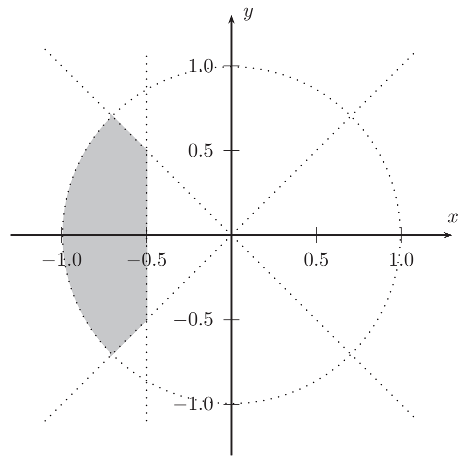

Fig. 1: Region $\mathcal{D}$ in Example 1.

matrix A as:

$$A = \begin{bmatrix} -2.369 & 5.297 & 5.225 \\ -1.351 & 2.481 & 3.445 \\ 0.684 & -1.148 & -2.112 \end{bmatrix}.$$

If we take the initial point as

$$\begin{aligned} z^0 &= (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, L^0)^T \\ &= (1, 0, 0, 1, 0, 1, 1)^T, \end{aligned}$$

then

$$P(x^0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying Algorithm 1, in the first step, we find $c_1(z^0) = -84.342$, $c_2(z^0) = 1$, and $\phi_M(x^0) = \lambda_{\text{max}}(M(x^0)) = 85.342 > 0$. Solving the LP problem (17) yields $z^1 = (-1, 1, 1, -1, -1, 0, -191.062)^T$. Calculations give the following Table 1.

Table 1. Iterative Results of Kelley's Method for Example 1

k	L^k	$c_1(z^k)$	$c_2(z^k)$
1	-19.772	-220.596	-2.732
2	-2.693	-21.071	-2.125
3	-1.353	-10.277	-0.361
⋮	⋮	⋮	⋮
32	-0.000826	-0.001011	0.000587
33	-0.000826	-0.000802	0.000526

$$\begin{aligned} z^{33} &= (x^{33}, L^{33})^T \\ &= (1, 0.322, 0.018, 1, -0.849, 0.818, -0.000826)^T. \end{aligned}$$

Since $\phi_M(x^{33}) = L^{33} - c_1(z^{33}) = -0.000024 < 0$ and $\lambda_{\text{max}}(P(x^0)) = c_2(z^{33}) = 0.000526 > 0$, the inequality (2) in Theorem 1 is satisfied for the matrix

$$P(x^{33}) = \begin{bmatrix} 1 & 0.322 & 0.018 \\ 0.322 & 1 & -0.849 \\ 0.018 & -0.849 & 0.818 \end{bmatrix},$$

which means $M(x^{33}) < 0$. Therefore, all the eigenvalues of the matrix A are within the $\mathcal{D}$ defined here, indicating that the matrix A is $\mathcal{D}$-stable.

Example 2 ([9]). Let $f_1(z) = z$, $f_2(z) = -z^2$ and $f_3(z) = -z^3$. Define

$$\mathcal{D}_1 = \{z \in \mathbb{C} : \ \mathrm{Re} f_1(z) < 0, \mathrm{Re}\ f_2(z) < 0 \},$$
$$\mathcal{D}_2 = \{z \in \mathbb{C} : \ \mathrm{Re} f_1(z) < 0, \mathrm{Re}\ f_3(z) < 0 \}.$$

Consider stability region $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ in Fig. 2.

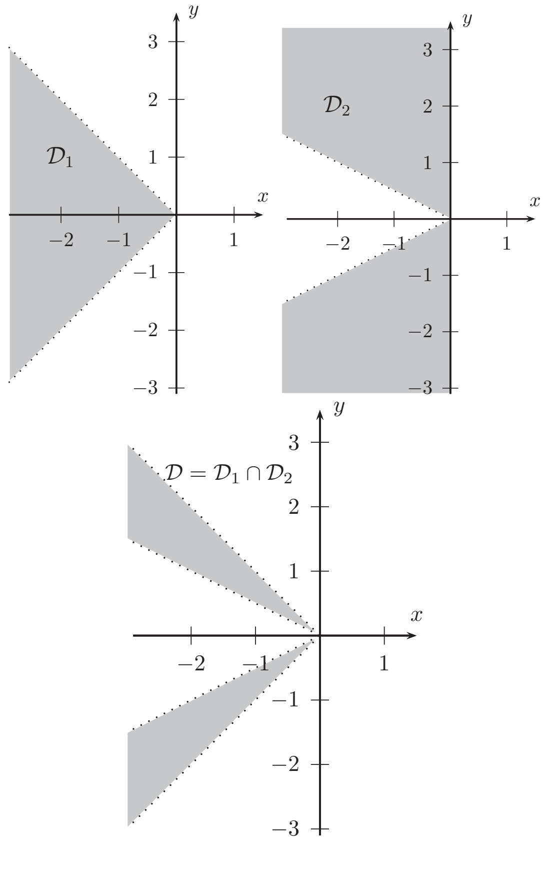

Fig. 2: Regions: $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$

In [9], the $\mathcal{D}$-stability of

$$A = \begin{bmatrix} -94.7 & -47.1 & -41.1 & -2.3 \\ 15.2 & -46.9 & 3.0 & -7.6 \\ 121.0 & 77.9 & 46.3 & 9.1 \\ -116.9 & 65.2 & -54.6 & -4.7 \end{bmatrix}$$

is considered. It has been solved with an algorithm based on the Schur decomposition method. Here we solve the $\mathcal{D}$-stability of this matrix using the Kelley's method. The polynomials

$$f_1(z) = z, \ f_2(z) = -z^2, \ f_3(z) = -z^3$$

define the boundaries of the $\mathcal{D}$ region, ensuring that the eigenvalues of the matrix A lie within this sector.

According to Theorem 2 in [9], to ensure the $\mathcal{D}$ stability of the matrix A, we need to find a symmetric positive definite matrix P such that the following matrix inequalities are satisfied:

$$\begin{aligned} f_1(A)^T P + P f_1(A) < 0, \\ f_2(A)^T P + P f_2(A) < 0, \\ f_3(A)^T P + P f_3(A) < 0, \end{aligned} \tag{19}$$

that is,

$$\begin{aligned} &A^T P + PA < 0 \\ &(-A^2)^T P + P(-A^2) < 0, \\ &(-A^3)^T P + P(-A^3) < 0. \end{aligned}$$

These inequalities ensure that the matrix A is $\mathcal{D}$-stable with a common positive definite matrix P.

Using the Kelley's method, we iterate to find a common positive definite matrix P that satisfies the stability conditions for the given matrices. Starting $z^0 = (1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1)$, that is $P(x^0)$ is a identity matrix, after 111 iterations, we find the solution (see Table 2). Calculations give

Table 2. Iterative Results of Kelley's Method for Example 2

k	L^k	$c_1(z^k)$	$c_2(z^k)$
0	-1.53×10^6	-3.603×10^5	1
1	-3.94×10^5	-2.087×10^6	-3.402
2	-2.34×10^5	-5.218×10^5	-2
3	-31924.888	-2.586×10^5	-0.523
⋮	⋮	⋮	⋮
110	-11.732012	-18.400783	0.006
111	-11.732012	-11.301216	0.007

$$\begin{aligned} z^{111} &= (x^{111}, L^{111})^T \\ &= (1, -0.305, 0.563, -0.081, 1, -0.091, \\ &\quad 0.240, 0.343, -0.009, 0.099, -11.732012)^T. \end{aligned}$$

Since $\phi_L(x^{111}) = L^{111} - c_1(z^{111}) = -0.430796 < 0$ and $c_2(z^{111}) = 0.007 > 0$, the positive definite matrix

$$P(x^{111}) = \begin{bmatrix} 1 & -0.305 & 0.563 & -0.081 \\ -0.305 & 1 & -0.091 & 0.240 \\ 0.563 & -0.091 & 0.343 & -0.009 \\ -0.081 & 0.240 & -0.009 & 0.099 \end{bmatrix}$$

is a common solution to (19) and $L(x^{111}, i) < 0$ for $i = 1, 2, 3$.

Using Kelley's method, we obtained a common positive definite matrix P that solves the matrix inequalities derived from A. As a result, matrix A is $\mathcal{D}$-stable. The results shown in Fig. 2 demonstrate the robustness of our solution by verifying that the eigenvalues of the matrix are within the defined stability region $\mathcal{D}$.

The examples above show how effective Kelley's method is in solving stability problems for matrices within complex $\mathcal{D}$ regions. Our proposed algorithm also has an advantage over other optimization methods in determining the common positive solution (if it exists) for the matrix inequalities. Kelley's cutting-plane method found a solution matrix that confirmed the stability of the given matrices within the defined $\mathcal{D}$ region. This was affirmed through multiple iterations, where the eigenvalues of the matrix remained within the defined stability region.

4 The $\mathcal{D}$-stability of One-parameter Matrix Families

Let $A_1, A_2 \in \mathbb{R}^n$ be $\mathcal{D}$-stable matrices. The set of their convex combinations is defined as

$$\mathcal{A} = \{A(\alpha) = (1-\alpha)A_1 + \alpha A_2 : \alpha \in [0,1]\}. \quad (20)$$

If $A(\alpha)$ is $\mathcal{D}$-stable for every $\alpha \in [0,1]$, then the matrix family $\mathcal{A}$ is called robustly $\mathcal{D}$-stable.

In this section, we will handle the problem of the robust stability of the one-parameter matrix family $\mathcal{A}$.

Theorem 5. *The family of matrices $\mathcal{A}$ in equation* (20) *is robustly $\mathcal{D}$-stable if and only if there exists a symmetric positive definite matrix $P(\alpha) > 0$ for each $\alpha \in [0,1]$, such that*

$$Q \otimes P(\alpha) + S \otimes (A(\alpha)P(\alpha)) + S^T \otimes (P(\alpha)A(\alpha)^T) + R \otimes (A(\alpha)P(\alpha)A(\alpha)^T) < 0. \quad (21)$$

Determining the $\mathcal{D}$-stability of the matrix family $\mathcal{A}$ (20) involves finding a matrix $P(\alpha) > 0$ that satisfies the matrix inequality in the Theorem 5 for every $\alpha \in [0,1]$, which can be a challenging task. Despite producing conservative results, we present the following theorem because of its applicability and relevance to our problem.

Theorem 6. *Let $\mathcal{A}$ be a given matrix family. If there exists a symmetric positive definite matrix $P > 0$ of dimension $n \times n$ such that for each $\alpha \in [0,1]$,*

$$Q \otimes P + S \otimes (A(\alpha)P) + S^T \otimes (PA(\alpha)^T) + R \otimes (A(\alpha)PA(\alpha)^T) < 0, \quad (22)$$

then the matrix family $\mathcal{A}$ is robustly $\mathcal{D}$-stable.

In the Theorem 6, the requirement for the existence of a positive definite matrix $P > 0$, which satisfies the matrix inequality (22) for every $\alpha \in [0,1]$, introduces a level of conservatism. However, this theorem can still be applied to solve the robust $\mathcal{D}$-stability problem of a one-parameter matrix family $\mathcal{A}$.

Let

$$M(x,\alpha) := Q \otimes P + S \otimes (A(\alpha)P) + S^T \otimes (PA(\alpha)^T) + R \otimes (A(\alpha)PA(\alpha)^T).$$

Define

$$\begin{aligned} \phi_M(x) &= \max_{0\le\alpha\le1} \lambda_{\max}(M(x,\alpha)) \\ &= \max_{0\le\alpha\le1, \|u\|=1} u^T M(x,\alpha)u. \end{aligned} \quad (23)$$

If there exists a point $\tilde{x}$ such that $P(\tilde{x}) > 0$ and $\phi_M(\tilde{x}) < 0$, then $\mathcal{A}$ is robustly $\mathcal{D}$-stable.

To overcome the computational challenge of calculating

$$\max_{0\le\alpha\le1} \lambda_{\max}(M(x,\alpha))$$

for $\alpha \in [0,1]$, we divide the interval $[0,1]$ into k parts and let

$$\alpha_i = \frac{i}{k} \quad (i = 0,1,2,\dots,k)$$

then

$$\phi_M(x) \approx F_M(x) := \max_{0\le i\le k, \|u\|=1} u^T M(x,\alpha_i)u.$$

For this value of k, if there exists a point $\tilde{x}$ such that $P(\tilde{x}) > 0$ and $F_M(\tilde{x}) < 0$, and if the inequality $\phi_M(\tilde{x}) < 0$ is satisfied, then we have obtained the solution to our problem. If the inequality is not satisfied, we proceed with a larger value of k and continue this procedure.

Example 3. Consider the set $\mathcal{D}$ given in Example 1. Let

$$A_1 = \begin{bmatrix} -2.369 & 5.297 & 5.225 \\ -1.351 & 2.481 & 3.445 \\ 0.684 & -1.148 & -2.112 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} -0.83 & 0.02 & -0.01 \\ -0.07 & -0.78 & -0.01 \\ 0.06 & -0.01 & -0.82 \end{bmatrix}$$

be $\mathcal{D}$-stable matrices (Fig. 3).

Starting from the initial point

$$\begin{aligned} z^0 &= (x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, L^0)^T \\ &= (1,0,0,1,0,1,1)^T, \end{aligned}$$

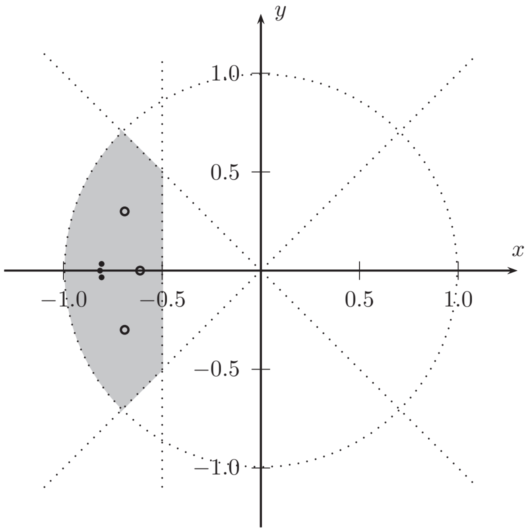

Fig. 3: The eigenvalues of matrix A_1 are shown with circles, and the eigenvalues of matrix A_2 are shown with points.

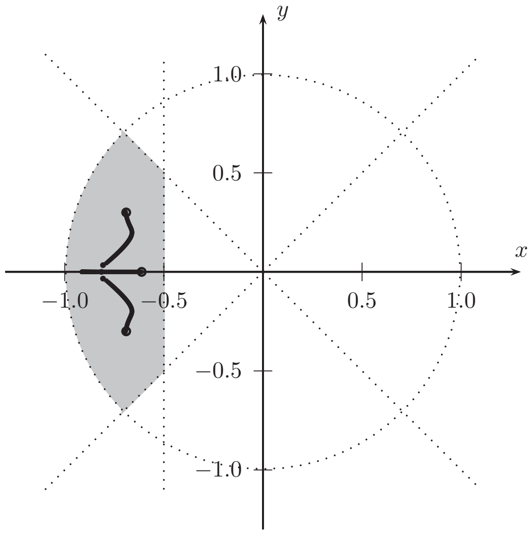

Fig. 4: The set of eigenvalues of the matrices in the set $\mathcal{A}$.

and applying Algorithm 1, in the first step we obtain

$$F_M(x^0) = \max_{0\le i\le k, \|u\|=1} u^T M(x, \alpha_i)u = 85.342 > 0.$$

When the LP problem (17) is solved, the point $z^1 = (-1, 1, 1, -1, -1, 0, -191.062)^T$ is obtained. Continuing this process, after 55 iterations we find

$$\begin{aligned} z^{55} &= (x_1^{55}, x_2^{55}, x_3^{55}, x_4^{55}, x_5^{55}, x_6^{55}, L^{55})^T \\ &= (1, -0.0072, 0.3432, 0.2489, -0.2329, \\ &\quad 0.3331, -0.0003)^T. \end{aligned}$$

For this point, $P(x^{55}) > 0$ and $F_M(x^{55}) = -0.000025 < 0$.

By examining the signs of the principal minors of $M(x^{55}, \alpha)$ for each $\alpha \in [0, 1]$, it is concluded that $M(x^{55}, \alpha) < 0$. Therefore, according to Theorem 5.2 the matrix family $\mathcal{A} = \{(1-\alpha)A_1 + \alpha A_2 : \alpha \in [0, 1]\}$ is robustly $\mathcal{D}$-stable (Fig. 4).

5 Conclusion

This study inspected the stability of a matrix within symmetric regions of the complex plane defined by quadratic matrix inequalities (QMI) and polynomial functions using Kelley's cutting-plane method. We proposed an algorithm to efficiently determine the $\mathcal{D}$-stability of matrices, with examples demonstrating its effectiveness.

We also addressed the robust $\mathcal{D}$-stability of one-parameter matrix families, providing theoretical results and computational techniques. To address the inherent conservatism in the $\mathcal{D}$-stability analysis, we plan to consider in future work the matrix $P(\alpha)$ as $P(\alpha) = (1-\alpha)P_1 + \alpha P_2$, where P_1 and P_2 are the matrices to be determined. This approach has the potential to reduce the conservatism associated with stability conditions and produce more accurate results.

References:

[1] S. Gutman and E. Jury, A general theory for matrix root-clustering in subregions of the complex plane, *IEEE Transactions on Automatic Control*, V.26, N.4, 1981, pp. 853–863.

[2] M. Chilali and P. Gahinet, H_∞ design with pole placement constraints: an LMI approach, *IEEE Transactions on Automatic Control*, V.41, N.3, 1996, pp. 358–367.

[3] J. Zrida, M.H. Bouazizi, Exact robust D-stability analysis for linear dynamical systems with polynomial parameter perturbation, *International Journal of Control*, V.95, N.11, 2022, pp.2885–2899.

[4] J. Pätzold, A. Schöbel, Approximate cutting plane approaches for exact solutions to robust optimization problems, *European Journal of Operational Research*, V.284, 2020, pp.20–30.

[5] A.M. Souza, R.C.L.F. Oliveira, P.L.D. Peres, Computation of stable convex combinations of a set of matrices through LMIs with application in H_∞ control of switched linear systems, *Systems & Control Letters*, V.189, 2024.

[6] B. Sereni, M.A.L. Beteto, E. Assunçõ, M.C.M. Teixeira, Pole placement LMI constraints for stability and transient performance of LPV systems with incomplete state measurement, *Journal of the Franklin Institute*, V.359, No.2, 2022, pp. 837-858.

[7] B.R. Barmish, *New Tools for Robustness of Linear Systems*, Macmillan, 1994.

[8] T. Büyükköroğlu, G. Çelebi, V. Dzhafarov, On the robust stability of polynomial matrix families, *Electronic Journal of Linear Algebra*, V.30, 2015, pp.905–915.

[9] T. Mori, H. Kokame, A parameter-dependent Lyapunov function for a polytope of matrices, *IEEE Transactions on Automatic Control*, V.45, N.8, 2000, pp.1516–1519.

[10] Ş. Yılmaz, T. Büyükköroğlu, On two problems for matrix polytopes, *Applied Mathematics*, V.5, 2014, pp.2650–2656.

[11] R.A. Horn and C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

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