Bounds on Initial Coefficients for Bi-Univalent Functions Linked to q-Analog of Le Roy-Type Mittag-Leffler Function

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Abstract: - This study introduces a new class of bi-univalent functions by incorporating the q-analog of Le Roy-type Mittag-Leffler functions alongside q-Ultraspherical polynomials. We formulate and solve the Fekete-Szegö functional problems for this newly defined class of functions, providing estimates for the coefficients $|\alpha_2|$ and $|\alpha_3|$ in their Taylor-Maclaurin series. Additionally, our investigation produces novel results by adapting the parameters in our initial discoveries.

Key-Words: Orthogonal polynomial; q-Ultraspherical polynomials, Analytic functions; Univalent functions, Bi-univalent functions, Fekete-Szegö problem, Subordination, q-calculus.

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1 Introduction

The realm of quantum calculus, also referred to as q-calculus, expands upon conventional calculus by integrating the principles of quantum mechanics. q-calculus, a branch of mathematics, introduces a novel parameter denoted as q, which extends classical calculus principles and methods. This area demonstrates a broad spectrum of applications spanning various fields such as mathematics, physics, and engineering. Within the scope of q-calculus, the theory of q-orthogonal polynomials (q-OP) holds particular significance and has been subject to extensive research.

The origins of the q-OP theory can be attributed to the investigations carried out in the 1940s and 1950s. [1], established a novel category of polynomials known as q-polynomials in his research. These polynomials exhibit a distinct recurrence relation that incorporates the q-analog of the factorial function. The theory of q-orthogonal polynomials was extended by generalizing the previously described polynomials, as referenced in [2].

The q-OP polynomials form a set of orthogonal polynomials, where orthogonality is defined with regard to a specific weight function that is dependent on the parameter q. These polynomials are widely used in diverse fields of mathematics and physics, such as number theory, combinatorics, statistical mechanics, and quantum mechanics. Various varieties of q-OP exist, such as q-Hermite, q-Jacobi, q-Laguerre, and q-Gegenbauer polynomials, among others. Every variant of q-OP possesses its own distinct recurrence relation, weight function, and orthogonality qualities. For a thorough examination, refer to the extensive study documented in ([3], [4], [5], [6], [7]).

Exploring q-OP has yielded significant advancements and methodologies in q-calculus, such as the q-analog of the binomial theorem, q-difference equations, and q-special functions. The theory of q-OP has been applied to analyze q-integrals and q-series, which are fundamental tools in the field of q-calculus. Recently, Jackson's q-exponential has been redefined as a series of regular exponentials with clear coefficients, making it self-contained, as referenced in [8]. This result has significant implications for the theory of q-orthogonal polynomials in the current context and should be fully acknowledged.

The theory of orthogonal polynomials has been thoroughly examined because of its wide-ranging applications in several branches of mathematics and physics. Orthogonal polynomials and their analogs have gained significance as a valuable tool for analyzing analytic functions in the complex plane, specifically bi-univalent functions, in recent years.

2 Preliminaries

Consider the set $\mathcal{A}$ consisting of functions Φ that can be expressed in the form

$$\Phi(\zeta) = \zeta + \sum_{n=2}^{\infty} \alpha_n \zeta^n, \tag{1}$$

where ζ be a complex number that lies within the open unit disk $\mathcal{O}$, and let Φ be an analytic function in $\mathcal{O}$. In addition, Φ must fulfill the normalization requirement $\Phi'(0) - 1 = 0 = \Phi(0)$. The subclass of $\mathcal{A}$ that consists of functions of Eq. (1) and are univalent in $\mathcal{O}$ is denoted by $\mathcal{S}$. For any function Φ in the subfamily $\mathcal{S}$, there exists an inverse function denoted as Φ^{-1} and defined by

$$\zeta = \Phi^{-1}(\Phi(\zeta)), \quad \varpi = \Phi(\Phi^{-1}(\varpi)),$$

and

$$|\varpi| < r_0(\Phi); \zeta \in \mathcal{O}.$$

where

$$\begin{aligned} \mathsf{h}(\varpi) = \Phi^{-1}(\varpi) = \varpi(1 - \varpi^3(\alpha_4 + 5\alpha_2^3 - 5\alpha_3\alpha_2) \\ + \varpi^2(-\alpha_3 + 2\alpha_2^2) - \varpi\alpha_2 + \cdots) \end{aligned} \tag{2}$$

The definition of the subclass Σ in the set $\mathcal{S}$ involves specifying the category of bi-univalent functions in $\mathcal{O}$, as expressed by equation (1). Examples of the class Σ functions include

$$\Phi_1(\zeta) = \frac{\zeta}{1-\zeta}, \quad \Phi_2(\zeta) = \log\left(\frac{1}{1-\zeta}\right)$$

and

$$\Phi_3(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right).$$

The inverse functions that correspond to the aforementioned functions:

$$\mathsf{h}_1(\varpi) = \frac{\varpi}{1+\varpi}, \quad \mathsf{h}_2(\varpi) = \frac{e^{2\varpi}-1}{e^{2\varpi}+1}$$

and

$$\mathsf{h}_3(\varpi) = \frac{e^{\varpi}-1}{e^{\varpi}}.$$

The implementation of differential subordination of analytical functions has the potential to offer considerable benefits to the domain of geometric function theory. The authors in [9], proposed the original differential subordination problem, which has subsequently been examined in greater detail in [10]. The book referenced in [11], provides a comprehensive overview of the advancements made in the field, along with their respective dates of publication.

This article presents an overview of q-calculus, initially introduced by Jackson and subsequently explored by numerous mathematicians, [12], [13], [14], [15], [16]. It focuses on introducing key concepts and definitions within the realm of q-calculus. Additionally, it highlights the significance of the q-difference operator, widely employed in scientific disciplines such as geometric function theory. Emphasizing that q lies within the interval $(0, 1)$, the study extensively draws on fundamental definitions and properties of $q$calculus, as documented in [7].

Definition 1. [12]. Let $0 < q < 1$. The q-bracket $[k]_q$ is formally defined as such

$$[\kappa]_q = \begin{cases} \frac{1-q^\kappa}{1-q}, & \text{if} \quad 0 < \mathsf{q} < 1, \ \kappa \in \mathbf{C} \setminus \{0\} \\ q^{\kappa-1} + \cdots + q^2 + q + 1 & \text{if} \quad \kappa \in \mathbf{N} \\ 1 \quad \text{if} \quad \mathsf{q} \to 0^+, \kappa \in \mathbf{C} \setminus \{0\} \\ \kappa \quad \text{if} \quad \mathsf{q} \to 1^-, \kappa \in \mathbf{C} \setminus \{0\} \end{cases}$$

Definition 2. [12]. The q-derivative, also known as the q-difference operator, of a function Φ is defined by

$$\partial_q \Phi(\zeta) = \begin{cases} \frac{\Phi(\zeta) - \Phi(q\zeta)}{\zeta - q\zeta}, & \text{if} \quad 0 < \mathsf{q} < 1, \ \zeta \neq 0 \\ \Phi'(0) \quad \text{if} \quad \zeta = 0 \\ \Phi'(\zeta) \quad \text{if} \quad \mathsf{q} \to 1^-, \zeta \neq 0 \end{cases}$$

Consider two complex parameters ε and ϱ such that the real part of ε and ϱ is greater than zero. The generalized Mittag-Leffler type function was initially proposed by [16], through

$$\mathcal{M}_{\varrho,\varepsilon}(\zeta) = \sum_{\kappa=0}^{\infty} \frac{\zeta^\kappa}{\Gamma(\varepsilon\,\kappa + \varrho)} \qquad (\zeta \in \mathbf{C}). \tag{3}$$

The study, [17], and independently, [18], have recently introduced a Mittag-Leffler function of the Le Roy type, defined by:

$$\mathcal{F}^{\gamma}_{\varrho,\varepsilon}(\zeta) = \sum_{\kappa=0}^{\infty} \frac{\zeta^{\kappa}}{(\Gamma(\varepsilon\,\kappa+\varrho))^{\gamma}} \qquad (\zeta \in \mathbf{C}). \qquad (4)$$

Assuming that $\Re e\{\varepsilon\} > 0$ and $\Re e\{\varrho\} > 0$, [19], introduced the q-Mittag-Leffler-type function, as

$$\mathcal{M}_{\varrho,\varepsilon}(\zeta;\mathsf{q}) = \sum_{\kappa=0}^{\infty} \frac{\zeta^{\kappa}}{\Gamma_{\mathsf{q}}(\varepsilon\,\kappa+\varrho)} \qquad (\zeta \in \mathbf{C}). \qquad (5)$$

The study, [20], recently proposed a normalization of the q-analog of the Le Roy-type Mittag-Leffler function, denoted by $\mathcal{M}^{\gamma}_{\varrho,\varepsilon}(\zeta;q)$ where ($\zeta \in \mathcal{O}$). This normalization is given by

$$\mathcal{M}^{\gamma}_{\varrho,\varepsilon}(z;q) = \zeta + \sum_{\kappa=2}^{\infty} \left(\frac{\Gamma_{\mathsf{q}}(\varrho)}{\Gamma_{\mathsf{q}}(\varepsilon\,(\kappa-1)+\varrho)} \right)^{\gamma} \zeta^{\kappa}, \qquad (6)$$

where $\Re e(\varrho) > 0$, $\varepsilon \in \mathbf{C} \setminus \{0,-1,-2,\cdots\}$. The gamma function, denoted by Γ_q, where $q \in (0,1)$, can be alternatively defined by

$$\Gamma_q(1+\zeta) = (1-q^{\varpi})(1-q)^{-1}\Gamma_q(\zeta). \qquad (7)$$

The linear operator ${}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon} : \mathcal{A} \to \mathcal{A}$ can be defined using the concept of convolution (or the Hadamard product) by

$$\begin{aligned} {}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta) &= \mathcal{M}^{\gamma}_{\varrho,\varepsilon}(z;q) * \Phi(\zeta) \\ &= \zeta + \textstyle\sum_{\kappa=2}^{\infty} \left(\frac{\Gamma_{\mathsf{q}}(\varrho)}{\Gamma_{\mathsf{q}}(\varepsilon\,(\kappa-1)+\varrho)} \right)^{\gamma} a_{\kappa}\zeta^{\kappa}, \\ &= \zeta + \left(\frac{\Gamma_{\mathsf{q}}(\varrho)}{\Gamma_{\mathsf{q}}(\varepsilon+\varrho)} \right)^{\gamma} a_2\zeta^2 + \left(\frac{\Gamma_{\mathsf{q}}(\varrho)}{\Gamma_{\mathsf{q}}(2\varepsilon+\varrho)} \right)^{\gamma} a_3\zeta^3 + \cdots . \end{aligned} \qquad (8)$$

where $q \in (0,1)$, $\gamma > 0$, $\Re e(\varrho) > 0$, $\varepsilon \in \mathbf{C} \setminus \{0,-1,-2,\cdots\}$ and Γ_q of the form (7).

The $\mathcal{G}^{(\$)}_q(\ell,\zeta)$, referred to as the q-UP, are a set of orthogonal polynomials that are defined on the interval $[-1,1]$. These polynomials are defined with respect to the weight function $(1-\ell^2)^{\$-\frac{1}{2}}$ on the same interval, and feature a q-analog. The study, [5], identified a category of q-generalized polynomials, commonly referred to as q-UP, which are essentially the following polynomials

$$\mathcal{G}^{(\lambda)}_q(\ell,\zeta) = \sum_{n=0}^{\infty} \mathbf{C}^{(\lambda)}_n(\ell;q)\zeta^n, \quad (\ell \in [-1,1],\ \zeta \in \mathcal{O}). \qquad (9)$$

The study, [6], in 2006, discovered the initial terms of UP's q-analog in 2006, which listed below:

$$\begin{aligned} &\mathbf{C}^{(\lambda)}_0(\ell;q) = 1 \\ &\mathbf{C}^{(\lambda)}_1(\ell;q) = 2[\lambda]_q\ell \\ &\mathbf{C}^{(\lambda)}_2(\ell;q) = 2\left([\lambda]_{q^2} + [\lambda]^2_q\right)\ell^2 - [\lambda]_{q^2} \end{aligned} \qquad (10)$$

Orthogonal polynomials have been utilized in the examination of bi-univalent functions. The utilization of orthogonal polynomials in the examination of bi-univalent functions has yielded significant outcomes and perspectives in the realm of geometric function theory. In contemporary literature, there has been a surge of interest among scholars in exploring subsets of bi-univalent functions that are linked to orthogonal polynomials, specifically those related to Ultraspherical and Chebyshev polynomials. Estimations for the initial coefficients of functions were discovered. Nevertheless, the issue of establishing precise coefficient limits for $|\alpha_n|$, $(n = 3,4,5,\cdots)$, is yet to be resolved, as indicated in several sources ([21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35]).

On the other hand, in 2023, [24], constructed various categories of analytic bi-univalent functions utilizing q-UP. The present study derives the Fekete–Szegö inequalities and coefficient bounds $|\alpha_2|$ and $|\alpha_3|$ for functions that are members of the aforementioned subclasses.

The main aim of this research is to commence an inquiry into the attributes of bi-univalent functions through the utilization of the q-analog of Le Roy-type functions and Mittag-Leffler functions that are associated with q-Ultraspherical polynomials. To achieve this goal, the following definitions are taken into account.

3 Definition and examples

This section introduces new subcategories of bi-univalent functions. The subclasses are established by utilizing the q-analog of Le Roy-type functions, namely the Mittag-Leffler functions that are subordinated to the q-UP. In this paper, it is assumed that $q \in (0,1)$ and $\ell \in \left(\frac{1}{2},1\right]$, unless explicitly stated otherwise.

Definition 3. For $0 \leq \vartheta \leq 1$ and $\eta \in \mathbf{C} \setminus \{0\}$, a bi-univalent function Φ of the form (1) is said to be in the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$ if the following subordinations are satisfied:

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta), \tag{11}$$

and

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta). \tag{12}$$

The expressions for $\mathsf{h}(\varpi)$, ${}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}$, and $\mathcal{G}_q^{(\lambda)}$ are obtained from equations (2), (8), and (9), respectively.

Example 1. Let $\vartheta = 1$, $\eta \in \mathbf{C} \setminus \{0\}$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(1, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$ if the following subordinations are satisfied:

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)+\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta), \tag{13}$$

and

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)+\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta). \tag{14}$$

The expressions for $\mathsf{h}(\varpi)$, ${}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}$, and $\mathcal{G}_q^{(\lambda)}$ are obtained from equations (2), (8), and (9), respectively.

Example 2. Let $\vartheta = 1$ and $\eta \in \mathbf{C} \setminus \{0\}$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(0, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$ if the following subordinations are satisfied:

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta),$$

and

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta).$$

The expressions for $\mathsf{h}(\varpi)$, ${}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}$, and $\mathcal{G}_q^{(\lambda)}$ are

obtained from equations (2), (8), and (9), respectively.

Example 3. Let $0 \leq \vartheta \leq 1$, $\eta \in \mathbf{C} \setminus \{0\}$ and $q \to 1^-$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_1^{(\lambda)}(\ell, \zeta))$ if the following subordinations are satisfied:

$$1+\frac{1}{\eta}\Big(\left({}_1\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)'+\vartheta\zeta\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)''-1\Big) \prec \mathcal{G}_1^{(\lambda)}(\ell,\zeta),$$

and

$$1+\frac{1}{\eta}\Big(\left({}_1\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)'+\vartheta\zeta\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)''-1\Big) \prec \mathcal{G}_q^{(\lambda)}(\ell,\zeta). \tag{15}$$

The expressions for $\mathsf{h}(\varpi)$, ${}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}$, and $\mathcal{G}_q^{(\lambda)}$ are obtained from equations (2), (8), and (9), respectively.

4 The natural initial Taylor coefficients of the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$

Initially, the estimates for the coefficients of the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$, as defined in Definition 3.1, are provided.

Theorem 1. Let $f \in \Sigma$ given by (1) belongs to the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$. Then

$$|a_2| \leq \frac{2|\eta[\lambda]_q|\left(\Gamma_q(\varrho+\varepsilon)\right)^\gamma \ell.\ \sqrt{2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma[\lambda]_q\ell}}{\sqrt{\left(\Gamma_q(\varepsilon)\right)^\gamma\left|\begin{array}{c}4[3]_q[\lambda]_q^2([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\ell^2- \\ [2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\times \\ \Big(2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}\Big)\end{array}\right|}},$$

and

$$|a_3| \leq \frac{2|[\lambda]_q\eta|\ell}{[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma + \left(\frac{4\eta|[\lambda]_q|\ell}{[2]_q(1+\vartheta)}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma}.$$

Proof. If Φ belongs to the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$. As per Definition 3.1, the presence of certain analytic functions ω and v can be established, satisfying the conditions $\omega(0) = v(0) = 0$, and $|\omega(\zeta)| < 1$, $|v(\varpi)| < 1$ for all $\zeta, \varpi \in \mathcal{O}$. Under these conditions, we can express Φ as follows

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)-1\Big) = \mathcal{G}_q^{(\lambda)}(\ell,\omega(\zeta)), \tag{16}$$

and

$$1+\frac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)-1\Big) = \mathcal{G}_q^{(\lambda)}(\ell,v(\varpi)), \tag{17}$$

By utilizing equations (16) and (17), we can derive the following expression.

$$\begin{aligned}&1+\tfrac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\Phi(\zeta)\right)-1\Big)\\ &=1+C_1^{(\lambda)}(\ell;q)c_1\zeta+\Big[C_1^{(\lambda)}(\ell;q)c_2+C_2^{(\lambda)}(\ell;q)c_1^2\Big]\zeta^2+\cdots,\end{aligned} \tag{18}$$

and

$$\begin{aligned}&1+\tfrac{1}{\eta}\Big(\partial_q\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)+\vartheta\zeta\partial_q^2\left({}_q\mathcal{F}^{\gamma}_{\varrho,\varepsilon}\mathsf{h}(\varpi)\right)-1\Big)\\ &=1+C_1^{(\lambda)}(\ell;q)d_1\varpi+\Big[C_1^{(\lambda)}(\ell;q)d_2+C_2^{(\lambda)}(\ell;q)d_1^2\Big])\varpi^2+\cdots.\end{aligned} \tag{19}$$

It is generally understood that if

$$|\omega(\zeta)| = \left|c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cdots\right| < 1, \quad (\zeta \in \mathcal{O}),$$

and

$$|v(\varpi)| = \left|d_1\varpi + d_2\varpi^2 + d_3\varpi^3 + \cdots\right| < 1, \quad (\varpi \in \mathcal{O}),$$

then, for all $\jmath \in \{1, 2, 3, \cdots\}$, we know

$$|c_\jmath| \le 1 \text{ and } |d_\jmath| \le 1. \tag{20}$$

In view of (1), (2), from (18) and (19), we obtain

$$1 + \frac{[2]_q(1+\vartheta)}{\eta}\left(\frac{\Gamma_q(\varrho)}{\Gamma_q(\varepsilon+\varrho)}\right)^\gamma a_2\zeta + \frac{[3]_q([2]_q\vartheta+1)}{\eta}\times \left(\frac{\Gamma_q(\varrho)}{\Gamma_q(2\varepsilon+\varrho)}\right)^\gamma a_3\zeta^2 + \cdots = 1 + C_1^{(\lambda)}(\ell;q)c_1\zeta + \left[C_1^{(\lambda)}(\ell;q)c_2 + C_2^{(\lambda)}(\ell;q)c_1^2\right]\zeta^2 + \cdots,$$

and

$$1 - \frac{[2]_q(1+\vartheta)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(\varrho+\varepsilon)}\right)^\gamma a_2\varpi + \frac{[3]_q([2]_q\vartheta+1)}{\eta}\times \left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(2\varrho+\varepsilon)}\right)^\gamma (2a_2^2 - a_3)\varpi^2 + \cdots = 1 + C_1^{(\lambda)}(\ell;q)d_1\varpi + \left[C_1^{(\lambda)}(\ell;q)d_2 + C_2^{(\lambda)}(\ell;q)d_1^2\right]\varpi^2 + \cdots.$$

By comparing the pertinent coefficients in (18) and (19), we arrive at the following.

$$\frac{[2]_q(1+\vartheta)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(\varrho+\varepsilon)}\right)^\gamma a_2 = C_1^{(\lambda)}(\ell;q)c_1, \tag{21}$$

$$-\frac{[2]_q(1+\vartheta)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(\varrho+\varepsilon)}\right)^\gamma a_2 = C_1^{(\lambda)}(\ell;q)d_1, \tag{22}$$

$$\frac{[3]_q([2]_q\vartheta+1)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(2\varrho+\varepsilon)}\right)^\gamma a_3 = C_1^{(\lambda)}(\ell;q)c_2 + C_2^{(\lambda)}(\ell;q)c_1^2, \tag{23}$$

and

$$\frac{[3]_q([2]_q\vartheta+1)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(2\varrho+\varepsilon)}\right)^\gamma (2a_2^2 - a_3) = C_1^{(\lambda)}(\ell;q)d_2 + C_2^{(\lambda)}(\ell;q)d_1^2. \tag{24}$$

It follows from (21) and (22) that

$$c_1 = -d_1, \tag{25}$$

and

$$a_2^2 = \frac{1}{2}\left(\frac{\eta\left[C_1^{(\lambda)}(\ell;q)\right]}{[2]_q(1+\vartheta)}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma}(c_1^2+d_1^2)$$
$$c_1^2 + d_1^2 = 2\left(\frac{[2]_q(1+\vartheta)}{\eta\left[C_1^{(\lambda)}(\ell;q)\right]}\right)^2\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(\varrho+\varepsilon)}\right)^{2\gamma} a_2^2. \tag{26}$$

Adding (23) and (24), we get

$$\frac{2[3]_q([2]_q\vartheta+1)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(2\varrho+\varepsilon)}\right)^\gamma a_2^2 = C_1^{(\lambda)}(\ell;q)(c_2+d_2) + C_2^{(\lambda)}(\ell;q)(c_1^2+d_1^2). \tag{27}$$

Substituting the value of $(c_1^2 + d_1^2)$ from (26), we obtain

$$a_2^2 = \frac{\eta^2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^3(c_2+d_2)}{2\left(\Gamma_q(\varepsilon)\right)^\gamma\left\{[3]_q([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2 - [2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\right\}},$$

Applying for the coefficients c_2 and d_2 and using (10), we obtain

$$|a_2| \le \frac{2|\eta[\lambda]_q|\left(\Gamma_q(\varrho+\varepsilon)\right)^\gamma \ell.\sqrt{2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma[\lambda]_q\ell}}{\sqrt{\left(\Gamma_q(\varepsilon)\right)^\gamma\left|4[3]_q[\lambda]_q^2([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\ell^2 - [2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2 - [\lambda]_{q^2}\right)\right|}},$$

By subtracting (24) from (23), and using $c_1^2 = d_1^2$, we get

$$\frac{2[3]_q([2]_q\vartheta+1)}{\eta}\left(\frac{\Gamma_q(\varepsilon)}{\Gamma_q(2\varrho+\varepsilon)}\right)^\gamma(a_3 - a_2^2) = C_1^{(\lambda)}(\ell;q)(c_2 - d_2). \tag{28}$$

Then, in view of (25) and (26), Eq. (28) becomes

$$a_3 = \left\{\frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma C_1^{(\lambda)}(\ell;q)(c_2-d_2) + \frac{1}{2}\left(\frac{\eta}{[2]_q(1+\vartheta)}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2(c_1^2+d_1^2).\right\}$$

Thus applying (10), we conclude that

$$a_3 \le \frac{2|[\lambda]_q\eta|\ell}{[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma + \left(\frac{4\eta|[\lambda]_q|\ell}{[2]_q(1+\vartheta)}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma}$$

This completes the proof of Theorem.

5 The Fekete-Szegö functional

The authors in [36], established a precise limit for the functional $\mu a_2^2 - a_3$. The limit was derived using real values of μ $(0 \le \mu \le 1)$ and has been commonly known as the classical Fekete-ö outcome. Establishing precise boundaries for a given function within a compact family of functions $f \in \mathcal{A}$, and for any complex μ, poses a formidable challenge. The Fekete-ö inequality for functions belonging to the class $\mathcal{B}_\Sigma(\vartheta, \eta, \varrho, \varepsilon, \gamma, \mathcal{G}_q^{(\lambda)}(\ell, \zeta))$ is examined in view of [37], finding.

Theorem 2. Let $\Phi \in \Sigma$ defined by (1) and

belongs to the class $\mathcal{B}_\Sigma(\vartheta,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$ and μ is real number. Then, we have
$\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{2\left|\eta[\lambda]_q\right|\ell}{[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma, & |\mathcal{H}(\mu)| \leq \frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \\ 4\left|\eta[\lambda]_q\mathcal{H}(\mu)\right|\ell, & |\mathcal{H}(\mu)| \geq \frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \end{cases}$$

where

$$\mathcal{H}(\mu)=\frac{\frac{(1-\mu)}{2\left(\Gamma_q(\varepsilon)\right)^\gamma}\eta^2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2}{\left\{\begin{array}{c}[3]_q([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2- \\ [2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\end{array}\right\}}.$$

Proof. For $f \in \mathcal{B}_\Sigma(\vartheta,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$ as is in (1), from the equations (27) and (28), we have

$$\begin{gathered} a_3-\mu a_2^2= \\ 2\left(\Gamma_q(\varepsilon)\right)^\gamma\left\{[3]_q([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2\right. \\ \left.-[2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\right\} \\ =C_1^{(\lambda)}(\ell;q)\left(\left[\mathcal{H}(\mu)+\frac{\eta C_1^{(\lambda)}(\ell;q)}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma\right]c_2\right. \\ \left.+\left[\mathcal{H}(\mu)-\frac{\eta C_1^{(\lambda)}(\ell;q)}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma\right]d_2\right) \end{gathered}$$

where

$$\mathcal{H}(\mu)=\frac{\frac{(1-\mu)}{2\left(\Gamma_q(\varepsilon)\right)^\gamma}\eta^2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2}{\left\{\begin{array}{c}[3]_q([2]_q\vartheta+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2- \\ [2]_q^2(1+\vartheta)^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\end{array}\right\}}.$$

Then, we conclude that $\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{\left|\eta C_1^{(\lambda)}(\ell;q)\right|}{[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma, & |\mathcal{H}(\mu)| \leq \frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \\ 2\left|C_1^{(\lambda)}(\ell;q)\right||\mathcal{H}(\mu)|, & |\mathcal{H}(\mu)| \geq \frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \end{cases}$$

Which completes the proof of Theorem 2.

6 Corollaries

The following corollaries, which roughly match Examples 1, 2 and 3, are produced by Theorems 1 and 2.

Corollary 1. If Φ is an element of Σ defined by (1) and belongs to the class $\mathcal{B}_\Sigma(1,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$, then we can state the following

$$|a_2| \leq \frac{2\left|\eta[\lambda]_q\right|\left(\Gamma_q(\varrho+\varepsilon)\right)^\gamma\ell.\sqrt{2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma[\lambda]_q\ell}}{\sqrt{\begin{array}{c}\left(\Gamma_q(\varepsilon)\right)^\gamma\left|4[3]_q[\lambda]_q^2([2]_q+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\ell^2-\right. \\ 4[2]_q^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\times \\ \left.\left(2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}\right)\right|\end{array}}},$$

$$a_3 \leq \frac{2\left|[\lambda]_q\eta\right|\ell}{[3]_q([2]_q+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma+\left(\frac{4\eta\left|[\lambda]_q\right|\ell}{2[2]_q}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma},$$

and $\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{2\left|\eta[\lambda]_q\right|\ell}{[3]_q([2]_q+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma, & |\mathcal{H}(\mu)| \leq \frac{\eta}{2[3]_q([2]_q+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \\ 4\left|\eta[\lambda]_q\mathcal{H}(\mu)\right|\ell, & |\mathcal{H}(\mu)| \geq \frac{\eta}{2[3]_q([2]_q\vartheta+1)}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma \end{cases}$$

where

$$\mathcal{H}(\mu)=\frac{\frac{(1-\mu)}{2\left(\Gamma_q(\varepsilon)\right)^\gamma}\eta^2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2}{\left\{\begin{array}{c}[3]_q([2]_q+1)\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2- \\ 4[2]_q^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\end{array}\right\}}.$$

Corollary 2. If Φ is an element of Σ defined by (1) and belongs to the class $\mathcal{B}_\Sigma(0,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$, then we can state the following

$$|a_2| \leq \frac{2\left(\Gamma_q(\varepsilon)\right)^{\frac{-\gamma}{2}}\left|\eta[\lambda]_q\right|\left(\Gamma_q(\varrho+\varepsilon)\right)^\gamma\ell.\sqrt{2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma[\lambda]_q\ell}}{\sqrt{\begin{array}{c}\left|4[3]_q[\lambda]_q^2\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\ell^2-[2]_q^2\left(\Gamma_q(\varepsilon)\right)^\gamma\times\right. \\ \left.\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}\right)\right|\end{array}}},$$

$$a_3 \leq \frac{2\left|[\lambda]_q\eta\right|\ell}{[3]_q}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma+\left(\frac{4\eta\left|[\lambda]_q\right|\ell}{[2]_q}\right)^2\left(\frac{\Gamma_q(\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^{2\gamma},$$

and
$\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{2\left|\eta[\lambda]_q\right|\ell}{[3]_q}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma & |\mathcal{H}(\mu)| \leq \frac{\eta}{2[3]_q}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma, \\ 4\left|\eta[\lambda]_q\mathcal{H}(\mu)\right|\ell, & |\mathcal{H}(\mu)| \geq \frac{\eta}{2[3]_q}\left(\frac{\Gamma_q(2\varrho+\varepsilon)}{\Gamma_q(\varepsilon)}\right)^\gamma, \end{cases}$$

where

$$\mathcal{H}(\mu)=\frac{(1-\mu)\eta^2\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2}{2\left(\Gamma_q(\varepsilon)\right)^\gamma\left\{\begin{array}{c}[3]_q\eta\left(\Gamma_q(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;q)\right]^2- \\ [2]_q^2\left(\Gamma_q(\varepsilon)\right)^\gamma\left(\Gamma_q(2\varrho+\varepsilon)\right)^\gamma C_2^{(\lambda)}(\ell;q)\end{array}\right\}}.$$

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Corollary 3. If Φ is an element of Σ defined by (1) and belongs to the class $\mathcal{B}_\Sigma(\vartheta,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_1^{(\lambda)}(\ell,\zeta))$, then we can state the following

$$|a_2| \leq \frac{2|\eta\lambda|\left(\Gamma(\varrho+\varepsilon)\right)^{\gamma}\ell.\sqrt{2\lambda\left(\Gamma(2\varrho+\varepsilon)\right)^{\gamma}\ell}}{\sqrt{\begin{array}{c}(\Gamma(\varepsilon))^{\gamma}\left|12\lambda^2([2]_q\vartheta+1)\eta\left(\Gamma(\varrho+\varepsilon)\right)^{2\gamma}\ell^2-\right.\\ \left.4\lambda(1+\vartheta)^2\left(\Gamma(\varepsilon)\right)^{\gamma}\left(\Gamma(2\varrho+\varepsilon)\right)^{\gamma}\left(2\left(1+\lambda\right)\ell^2-1\right)\right|\end{array}}},$$

$$a_3 \leq \frac{2|\lambda\eta|\ell}{3(2\vartheta+1)}\left(\frac{\Gamma(2\varrho+\varepsilon)}{\Gamma(\varepsilon)}\right)^{\gamma}+\left(\frac{4\eta\lambda\ell}{2(1+\vartheta)}\right)^{2}\left(\frac{\Gamma(\varrho+\varepsilon)}{\Gamma(\varepsilon)}\right)^{2\gamma},$$

and $\left|a_3-\mu a_2^2\right| \leq$

$$\begin{cases} \frac{2|\eta\lambda|\ell}{3(2\vartheta+1)}\left(\frac{\Gamma(2\varrho+\varepsilon)}{\Gamma(\varepsilon)}\right)^{\gamma} & |\mathcal{H}(\mu)| \leq \frac{\eta}{6(2\vartheta+1)}\left(\frac{\Gamma(2\varrho+\varepsilon)}{\Gamma(\varepsilon)}\right)^{\gamma} \\ 4\left|\eta\lambda\mathcal{H}(\mu)\right|\ell, & |\mathcal{H}(\mu)| \geq \frac{\eta}{6(2\vartheta+1)}\left(\frac{\Gamma(2\varrho+\varepsilon)}{\Gamma(\varepsilon)}\right)^{\gamma} \end{cases}$$

where

$$\mathcal{H}(\mu) = \frac{(1-\mu)\eta^2\left(\Gamma(2\varrho+\varepsilon)\right)^{\gamma}\left(\Gamma(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;1)\right]^2}{2\left(\Gamma(\varepsilon)\right)^{\gamma}\left\{\begin{array}{c}3(2\vartheta+1)\eta\left(\Gamma(\varrho+\varepsilon)\right)^{2\gamma}\left[C_1^{(\lambda)}(\ell;1)\right]^2-\\ 4(1+\vartheta)^2\left(\Gamma(\varepsilon)\right)^{\gamma}\left(\Gamma(2\varrho+\varepsilon)\right)^{\gamma}C_2^{(\lambda)}(\ell;1)\end{array}\right\}}.$$

7 Conclusion

In this study, we have explored the coefficient challenges associated with each of the innovative subclasses of bi-univalent functions as defined in Definitions 3.1 within the open unit disk $\mathcal{O}$. These subclasses encompass $\mathcal{B}_\Sigma(\vartheta,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$, $\mathcal{B}_\Sigma(1,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$, $\mathcal{B}_\Sigma(0,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_q^{(\lambda)}(\ell,\zeta))$, and $\mathcal{B}_\Sigma(\vartheta,\eta,\varrho,\varepsilon,\gamma,\mathcal{G}_1^{(\lambda)}(\ell,\zeta))$. We have provided estimates for the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, as well as evaluations for the Fekete-Szego functional problem for functions within each of these bi-univalent function classes.

Upon specializing the parameters in our primary findings, we have identified several additional new results. It is anticipated that the q-defferintegral operator will have broad applications across various scientific domains, including mathematics and technology.

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