Optimal System for non-linear Burger equation $u_t = u_{xx} + uu_x$.

TSHIDISO MASEBE[a] , PETER MATHYE[b]
TUT
MSBE Department
2 Aubrey Matlala Rd, Soshanguve H
SOUTH AFRICA

[a]Orcid no. 0000-0002-3792-5213
[b]Orcid no. 0009-0007-4829-0827

Abstract: The paper discusses the optimal system for a nonlinear Burger equation whose coefficients are dependent on first order spatial derivatives. The main purpose for the project is to determine the optimal system for the operators accepted by the equation. We construct the principal Lie algebra, calculate transformations for the generators which provide one-parameter group of transformations for the operator using Lie equations. We construct optimal systems for the equation where the method requires a simplification of a vector to a general form for each of the transformations of the generators. These are finally used to determine invariant solutions for some operators.

Key–Words: Principal Lie Algebra, Transformations, Invariant solution, One-dimensional optimal systems.

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1 Introduction

The Burger's equation

$$u_t = u_{xx} + uu_x \tag{1}$$

is a basic non-linear partial differential equation that is used to model propagation of shocks and solitons, [1]. It also appears in many other applications including plasma physics, non-linear harmonics and traffic flow, [2]. Lie group method is one of most efficient computational methods to obtain exact analytic as well as invariant solutions of nonlinear partial differential equations. It was pioneered by Sophus Lie in the 19th century (1849-1899), [2].An optimal represent the best or most favoured. In the context of the project, the optimal system seek to determine the minimal representation of the operators accepted by the nonlinear Burger equation. An optimal system of one-dimensional subalgebras is constructed using Lie vectors. Optimal system of symmetry subalgebras is important in producing possible invariant solutions through through Lie symmetry simplification or reduction, [3]. The results of other work on symmetry method have been captured in several outstanding literary, works, [4], [5], [6], [7].

The optimal systems for a general Burger's equation

$$u_t = f(x,u)u_x^2 + g(x,u)u_{xx}$$

was determined by the method of symmetry group classification, [8]. The present work discusses the optimal system of the nonlinear partial differential equation (1). In this work we use the results of one-dimensional optimal systems to calculate the invariant solutions of some examples. The method followed in the construction of the one-dimensional optimal systems is found in [3].

In this paper while constructing the principal Lie algebra, we also show how to determine the Lie point symmetries of (1). We proceed to construct transformations for the generators which provide one-parameter group of transformations for the operator using Lie equations, [9], [10]. We also show the method of determining invariant solutions,[6], [7]. The paper also illustrates the construction of one-dimensional optimal systems of principal Lie algebras L_5. We conclude by calculating invariant solutions of some one-dimensional subalgebras of each extended algebra L_5.

2 Symmetries of the Burgers equation

The Burgers equation is given by (1) in which the dependent variable is u and independent variables t and x.

2.1 Prolongation formulas

Given that x and t are two independent variables, and u a differential variable, then the total derivatives are defined by

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots . \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots . \end{aligned}$$

The infinitesimal generator X is given by

$$X = T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + (fu+g)\frac{\partial}{\partial u}, \quad (2)$$

where $X^{[2]}$ is the second prolongation of X and is given by

$$\begin{aligned} X^{(2)} &= X + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} \quad (3) \\ &+ \zeta_{tx}^2\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}}. \end{aligned}$$

The coefficients $\zeta_x^1, \zeta_t^1, \zeta_{tt}^2, \zeta_{tx}^2$ and ζ_{xx}^2 are given by

$$\begin{aligned} \zeta_x^1 &= D_x(fu+g) - u_xD_x(\xi) - u_tD_x(T) \\ &= g_x + uf_x + u_x(f-\xi_x) - u_tT_x, \\ \zeta_t^1 &= D_t(fu+g) - u_xD_t\xi - u_tD_tT \\ &= g_t + uf_t + u_t(f-T_t) - u_x\xi_t. \end{aligned}$$

$$\begin{aligned} \zeta_{xx}^2 = \quad & D_x(\zeta_x^1) - u_{xx}D_x(\xi) - u_{xt}D_x(T) \\ = \quad & g_{xx} + uf_{xx} + u_x(2f_x - \xi_{xx}) - u_tT_{xx} \\ & + u_{xx}(f - 2\xi_x) - 2u_{tx}T_x, \\ \zeta_{tx}^2 = \quad & D_t(\zeta_x^1) - u_{xx}D_t(\xi) - u_{xt}D_t(T) \\ = \quad & g_{xt} + uf_{xt} + u_x(f_t - \xi_{xt}) - u_t(f_x - T_{xt}) \\ & + \xi_tu_{xx} + u_{tx}(f - \xi_x - T_t) - u_{tt}T_x, \\ \zeta_{tt}^2 = \quad & D_t(\zeta_t^1) - u_{xt}D_t(\xi) - u_{tt}D_t(T) \\ = \quad & g_{tt} + uf_{tt} + u_x(\xi_{tt}) - u_t(2f_t - T_{tt}) \\ & + u_{tt}(f - 2T_t) - 2u_{tx}\xi_t. \end{aligned}$$

2.2 Determination of symmetries Burgers equation

We solve the determining equations for symmetries of the Burgers equation (1). The determining equation is determined from the invariance condition

$$\begin{aligned} &\Big(T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + (fu+g)\frac{\partial}{\partial u} \\ &+ \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} \\ &+ \zeta_{xx}^2\frac{\partial}{\partial_{xx}}\Big)\Big(u_t - u_{xx} - uu_x\Big)\Big|_{u_t = u_{xx}+uu_x} = 0, \end{aligned}$$

or equivalently

$$\left(\zeta_t^1 - \zeta_{xx}^2 - u\zeta_x^1 - u_x(fu+g)\right)\Big|_{u_t = u_{xx}+uu_x} = 0, \quad (4)$$

After substituting for ζ_t^1,ζ_x^1 , ζ_{xx}^2 and $u_t = u_{xx} + uu_x$ in equation (4), we obtain

$$\begin{aligned} & g_t + uf_t + (u_{xx} + uu_x)(f - T_t) \quad (5) \\ & -u_x\xi_t - g_{xx} - uf_{xx} \\ & -u_x(2f_x - \xi_{xx}) + (u_{xx} + uu_x)T_{xx} \\ & -u_{xx}(f - 2\xi_x) + 2u_{tx}T_x \\ & -ug_x - u^2f_x - uu_x(f - \xi_x) \\ & +(u_{xx} + uu_x)T_x - uu_xf - u_xg = 0. \end{aligned}$$

Separation of coefficients in equation (5) yields

$$\begin{aligned} C &: \ g_t = 0, & (6) \\ u &: \ f_t - f_{xx} - g_x = 0, & (7) \\ u^2 &: \ f_x = 0, & (8) \\ u_x &: \ -uT_t + \xi_{xx} + u\xi_x - uf - g - \xi_t = 0, & (9) \\ u_{xx} &: \ T_x + 2\xi_x - T_t + T_{xx} = 0. & (10) \\ u_{tx} &: \ T_x = 0. & (11) \end{aligned}$$

Integrating (11) with respect to x results into

$$T = a(t) \quad (12)$$

Substituting for T in (10) and integrating with respect to x results in that

$$\xi = \frac{1}{2}a_tx + b(t), \quad (13)$$

Differentiating (13) with respect to t we have that

$$\xi_t = \frac{1}{2}a_{tt}x + b_t. \quad (14)$$

The determining equation (9) splits into

$$\begin{aligned} \xi_{xx} - g - \xi_t = 0 \quad & (15) \\ -T_t + \xi_x - f = 0 \quad & (16) \end{aligned}$$

After differentiating equation (14) with respect to t, and applying equation (6) we obtain that

$$\xi_{tt} = \frac{1}{2}a_{ttt}x + b_{tt} = 0, \quad (17)$$

whence $a_{ttt} = 0$ and $b_{tt} = 0$, and thus we have that

$$a(t) = C_1t^2 + 2C_2t + C_3, \qquad b(t) = C_4t + C_5. \quad (18)$$

It follows from equation (15) that

$$g = -\xi_t = -\frac{1}{2}a_{tt}x - b_t = -C_1x - C_4 \quad (19)$$

We also have equation (16) which determines that the function f is given by

$$f = T_t - \xi_x = -\frac{1}{2}a_t(t) = -C_1 t - C_2 \tag{20}$$

The infinitesimals are

$$\begin{aligned} T &= C_1 t^2 + 2C_2 t + C_3, && (21)\\ \xi &= C_1 tx + C_2 x + C_4 t + C_5, && (22)\\ \eta &= -(C_1 t + C_2)u - C_1 x - C_4 && (23) \end{aligned}$$

2.2.1 Symmetries

For the symbol of infinitesimal transformation or the generator,

$$X = (T)\frac{\partial}{\partial t} + (\xi)\frac{\partial}{\partial x} + (fu + g)\frac{\partial}{\partial u}$$

the corresponding symmetries are given by

$$\begin{aligned} X_1 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}, && (24)\\ X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, && (25)\\ X_3 &= \frac{\partial}{\partial t}, && (26)\\ X_4 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, && (27)\\ X_5 &= \frac{\partial}{\partial x}. && (28) \end{aligned}$$

2.3 Commutator Table

Considering the operators

$$\begin{aligned} X_a &= T_a\frac{\partial}{\partial t} + \xi_a\frac{\partial}{\partial x} + \eta_a\frac{\partial}{\partial u}\\ X_b &= T_b\frac{\partial}{\partial t} + \xi_b\frac{\partial}{\partial x} + \eta_b\frac{\partial}{\partial u} \end{aligned}$$

The commutator $[X_a, X_b]$ of operators (24) to (28) is a linear operator defined by the formula

$$[X_a, X_b] = X_a X_b - X_b X_a$$

Furthermore we define

$$\begin{aligned} [X_a, X_b] &= (X_a(T_b) - X_b(T_a))\frac{\partial}{\partial t}\\ &+ (X_a(\xi_b) - X_b(\xi_a))\frac{\partial}{\partial x}\\ &+ (X_a(\eta_b) - X_b(\eta_a))\frac{\partial}{\partial u} \end{aligned}$$

As an illustration we determine the commutator

$$\begin{aligned} [X_2, X_3] &= (X_2(1) - X_3(2t))\frac{\partial}{\partial t} && (29)\\ &+ (X_2(0) - X_3(x))\frac{\partial}{\partial x}\\ &+ (X_2(0) - X_3(0))\frac{\partial}{\partial u}\\ &= -2\frac{\partial}{\partial t}\\ &= -2X_3 \end{aligned}$$

The complete drawn out Table of commutators is given Table 1.

Table 1: Commutator Table of operators

$[,]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-2X_1$	$-X_2$	0	$-X_4$
X_2	$2X_1$	0	$-2X_3$	X_4	$-X_5$
X_3	X_2	$2X_3$	0	X_5	0
X_4	0	$-X_4$	$-X_5$	0	0
X_5	X_4	X_5	0	0	0

The Lie Algebra L_5 spanned by the symmetries (24 - 28) provide a possibility of finding invariant solutions of equation (1) based on any one-dimensional subalgebra of the algebra L_5, i.e. on any operator $X \in L_5$. However, there are an infinite number of one-dimensional subalgebra of the algebra L_5, since an arbitrary operator from L_5 is expressed as

$$X = l^1 X_1 + l^2 X_2 + ... + l^5 X_5 \tag{30}$$

which depends upon arbitrary constants $l^1, l^2, ..., l^5$.

2.4 Construction of an optimal system of one-dimensional subalgebras.

The construction of optimal system of one-dimensional subalgebras of Lie Algebra L_5 follow from the method in [3]. The transformations of the symmetry group with Lie algebra L_5 provide the 5 - parameter group of transformations of the operators $X \in L_5$ or, equivalently, linear transformations of the vector

$$l = (l^1, l^2, ..., l^5) \tag{31}$$

To determine linear transformations we use their generators

$$E_\alpha = c^i_{\alpha\beta} X_i \quad i = 1, 2, 3, 4, 5.$$

and the structure constant of the Lie Algebra L_5 is given by $c^i_{\alpha\beta}$ which is defined by

$$[X_\alpha, X_\beta] = c^i_{\alpha\beta} X_i$$

Case 1: For $\alpha = 1$, we have

$$E_1 = c^i_{1\beta} l^\beta \frac{\partial}{\partial l^i}$$

where $c^i_{\alpha\beta}$ the structure constant of the Lie Algebra L_5 is given by

$$[X_1, X_\beta] = c^i_{1\beta} X_i$$

Setting $\beta = 2$, we have $i = 1$. That is row (1) column (2) from the commutator table, we have that

$$[X_1, X_2] = c^1_{12} X_1$$

The non-vanishing structure constant is

$$c^1_{12} = -2$$

Setting $\beta = 3$, we have $i = 2$. That is row (1) column (3) from the commutator table, we have that

$$[X_1, X_3] = c^2_{13} X_2$$

The non-vanishing structure constant is

$$c^2_{13} = -1$$

Setting $\beta = 5$, we have $i = 4$. That is row (1) column (5) from the commutator table, we have that

$$[X_1, X_5] = c^4_{15} X_4$$

The non-vanishing structure constant is

$$c^4_{15} = -1$$

From equation (2.4) we have that

$$E_1 = c^1_{12} l^2 X_1 + c^2_{13} l^3 X_2 + c^5_{15} l^5 X_4$$

Thus we have that

$$E_1 = -2l^2 \frac{\partial}{\partial l^1} - l^3 \frac{\partial}{\partial l^2} - l^5 \frac{\partial}{\partial l^4} \qquad (32)$$

Case 2: For $\alpha = 2$, we have

$$E_2 = c^i_{2\beta} l^\beta \frac{\partial}{\partial l^i}$$

where $c^i_{\alpha\beta}$ the structure constant of the Lie Algebra L_5 is given by

$$[X_2, X_\beta] = c^i_{2\beta} X_i$$

Setting $\beta = 2$, we have $i = 1$. That is row (2) column (1) from the commutator table, we have

$$[X_2, X_1] = c^1_{21} X_1$$

The non-vanishing structure constant is

$$c^1_{21} = 2$$

Setting $\beta = 3$, we have $i = 3$. That is row (2) column (3) from the commutator table, we have that

$$[X_2, X_3] = c^3_{23} X_3$$

The non-vanishing structure constants are

$$c^3_{23} = -2$$

Setting $\beta = 4$, we have $i = 4$. That is row (2) column (4) from the commutator table, we have that

$$[X_2, X_4] = c^4_{24} X_4$$

The non-vanishing structure constant is

$$c^4_{24} = 1$$

Setting $\beta = 5$, we have $i = 5$. That is row (2) column (5) from the commutator table, we have that

$$[X_2, X_5] = c^5_{25} X_5$$

The non-vanishing structure constant is

$$c^5_{25} = -1$$

From equation (2.4) we have that

$$E_2 = c^1_{21} l^1 X_1 + c^3_{23} l^3 X_3 + c^4_{24} l^4 X_4 + c^5_{25} l^5 X_5$$

Thus we have that

$$E_2 = 2l^1 \frac{\partial}{\partial l^1} - 2l^3 \frac{\partial}{\partial l^3} + l^4 \frac{\partial}{\partial l^4} - l^5 \frac{\partial}{\partial l^5}$$

Case 3: For $\alpha = 3$, we have

$$E_3 = c^i_{3\beta} l^\beta \frac{\partial}{\partial l^i}$$

where $c^i_{\alpha\beta}$ the structure constant of the Lie Algebra L_5 is given by

$$[X_3, X_\beta] = c^i_{3\beta} X_i$$

Setting $\beta = 1$, we have $i = 2$ That is row (3) column (1) from the commutator table, we have that

$$[X_3, X_1] = c^2_{31} X_2$$

The non-vanishing structure constant is

$$c_{31}^{2} = 1$$

Setting $\beta = 2$, we have $i = 3$. That is row (3) column (2) from the commutator table, we have that

$$[X_3, X_2] = c_{32}^{3} X_3$$

The non-vanishing structure constant is

$$c_{32}^{3} = 2$$

Setting $\beta = 4$, we have $i = 5$. That is row (3) column (4) from the commutator table, we have that

$$[X_3, X_4] = c_{34}^{5} X_5$$

The non-vanishing structure constant is

$$c_{34}^{5} = 1$$

From equation (2.4) we have that

$$E_3 = c_{31}^{2} l^1 X_2 + c_{32}^{3} l^2 X_3 + c_{34}^{5} l^4 X_5$$

Thus we have that

$$E_3 = l^1 \frac{\partial}{\partial l^2} + 2l^2 \frac{\partial}{\partial l^3} + l^4 \frac{\partial}{\partial l^5}$$

Case 4: For $\alpha = 4$, we have

$$E_4 = c_{4\beta}^{i} l^{\beta} \frac{\partial}{\partial l^i}$$

where $c_{\alpha\beta}^{i}$ the structure constant of the Lie Algebra L_5 is given by

$$[X_4, X_\beta] = c_{4\beta}^{i} X_i$$

Setting $\beta = 2$, we have $i = 4$. That is row (4) column (2) from the commutator table, we have that

$$[X_4, X_2] = c_{42}^{4} X_4$$

The non-vanishing structure constants are

$$c_{42}^{4} = -1$$

Setting $\beta = 3$, we have $i = 5$. That is row (4) column (3) from the commutator table, we have that

$$[X_4, X_3] = c_{43}^{5} X_5$$

The non-vanishing structure constant is

$$c_{43}^{5} = -1$$

From equation (2.4) we have that

$$E_4 = c_{42}^{4} l^2 X_4 + c_{43}^{5} l^3 X_5$$

Thus we have that

$$E_4 = -l^2 \frac{\partial}{\partial l^4} - l^3 \frac{\partial}{\partial l^5}$$

Case 5: For $\alpha = 5$, we have

$$E_5 = c_{5\beta}^{i} l^{\beta} \frac{\partial}{\partial l^i}$$

where $c_{\alpha\beta}^{i}$ the structure constant of the Lie Algebra L_5 is given by

$$[X_5, X_\beta] = c_{5\beta}^{i} X_i$$

Setting $\beta = 1$, we have $i = 4$. That is row (5) column (1) from the commutator table, we have that

$$[X_5, X_1] = c_{51}^{4} X_4$$

The non-vanishing structure constants are

$$c_{51}^{4} = 1$$

Setting $\beta = 2$, we have $i = 5$. That is row (5) column (2) from the commutator table, we have that

$$[X_5, X_2] = c_{52}^{5} X_5$$

] The non-vanishing structure constant is

$$c_{52}^{5} = 1$$

From equation (2.4) we have that

$$E_5 = c_{51}^{4} l^1 X_4 + c_{52}^{5} l^2 X_5$$

Thus we have that

$$E_5 = l^1 \frac{\partial}{\partial l^4} + l^2 \frac{\partial}{\partial l^5}$$

In summary we have the following linear transformations

$$E_1 = -2l^2 \frac{\partial}{\partial l^1} - l^3 \frac{\partial}{\partial l^2} - l^5 \frac{\partial}{\partial l^4}, \quad (33)$$

$$E_2 = 2l^1 \frac{\partial}{\partial l^1} - 2l^3 \frac{\partial}{\partial l^3} + l^4 \frac{\partial}{\partial l^4} - l^5 \frac{\partial}{\partial l^5}, \quad (34)$$

$$E_3 = l^1 \frac{\partial}{\partial l^2} + 2l^2 \frac{\partial}{\partial l^3} + l^4 \frac{\partial}{\partial l^5}, \quad (35)$$

$$E_4 = -l^2 \frac{\partial}{\partial l^4} - l^3 \frac{\partial}{\partial l^5}, \quad (36)$$

$$E_5 = l^1 \frac{\partial}{\partial l^4} + l^2 \frac{\partial}{\partial l^5} \quad (37)$$

2.5 Construction of Lie Equations

We determine the transformations provided by the generators (33) to (37). For the generator E_1, the Lie equations for the parameter a_1 are written

$$\frac{d\bar{l}^1}{da_1} = -2\bar{l}^2 \quad \frac{d\bar{l}^2}{da_1} = -\bar{l}^3 \qquad (38)$$
$$\frac{d\bar{l}^4}{da_1} = -\bar{l}^5 \quad \frac{d\bar{l}^5}{da_1} = 0 \quad \frac{d\bar{l}^3}{da_1} = 0$$

We integrate all five equations of (38) using the initial condition

$$\bar{l}|_{a_1=0} = l \qquad (39)$$

We proceed as follows. For E_1 we have that

$$\begin{aligned}
&\tfrac{d\bar{l}^3}{da_1} = 0 \\
\Rightarrow\ &\bar{l}^3 = l^3; \quad \tfrac{d\bar{l}^5}{da_1} = 0 \\
\Rightarrow\ &\bar{l}^5 = l^5; \\
\tfrac{d\bar{l}^2}{da_1} = -\bar{l}^3 \quad \Rightarrow\ &\bar{l}^2 = -l^3 a_1 + l^2; \\
&\tfrac{d\bar{l}^1}{da_1} = -2[-l^3 a_1 + l^2] \\
\Rightarrow\ &\int d\bar{l}^1 = \int \{2l^3 a_1 - 2l^2\} da_1 \\
\Rightarrow\ &\bar{l}^1 = l^3 a_1^2 - 2a_1 l^2 + l^1; \\
\Rightarrow\ &\bar{l}^4 = -l^5 a_1 + l^4; \quad \bar{l}^5 = l^5
\end{aligned}$$

Thus for E_1 we have the following transformations

$$\bar{l}^1 = l^3 a_1^2 - 2a_1 l^2 + l^1; \ \bar{l}^2 = -l^3 a_1 + l^2; \qquad (40)$$
$$\bar{l}^3 = l^3 \ \bar{l}^4 = -l^5 a_1 + l^4; \ \bar{l}^5 = l^5$$

For the generator E_2 we have the transformations

$$\bar{l}^1 = l^1 a_2^2; \ \bar{l}^2 = l^2; \ \bar{l}^3 = l^3 a_2^{-2} \qquad (41)$$
$$\bar{l}^4 = l^4 a_2; \ \bar{l}^5 = -l^5 a_2.$$

For the generator E_3 we have the transformations

$$\bar{l}^1 = l^1; \ \bar{l}^2 = l^1 a_3 + l^2; \ \bar{l}^3 = l^1 a_3^2 + 2l^2 a_3 + l^3; \ (42)$$
$$\bar{l}^4 = l^4; \ \bar{l}^5 = l^4 a_3 + l^5.$$

For the generator E_4 we have the transformations

$$\bar{l}^1 = l^1; \ \bar{l}^2 = l^2; \ \bar{l}^3 = l^3; \qquad (43)$$
$$\bar{l}^4 = -l^2 a_4 + l^4; \ \bar{l}^5 = -l^3 a_4 + l^5.$$

For the generator E_5 we have the transformations

$$\bar{l}^1 = l^1; \ \bar{l}^2 = l^2; \ \bar{l}^3 = l^3; \qquad (44)$$
$$\bar{l}^4 = l^1 a_5 + l^4; \ \bar{l}^5 = l^2 a_5 + l^5.$$

2.6 One functionally independent invariant

The assertion that the 5×5 matrix $||c^{\lambda}_{\mu\nu} l^{\nu}||$ of coefficients of operators (33) to (37) has rank four, means that the transformations (40) to (44) have precisely one functionally independent invariant. The integration of the equations (33) to (37)

$$E_{\mu}(J) = 0 \quad \mu = 1, 2, 3.., 5 \qquad (45)$$

will help determine the invariant. From equation (33) we have

$$E_1(J) = -2l^2 \frac{\partial J}{\partial l^1} - l^3 \frac{\partial J}{\partial l^2} - l^5 \frac{\partial J}{\partial l^4} = 0 \qquad (46)$$

The characteristic equation of equation (46) is given by

$$\frac{dl^1}{-2l^2} = \frac{dl^2}{-l^3} = \frac{dl^4}{-l^5} \qquad (47)$$

Solving the linear equation

$$\frac{dl^1}{-2l^2} = \frac{dl^2}{-l^3} \qquad (48)$$

yields that

$$C = (l^2)^2 - l^1 l^3$$

Similarly from (35) we have

$$E_3(J) = l^1 \frac{\partial J}{\partial l^2} + 2l^2 \frac{\partial J}{\partial l^3} + l^4 \frac{\partial J}{\partial l^5} = 0 \qquad (49)$$

The characteristic equation of equation (49) is given by

$$\frac{dl^2}{l^1} = \frac{dl^3}{2l^2} = \frac{dl^5}{l^4} \qquad (50)$$

Solving the linear equation

$$\frac{dl^2}{l^1} = \frac{dl^3}{2l^2} \qquad (51)$$

yields that

$$C = (l^2)^2 - l^1 l^3$$

A similar integration with equations (34) and (36) will yield that $l^1 l^3 = l^4 l^5 = C, l^4 l^3 - l^1 l^5 = C$ etc. The logical conclusion is that the transformations (40) to (44) have one functionally independent invariant given by

$$J = (l^2)^2 - l^1 l^3 \qquad (52)$$

This invariant helps to simplify the vector used to determine the optimal system. We can exclude the operator X_1 from the operators providing for the optimal system where possible. This is done by eliminating $\bar{l}^1$ from the transformations of the optimal system from transformation (40). To accomplished this we solve the quadratic equation

$$l^3 a_1^2 - 2l^2 a_1 + l^1 = 0$$

for a_1 from the transformation (40). The results is that

$$a_1 = \frac{l^2 \pm \sqrt{J}}{l^3} \tag{53}$$

where J is given by equation (52). We apply equation (53) only if $J \geq 0$.

The method to determine the optimal system requires the simplification of the vector (31) by means of transformations (40) to (44).

2.7 Cases

The construction of optimal system is divided into several cases.

(1) **The case** $l^3 = 0$. We subdivide this into cases namely (a) $l^3 = 0$, $l^2 \neq 0$ and (b) $l^3 = 0$, $l^2 = 0$.

(a) We discuss the case when $l^3 = 0, \;\; l^2 \neq 0$.
The vector (30) takes the form

$$(l^1, l^2, 0, l^4, l^5), \quad \text{where } \; l^2 \neq 0.$$

We use l^2 to reduce the given vector (31). From equation (40) if we set $a_1 = \frac{l^1}{2l^2}$, then $l^1 = 0$. The vector reduces to

$$(0, l^2, 0, l^4, l^5)$$

Since $l^2 \neq 0$ then from equation (43) we have $\bar{l}^4 = -l^2 a_4 + l^4$. Setting $a_4 = \frac{l^4}{l^2}$, we have that $l^4 = 0$. If we let $a_5 = \frac{l^4}{l^2}$ from equation (44) in $\bar{l}^5 = l^2 a_5 + l^5$, we get $l^5 = 0$. The vector (31) reduces to

$$(0, l^2, 0, 0, 0)$$

Since $l^2 \neq 0$ we can divide the vector (31) by l^2 and obtain the following representation for the optimal system

$$X_2 \tag{54}$$

(b) The case $l^3 = 0, \;\; l^2 = 0$. results in the the vector (31) taking the form

$$(l^1, 0, 0, l^4, l^5)$$

If $l^1 \neq 0$ we use transformation (43) with $a_4 = -\frac{l^4}{l^1}$, and have that $l^4 = 0$. The vector (31) reduces to

$$(l^1, 0, 0, 0, l^5, 0)$$

. If $l^5 \neq 0$, we can assume that $l^5 = 1$, use transformation (41) and make $l^1 \pm 1$. Taking into account the possibility that $l^5 = 0$, we thus obtain the following representation for the optimal system

$$X_5, \;\; X_1 + X_5, \;\; X_1 - X_5 \tag{55}$$

If $l^5 = 0$ and $l^1 \neq 0$, we set $l^1 = 1$. We apply transformation (44) with $a_5 = -l^4$ and obtain the vector

$$(1, 0, 0, 0, 0)$$

If $l^1 = 1$, we get the vector

$$(0, 0, 0, 1, 0)$$

The contribution to the optimal system is provided by the vectors

$$X_1, \;\; X_4 \tag{56}$$

(2) **The case** $l^3 \neq 0, \;\; J > 0$.
We now define a_1 in terms of equation (2.6) and eliminate l^1. We thus have the vector given by

$$(0, l^2, l^3, l^4, l^5), \quad \text{where } \; l^3 \neq 0.$$

J is an invariant for the transformations (40) to (44), and the condition $J > 0$ with $l^3 \neq 0$ implies that $l^2 \neq 0$. We use the transformation (44) with $\bar{l}^5 = l^2 a_5 + l^5$, and set $a_5 = -\frac{l^5}{l^2}$ and have $l^5 = 0$. We also use the transformation (43) with $\bar{l}^4 = -l^2 a_4 + l^4$ and set $a_4 = \frac{l^4}{l^2}$ resulting in $l^4 = 0$. We set $a_3 = -\frac{l^3}{l^2}$ in the transformation (42) where $\bar{l}^3 = 2l^2 a_3 + l^3$, and get $l^3 = 0$. The vector (31) reduces to

$$(0, l^2, 0, 0, 0).$$

The representation for the optimal system is similar to equation (54).

(3) **The case** $l^3 \neq 0, \;\; J = 0$.
The case reduces equation (52) to $a_1 = \frac{l^2}{l^3}$. When we apply transformation (40) we have that $l^2 = 0$. Due to the invariance of $J = (l^2)^2 - l^1 l^3$ with $l^2 = 0$, it follows that $l^1 = 0$. The vector (31) becomes

$$(0, 0, l^3, l^4, l^5)$$

We set $a_4 = \frac{l^5}{l^3}$, in the transformation (43) and have that $l^5 = 0$. This simplifies the vector to

$$(0, 0, l^3, l^4, 0, 0)$$

If $l^4 \neq 0$, we apply the transformation (41) and approximate a_2 to make $l_4 = \pm 1$ and $l^3 = 1$ with the possibility that $l^4 = 0$ to obtain the representation for the optimal system as

$$X_3 \;\; X_3 + X_4 \;\; X_3 - X_4 \tag{57}$$

(3) **The case** $l^3 \neq 0, \;\; J < 0$.
The condition $J = (l^2)^2 - l^1 l^3 < 0$ implies that $l^1 \neq 0$ since $l^3 \neq 0$. From the transformation (40) with $a_1 = \frac{l^2}{l^3}$ we get $l^2 = 0$. In a similar fashion from the transformation (43) with $a_4 = \frac{l^5}{l^3}$ we get $l^5 = 0$. We also apply transformation (44) with $a_5 = -\frac{l^4}{l^1}$ and get $l^4 = 0$. The vector (31) reduces to

$$(l^1, 0, l^3, 0, 0).$$

The condition $J < 0$, with $l^2 = 0$ suggests that l^1 and l^3 should have a common sign. We can approximate a_2 in transformation (41) such that when we divide the vector (31) by an appropriate constant we get that $l^1 = l^3 = 1$. The representation for the optimal system as given by

$$X_1 + X_3 \tag{58}$$

We finally collect all the operators (24) to (28) together with the operators (54), (55), (56), (57) and (58) to form the optimal system

$$\begin{gathered} X_1, \;\; X_2, \;\; X_3, \;\; X_4, \;\; , X_5, \\ X_1 + X_5, \;\; X_1 - X_5, \\ X_3 - X_4, \;\; X_3 + X_4, \\ X_1 + X_3 \end{gathered} \tag{59}$$

3 Invariant Solutions for equation (59)

A useful feature of a symmetry is that it preserves the solutions of a differential equation. This means that if a differential equation has a symmetry then the solutions of the differential equation remain unchanged under symmetry transformations. The symmetry transformations merely permute the integral curves among themselves. Such integral curves are termed invariant solutions. To construct an optimal system of invariant solutions we have to determine the invariant solution for each of the operators of the optimal system (59).

3.1 Invariant Solution for the operator X_1

The operator $X_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}$ has the characteristic equation given by

$$\frac{dt}{t^2} = \frac{dx}{tx} = -\frac{du}{(x+tu)} \tag{60}$$

There are two linear equations that are formed from the characteristic equation (60). The first such linear equation is

$$\frac{dt}{t^2} = \frac{dx}{tx} \tag{61}$$

Integrating equation (61) yields that $\frac{x}{t} = C_1$ where C_1 is the constant of integration. Hence one of the invariants is

$$\frac{x}{t} = \lambda_1 \tag{62}$$

Checking the invariant λ_1 we have that

$$X_1(\lambda_1) = t^2\frac{\partial(\frac{x}{t})}{\partial t} + tx\frac{\partial(\frac{x}{t})}{\partial x} - (x+tu)\frac{\partial(\frac{x}{t})}{\partial u} = -x + x = 0$$

Thus the operator satisfies the invariant condition.
The second linear equation is given as

$$\frac{dt}{t^2} = -\frac{du}{(x+tu)} \tag{63}$$

The equation (63) simplifies to the first order Ordinary differential equation given by

$$\frac{du}{(\frac{x}{t}+u)} + \frac{dt}{t} = 0 \tag{64}$$

but from equation (62) we have that $\frac{x}{t} = \lambda_1$, which we replace in equation (64) and arrive at the equation

$$\frac{du}{(\lambda_1 + u)} + \frac{dt}{t} = 0. \tag{65}$$

Solving the equation (65) we obtain the second invariant given by the equation

$$v = tu + x \tag{66}$$

Checking the invariant v we have that

$$\begin{aligned} X_1(v) &= t^2\frac{\partial(tu+x)}{\partial t} \\ &+ tx\frac{\partial(tu+x)}{\partial x} \\ &- (x+tu)\frac{\partial(tu+x)}{\partial u} \\ &= t^2u + tx - t^2u - tx \\ &= 0 \end{aligned} \tag{67}$$

Thus the operator satisfies the invariant condition. We designate one of the invariants to be a function of the other i.e.

$$v = \phi(\lambda_1)$$

The invariant solution is given by

$$u = \frac{\phi(\lambda_1)}{t} - \lambda_1 \tag{68}$$

We substitute for

$$\begin{aligned} u_t &= \frac{1}{t^2}(x - \phi(\lambda_1) - \lambda_1 \phi'(\lambda_1)) \\ u_x &= \frac{1}{t^2}\phi'(\lambda_1) - \frac{1}{t} \\ u_{xx} &= \frac{1}{t^3}\phi''(\lambda_1) \end{aligned} \tag{69}$$

into the equation (1) and we obtain that

$$u_t - u_{xx} - uu_x = \phi''(\lambda_1) + \phi'(\lambda_1))\phi = 0 \tag{70}$$

which when integrated once yields that

$$\phi'(\lambda_1) + \frac{1}{2}\phi^2(\lambda_1) = \frac{1}{2}C_1 \tag{71}$$

This gives that

$$\frac{d\phi(\lambda_1)}{d\lambda_1} = \frac{1}{2}(C_1 - \phi^2(\lambda_1)) \tag{72}$$

This implies that

$$\int \frac{d\phi(\lambda_1)}{C_1 - \phi^2(\lambda_1)} = \int \frac{1}{2} d\lambda_1 \tag{73}$$

which gives that

$$\int \frac{d\phi(\lambda_1)}{C_1 - \phi^2(\lambda_1)} = \frac{1}{2}\lambda_1 + A \tag{74}$$

with A a constant. For $C_1 = 0$, we obtain that

$$\phi(\lambda_1) = \frac{2}{\lambda_1 + 2A} \tag{75}$$

We conclude that the invariant solution is given by

$$u(t,x) = \frac{2}{x + 2At} - \frac{x}{t} \tag{76}$$

4 Conclusion

The purpose of the project was to gain an insight into the method of optimal system a non linear equation using the simplification of a vector. The challenges were how to simplify the vector used to determine the optimal system. However the determination of the rank the coefficients matrix of operators helped solve the problem. From the present project, the method of finding optimal systems of one-dimensional subalgebras, proved to be effective. We would like to explore them further and even for higher dimensional subalgebras. Future projects would also include extending on the current one to determine an optimal system of the invariant solutions for the equation (1).

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