# A New Trend of Bipolar-Valued Fuzzy Cartesian Products, Relations, and Functions

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*Abstract*: - A bipolar-valued fuzzy set (BVFS) is a generalization of the fuzzy set (FS). It has been applied to a wider range of problems that cannot be represented by FS. New forms of the bipolar-valued fuzzy Cartesian product (BVFCP), bipolar-valued fuzzy relations (BVFRs), bipolar-valued fuzzy equivalence relations (BVFERs), and Bipolar-valued fuzzy functions (BVFFs) are constructed to be a cornerstone of creating new approach of BVF group theory. Unlike other approaches, the definition of BVFCP "A×B" is exceptionally helpful at reclaiming again the subset A and B by using a fitting lattice. Also, the present approach reduced the calculations and numerical steps in contrast to fuzzy and classical BVF cases. Results relating to those on relations, equivalence relations, and functions in the fuzzy cases are proved for BVFRs, BVFERs, and BVFFs.

*Key-Words:* - Bipolar Valued Fuzzy Cartesian Product, Bipolar Valued Fuzzy Relation, Bipolar Valued Fuzzy Equivalence Relations, Bipolar valued Fuzzy Functions, Fuzzy Cartesian Product, Fuzzy Relation, Fuzzy Equivalence Relations, Fuzzy Functions.

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## 1 Introduction

In 1965, Fuzzy sets, [1], were introduced as a generalization of an ordinary set. The concepts of similarity relations and fuzzy orderings were fundamental in many branches of pure and practical research, [2]. The most modern electronic machines help people save energy, time, water, and effort by using the notion of fuzzy sets. Regarding people's needs the notions of fuzzy sets, [1], and fuzzy logic control systems were established and applied in the deepness of industry. Several applications are used and applied fuzzy logic control in the industry such as washing machines, subway trains, cars, coffee machines, etc, [3], [4], [5], [6], [7], [8], [9], [10].

Some limitations and obstacles in conveying human information and inherited experience to mathematical tools led to the incorporation of the perception of FS. For instance, [11], established the notion of BVFS as an enlargement of FSs in which the codomain of membership degree is expanded from unit interval "[0, 1]" to the "[-1, 1]". In BVFS, the "0" value of the membership degree represents elements that are irrelevant to the identical property, the value of the membership degree lies in (0, 1] represents elements that partially satisfy the property, While the value of membership degree lies in [-1, 0) represents elements that partially satisfy to essentially connected with opposite-property. In terms of BVFS applications, numerous researchers proposed applications for decision-making problems, In [12] applied bipolar-valued rough fuzzy sets to decision information systems. In 2022, [13], introduced the assessment for choosing the best alternative fuel by using bipolar-valued fuzzy sets. Also, In 2019, [14], incorporated Hesitant fuzzy sets and bipolar-valued fuzzy sets to solve the problem of multi-attribute group decision-making. In 2020, [15], defined bipolar Fuzzy Graphs with Applications.

In 2023, [16], defined the bipolar interval-valued fuzzy hypergraph. Their notion can represent fuzzy data structure. Their research discovered the inner relationship of fuzzy data and gave some characterizations of it. In 2018, [17], introduced the notion of bipolar fuzzy matroids and applied it to graph theory and linear algebra. Also, they applied several applications in decision support systems and network analysis by using bipolar fuzzy matroids. The first generalization of bipolar fuzzy sets to the realm of complex numbers is highlighted by introducing the notion of bipolar complex fuzzy sets, [18].

Functions are unique types of relations in standard set theory, while relations are subsets of Cartesian products. Therefore, the standard theory of relations and functions heavily relies on the concept of the Cartesian product. Several researchers, [19], [20], have dealt with bipolar-valued fuzzy relations without referring to what could be termed bipolar-valued fuzzy Cartesian products; this concept has not yet been properly explained.

In this stage, the priority of finding the basic notions of mathematics such as Cartesian products, relations, and equivalence relations based on BVFS becomes essential. Several researchers, [21], introduced fuzzy Cartesian products, fuzzy relations, fuzzy equivalence relations, and fuzzy functions. Also [22], [23], started their works by introducing Cartesian products, relations, equivalence relations, and functions under intuitionistic fuzzy sets following the, [21], approach. Later, [24], incorporated the complex fuzzy sets and group theory by defining the complex fuzzy Cartesian products, relations, and functions according to [21]. All the mentioned researchers extended their studies to the field of algebra and used the [25], approach in fuzzy sets, intuitionistic fuzzy sets, and complex fuzzy sets, respectively, to build the fuzzy space, intuitionistic fuzzy space, and complex fuzzy space. Therefore, our contribution to defining the bipolar fuzzy Cartesian products, relations, and functions can be straightforward by following the [21], approach under a new set of bipolar-valued fuzzy sets. The problem appears in building a reasonable and rational structure of BVFCP, BVFRs, and BVFERs. Therefore, we start from ordinary set principles. Ordinary functions are considered a kind of ordinary relations as well as ordinary relations are a collection of elements contained in ordinary Cartesian products in ordinary set theory, [26]. Thus, the Cartesian products highlighted a magnificent need to build the basic theories of functions and relations. The notion of fuzzy relations was applied in several types of research, [27], [28], [29], without referring to the notion of fuzzy Cartesian products (FCPs). Later, the generalized notions of FCPs, fuzzy relations (FR), and fuzzy equivalent relations (FERs) were reasonably achieved by [21]. In the same manner, the notion of BVFCPs, BVFRs, and BVFER are not yet correctly accomplished.

In 1991, [21], have avoided the inconvenience of retrieving the fuzzy subsets A and B from the A×B defined by [30], and reduced fuzzy Cartesian products to the ordinary Cartesian product. In, Reference, [21], They proposed FRs and FERs based on his new finding of fuzzy Cartesian products. [31], got several findings by employing the concept of fuzzy relations. Consequently, [25], introduced a new method to fuzzy group theory based on fuzzy space and fuzzy binary operations. His method was judged to reformulate and generalize the fuzzy subgroups, [32]. The other authors used and applied, [33], approach to introducing Fuzzy ideals and bi-ideal in fuzzy semigroups, fuzzy normal subgroups, [34], intuitionistic fuzzy spaces, and intuitionistic fuzzy groups, [22], [23], complex fuzzy groups, [24] and others.

In this research, a reasonable development of bipolar-valued fuzzy Cartesian products is proposed. This notion avoids the inconvenience that appeared in [30] and can reduce BVFCP to fuzzy Cartesian products and consequently to an ordinary Cartesian product. After that, some reasonable notions such as BVFR and BVFER are introduced. Some results corresponding to those on crisp relations, fuzzy relations, and fuzzy equivalence relations are studied and proved for BVFRs, BVFERs and BVFFs.

## 2 Preliminaries

In this section, we recall some main theorems and notions related to the present results.

**Definition 2.1** [1], A fuzzy set $A$ can be written as a membership function $\eta_A(u)$ maps a universe of discourse $U$ to a unit interval $[0, 1] = I$.

**Definition 2.2** [12]Let $U$ be a nonempty set. Then, a set $H=(H^{-},H^{+})$ is called a bipolar-valued fuzzy set in $U$, where $H^{+}:U\rightarrow[0,1]$ and $H^{-}:U\rightarrow[-1,0]$.

**Definition 2.3** [12] Let $U$ be a nonempty set, and let $H,T\in BPF(U)$.

- **(i)** $H$ is contained in $T$, denoted by $H\subset T$, if $H^{+}(u)\leq T^{+}(u)$ and $H^{-}(u)\geq T^{-}(u),\forall u\in U$.
- **(ii)** The form $H^{c}=((H^{c})^{-},(H^{c})^{+})$ represents the complement of $H$, and it is a BFS in $U$ defined as: $H^{c}(u)=\left(-1-H^{+}(u),1-H^{+}(u)\right),\forall u\in U$, where $(H^{c})^{+}(u)=1-H^{-}(u),(H^{c})^{-}(u)=-1-H^{-}(u)$.
- **(iii)** The form $H\cap T$ represents the intersection of $H$ and $T$, and it is a BFS in $U$ defined as: $(H\cap T)(u)=(H^{-}(u)\vee T^{-}(u),H^{+}(u)\wedge T^{+}(u)),\forall u\in U$.
- **(iv)** The form $H\cup T$ represents the union of $H$ and $T$, and it is a BFS in $U$ defined as: $(H\cup T)(u)=(H^{-}(u)\wedge T^{-}(u),H^{+}(u)\vee T^{+}(u)),\forall u\in U$.

**Definition 2.4** [21] The FCP of two ordinary sets $U$ and $V$, $U\,\overline{\times}\,V$, is the collection of all M-fuzzy subsets of $U\times V$, where, $U\,\overline{\times}\,V=M^{U\times V}$

An element of $U\,\overline{\times}\,V$ is then a function $C:U\times V\rightarrow M$, or

$$C=\{\left((u,v),(\delta_1,\delta_2)\right):(u,v)\in U\times V,(\delta_1,\delta_2)=C(u,v)\in M=I\times I\}.$$

The FCP of a fuzzy subset $H=\{(u,\delta)\}$ of $U$ and a fuzzy subset $T=\{(v,\gamma)\}$ of $V$ is the M-fuzzy subset $H\times T$ of $U\times V$ defined by:

$$H\,\overline{\times}\,T=\{\left((u,v),(\delta,\vartheta)\right):u\in U,v\in V\}$$

It is clear that $H\,\overline{\times}\,T$ is an element of $U\,\overline{\times}\,V$ for every $H\in J^{U}$ and $T\in J^{V}$.

**Definition 2.5** [21] An FR $\rho$ maps $U$ to $V$ is a subset of the FCP $U\,\underline{\overline{\times}}\,V$. Then $\rho$ is a collection of M-fuzzy subsets $C:U\times V\rightarrow M$. An FR maps $U$ to $V$ is called an FR in $U$.

**Definition 2.6** [21] Let $\rho$ be an FR in $U$, that is $\rho\subset U\,\underline{\overline{\times}}\,U$. Then $\rho$ is called:

**-Reflexive** in $U$ if and only if $\forall u\in U$ and $\delta\in I$, $\exists H\in\rho$ such that $((u,u),(\delta,\delta))\in H\in\rho$.

**-Symmetric** if and only if whenever $((u,v),(\delta,\vartheta))\in H\in\rho,\exists T\in\rho$ such that $((v,u),(\vartheta,\delta))\in T\in\rho$.

**-Transitive** if and only if whenever $((u,v),(\delta,\vartheta))\in H\in\rho$ and $((v,z),(\vartheta,\alpha))\in T\in\rho$, $\exists C\in\rho$ such that $((u,z),(\delta,\alpha))\in C\in\rho$.

An FR in $U$ is called a FER in $U$ if and only if it is satisfies all axioms above.

**Definition 2.7** A fuzzy function from $M$ to $N$ is a fuzzy relation $G$ from $M$ to $N$ that meets the conditions given below:

**(i)** For every element $m\in M$ and membership grade $e\in L$, there exist unique elements $n\in N$ and membership grade $t\in L$ such that $(m,n,e,t)$ belongs to some $A\in G$.

**(ii)** If $(m,n,e_1,t_1)\in A\in G$ and $(m,n',e_2,t_2)\in B\in G$, then $n=n'$.

**(iii)** If $(m,n,e_1,t_1)\in A\in G$ and $(m,n,e_2,t_2)\in B\in G$, then $e_1>e_2$ indicates $t_1\geq t_2$.

**(iv)** If $(m,n,e,t)\in A\in G$, then $e=0$ indicates $t=0$ and $e=1$ indicates $t=1$.

Conditions (i) and (ii) lead to the conclusion that there exists a unique ordinary function $G{:}M\rightarrow N$ and for each element $m\in M$, there exists a unique ordinary function $g_m: L\rightarrow L$. Conditions (iii) and (iv) are equivalent to the following conditions:

(***a***) $g_m$ shows nondecreasing behaviour on the set $L$.

(***b***) $g_m(0)=0$ and $g_m(1)=1$.

# 3 New Cartesian Product between BVFSs

In this section, the form of BVFCP is discussed, and the structure of a suitable lattice is presented below. The main definition is formulated in Definition 3.1. the difference between our approach and the previous approach is illustrated by Example 3.1. A justification after the example is discussed in detail. Lastly, some relations of BVFCP union and intersection are described in Proposition 3.1.

The totally ordered set $W=[-1,0]\times[0,1]$ is a lattice concerning infimum $\wedge$ and supremum $\vee$ operations. Then $W$ is distributive but not complemented lattice. Here a partial order "$\leq$" on $W$, is defined on $W\times W=K$ as follows:

**(i)** $[(\delta_1{}^{-},\delta_1{}^{+}),(\delta_2{}^{-},\delta_2{}^{+})]\leq[(\vartheta_1{}^{-},\vartheta_1{}^{+}),(\vartheta_2{}^{-},\vartheta_2{}^{+})]$ iff $\delta_1{}^{-}\geq\vartheta_1{}^{-},\delta_2{}^{-}\geq\vartheta_2{}^{-}$, and $\delta_1{}^{+}\leq\vartheta_1{}^{+},\delta_2{}^{+}\leq\vartheta_2{}^{+}$, whenever $\vartheta_1{}^{-},\vartheta_1{}^{+}\neq 0$ and $\vartheta_2{}^{-},\vartheta_2{}^{+}\neq 0$.

**(ii)** $[(0,0),(0,0)]=[(\vartheta_1{}^{-},\vartheta_1{}^{+}),(\vartheta_2{}^{-},\vartheta_2{}^{+})]$ whenever $\vartheta_1{}^{-},\vartheta_1{}^{+}=0$ or $\vartheta_2{}^{-},\vartheta_2{}^{+}=0$.

for every $\left[(\delta_1{}^{-},\delta_1{}^{+}),(\delta_2{}^{-},\delta_2{}^{+})\right]$, and $[(\vartheta_1{}^{-},\vartheta_1{}^{+}),(\vartheta_2{}^{-},\vartheta_2{}^{+})]\in K$.

The Cartesian product $K=W\times W$ is then a distributive but not complemented vector lattice. The infimum and supremum operations in $K$ are characterized as follows:

1. The operation of infimum in $K$ is characterized by

$$[(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)] \wedge [(\vartheta_1{}^-, \vartheta_1{}^+), (\vartheta_2{}^-, \vartheta_2{}^+)] = [(\delta_1{}^-, \delta_2{}^-) \vee (\vartheta_1{}^-, \vartheta_2{}^-)],$$
$$[(\delta_1{}^+, \delta_2{}^+) \wedge (\vartheta_1{}^+, \vartheta_2{}^+)] = [((\delta_1{}^- \vee \vartheta_1{}^-), (\delta_2{}^- \vee \vartheta_2{}^-)), ((\delta_1{}^+ \wedge \vartheta_1{}^+), (\delta_2{}^+ \wedge \vartheta_2{}^+))].$$
$$= [((\delta_1{}^- \vee \vartheta_1{}^-), (\delta_1{}^+ \wedge \vartheta_1{}^+)), ((\delta_2{}^- \vee \vartheta_2{}^-), (\delta_2{}^+ \wedge \vartheta_2{}^+))].$$

2. The operation of supremum in $K$ is characterized by

$$[(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)] \vee [(\vartheta_1{}^-, \vartheta_1{}^+), (\vartheta_2{}^-, \vartheta_2{}^+)] = [(\delta_1{}^-, \delta_2{}^-) \wedge (\vartheta_1{}^-, \vartheta_2{}^-)],$$
$$[(\delta_1{}^+, \delta_2{}^+) \vee (\vartheta_1{}^+, \vartheta_2{}^+)] \leq [((\delta_1{}^- \wedge \vartheta_1{}^-), (\delta_2{}^- \wedge \vartheta_2{}^-)), ((\delta_1{}^+ \vee \vartheta_1{}^+), (\delta_2{}^+ \vee \vartheta_2{}^+))]$$
$$= [((\delta_1{}^- \wedge \vartheta_1{}^-), (\delta_1{}^+ \vee \vartheta_1{}^+)), ((\delta_2{}^- \wedge \vartheta_2{}^-), (\delta_2{}^+ \vee \vartheta_2{}^+))],$$
$$\forall [(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)], [(\vartheta_1{}^-, \vartheta_1{}^+), (\vartheta_2{}^-, \vartheta_2{}^+)] \in K.$$

Note that the equality holds in the part "2" when $\delta_j{}^-$and $\delta_j{}^+, \neq 0 \neq \vartheta_j{}^-$ and $\vartheta_j{}^+$.

A $K$-bipolar valued fuzzy subset associate values of membership function from $U$ to the lattice $K = W \times W$ e, is thus a function from $U$ to $K$.

In this research, the form $\{(u, (H^-(u), H^+(u))) : u \in U\}$ or, simply, $\{(u, (\delta^-, \delta^+))\}$, where $H^-(u) = \delta^-$, $H^+(u) = \delta^+$, are used to represent a BVF subset $H$ of $U$. Also, a K-BVF subset of $U$, a BVF subset of $U \times V$ and a K-BVF subset of $U \times V$ are represented by $\{(u, [(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)])\}, \{((u, v), (\delta^-, \delta^+))\}$ and $\{((u, v), [(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)])\}$, respectively. To each BVF subset $\{(u, (\delta_1{}^-, \delta_1{}^+))\}$ of $U$ and BVF subset $\{(v, (\delta_2{}^-, \delta_2{}^+))\}$ of $V$ there maps an K-BVF subset $\{((u, v), [(\delta_1{}^-, \delta_1{}^+), (\delta_2{}^-, \delta_2{}^+)])\}$ of $U \times V$. Also, the representation $(u, (\delta^-, \delta^+)) \in H$; where $H \in W^U$, where, $H^-(u) = \delta^-$ *and* $H^+(u) = \delta^+$.

**Definition 3.1** The BVFCP of two ordinary sets $U$ and $V$, denoted by $U \overline{\times} V$, is the collection of all $K$-BVF subsets of $U \times V$ that is $U \overline{\times} V = K^{U \times V}$,

An element of $U \overline{\times} V$ is then a function $M: U \times V \to K$, or

$$M = \{((u, v), [(\delta^-, \delta^+), (\vartheta^-, \vartheta^+)]) : (u, v) \in U \times V, [(\delta^-, \delta^+), (\vartheta^-, \vartheta^+)] = M(u, v) \to K\}.$$

The BVFCP of a BVF subset $H = \{(u, (\delta^-, \delta^+))\}$ of $U$ and a BVF subset $T = ((v, (\vartheta^-, \vartheta^+))\}$ of $V$ is the K-BVF subset $H \underline{\times} T$ of $U \times V$ defined by:

$$H \underline{\times} T = \{((u, v), ((H^-(u), H^+(u)), (T^-(v), T^+(v)))) : u \in U, v \in V\} \equiv \{((u, v), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+)))\}.$$

Therefore, $H \underline{\times} T$ is an element of $U \overline{\times} V$, $\forall H \in W^U$ and $\forall T \in W^V$.

**Example 3.1** In this example we are going to compare the ordinary approach with the present approach, suppose a BVF subset $H = \{(u_1, (-.2, .6)), (u_2, (-.4, .3)), (u_3, (-.8, .9))\}$ of $U$ and a BVF subset $T = ((v_1, (-.5, .8)), (v_2, (-.3, 1))\}$ of $V$. Then the K-BVF subset $H \underline{\times} T$ of $U \times V$ defined by:

$$H \underline{\times} T = \{((u_1, v_1), ((-.2, .6), (-.5, .8))), ((u_1, v_2), ((-.2, .6), (-.3, 1))), ((u_2, v_1), ((-.4, .3), (-.5, .8))), ((u_2, v_2), ((-.4, .3), (-.3, 1))), ((u_3, v_1), ((-.8, .9), (-.5, .8))), ((u_3, v_2), ((-.8, .9), (-.3, 1)))\}$$

But the ordinary (classical) cartesian product of two subsets BVFS $H \times T$ of $U \times V$ defined by:

$$H \times T = \{((u_1, v_1), (t^-(-.2, -.5), t^+(.6, .8))), ((u_1, v_2), (t^-(-.2, -.3), t^+(.6, 1))), ((u_2, v_1), (t^-(-.4, -.5), t^+(.3, .8))), ((u_2, v_2), (t^-(-.4, -.3), t^+(.3, 1))), ((u_3, v_1), (t^-(-.8, -.5), t^+(.9, .8))), ((u_3, v_2), (t^-(-.8, -.3), t^+(.9, 1)))\}$$
$$= \{((u_1, v_1), (-.2, .6)), ((u_1, v_2), (-.2, .6)), ((u_2, v_1), (-.4, .3)), ((u_2, v_2), (-.3, .3)), ((u_3, v_1), (-.5, .8\})), ((u_3, v_2), (-.3, .9))\}.$$

where $t = (t^-, t^+)$ is any BVF t-norm, here we apply $t = (max, min)$.

In Example 3.1, the difference clearly appeared by providing the ability to recall the values of positive and negative membership functions for both objects $u_i \in U$ and $v_j \in V$, after computing the K-BVF subset $H \underline{\times} T$ of $U \times V$. In contrast to the ordinary case of BVFS, the details were omitted and new values appeared. The presented approach identifies and recalls the original information before computations and simplification of the information in any application.

**Remarks**: (1) When the ordinary sets U and V are considered as bipolar-valued fuzzy subsets of themselves, i.e. $U = \{(u, (0, 1): u \in U\}$, $V = \{(v, (0, 1): v \in V\}$, the notions of bipolar-valued fuzzy Cartesian product and both fuzzy Cartesian products and ordinary Cartesian products of $U$ and $V$ equal, i.e. $U \underline{\times} V = U \times V$.
(2) It is easy to generalize the previous definition and statements by substituting a random completely distributive lattice for the $W$. The following proposition can be easily verified if one considers the properties of the lattice $K$.

**Proposition 3.1** For all nonempty BVF subsets $H, T$ of $U$ and nonempty fuzzy subsets $C$, $D$ of $V$, we have:

$$H \underline{\times} (C \cap D) = (H \underline{\times} C) \cap (H \underline{\times} D), \quad (1)$$

$$H \underline{\times} (C \cup D) = (H \underline{\times} C) \cup (H \underline{\times} D), \quad (2)$$

$$(H \cap T) \underline{\times} (C \cap D) = (H \underline{\times} C) \cap (T \underline{\times} D), \quad (3)$$

$$(H \cup T) \underline{\times} (C \cup D) \supset (H \underline{\times} C) \cup (T \underline{\times} D), \quad (4)$$

$$H \underline{\times} C \subset T \underline{\times} D \Longrightarrow H \subset T \; and \; C \subset D, \quad (5)$$

**Proof**. Trivial.

## 4 Bipolar Valued Fuzzy- Relations and Equivalence Relations

In this section, the main definitions of BVFR and BVER are proposed and discussed. The most fundamental theorems and results on ordinary relations, [26] and fuzzy relations, [21], are investigated and developed to be reasonable under the notion of BVFR and BVFER.

**Definition 4.1** A BVFR $\beta$ maps $U$ to $V$ is a subset of the BVFCP $U \overline{\times} V$. In other words, $\beta$ is a member of K-BVF subsets $M : U \times V \rightarrow K$. A BVFR from $U$ to $U$ is said to be a BVFR in $U$.
**Note 4.1** From Definition 3.1, we may see that the BVFCP $U \overline{\times} V$ is itself a BVFR from $U$ to $V$.
**Note 4.2** The BVFCP $U \overline{\times} U$ is called the complete BVFR in $U$.
**Note 4.3** The BVFR $\emptyset \underline{\times} \emptyset = \emptyset$ is called the null BVFR.
**Note 4.4** The identity BVFR lies between complete and null BVFR, denoted by $\Delta_U$. that is $\Delta_U = \{((u, u), ((\delta^-, \delta^+), (\delta^-, \delta^+))): u \in U\}$ contains in K-BVF subset.

**Definition 4.2** Let $\beta_1$ and $\beta_2$: $U \rightarrow V$ to $V$ be two BVFRs. We call that $\beta_2$ is containing $\beta_1$, denoted by $\beta_1 \subset \beta_2$ if and only if when $((u, v), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+))) \in H \in \beta_1$, there exists $B \in \beta_2$ such that $((u, v), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+))) \in T \in \beta_2$. If $\beta_1 \subset \beta_2$ and $\beta_2 \subset \beta_1$, then $\beta_1$ and $\beta_2$ are equal, that is $\beta_1 = \beta_2$ .
**Note. 4.5** We may associate each K-BVF subset $M = \{((u, v), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+)))\}$ of $U \times V$ to a K-BVF subset $M^{-1}$ of $V \times U$ defined by
$M^{-1} = \{((v, u), ((\vartheta^-, \vartheta^+), (\delta^-, \delta^+)))\}$.

**Definition 4.3** Let $\beta: U \rightarrow V$ be a BVFR. The inverse of $\beta$ = $\beta^{-1}: V \rightarrow U$ is the BVFR defined by $\beta^{-1} = \{ M^{-1} : M \epsilon \beta\}$.

**Definition 4.4** Let $\beta: U \rightarrow V$ and $\gamma: V \rightarrow Z$ be two BVFRs. The composition of $\beta$ and $\gamma$, denoted $\gamma \circ \beta: U \rightarrow Z$, is a BVFR defined by
$\gamma \circ \beta = \{((u, z), ((\delta^-, \delta^+), (\alpha^-, \alpha^+))) \in M : M \in U \overline{\times} Z\}$. Where a K-BVF subsets $M \in U \overline{\times} Z$ defined by:

$((u, z), ((\delta^-, \delta^+), (\alpha^-, \alpha^+))) \in M$ if and only if $\exists (v, (\vartheta^-, \vartheta^+)) \in V \times W$ such that $((u, v), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+))) \in A$ and $((v, z), ((\vartheta^-, \vartheta^+), (\alpha^-, \alpha^+))) \in B$ for some $\beta$ and $B \in \gamma$.
**Example 4.1** Suppose I, N, A, Z, M and Q represent names of cities and there are three sets labelled as $W = \{M, Q\}$, $V = \{N, Z\}$, and $U = \{I, A\}$. Then the ordinary Cartesian product of $W \times U$, and $U \times V$ are defined as $W \times U = \{(M, I), (M, A), (Q, I), (Q, A)\}$, and $U \times V = \{(I, N), (I, Z), (A, N), (A, Z)\}$. For example, let $\beta_1(W, U)$ be a relation called "the first city is warmer than the second city" and let $\beta_2(U, V)$ be a relation called "the first city is more modern than the second city". The relations $\beta_1(W, U)$ and $\beta_2(U, V)$ can be presented by the opinion of tourists who have visited and/or had enough knowledge about these cities. The following relational matrices may evaluate the relations $\beta_1, \beta_2$ by using a bipolar fuzzy mathematical method as:

$$\beta_1(W, U) = \begin{bmatrix} & \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{M} & ((-.4, .6), (-.7, .4)) & ((-.4, .6), (-.1, .5)) \\ \boldsymbol{Q} & ((-.6, .9), (-.7, .4)) & ((-.6, .9), (-.1, .5)) \end{bmatrix}$$

For illustration, the relation between cities $M$ and $I$ (the element $(M, I) \in W \times U$) may be represented with membership value of

$\big((-.4,.6),(-.7,.4)\big) \in K$, where the values "0.6", and "0.4" have pointed the opinion of tourists that the cities *M*, and *I*, respectively, are warm (satisfied the property) and the values "− 0.4", and "− 0.7" have pointed the opinion of tourists that the cities M, and I, respectively, are not warm (not satisfied the property).

$\beta_2(U,V)=$

$$\begin{bmatrix} & \boldsymbol{N} & \boldsymbol{Z} \\ \boldsymbol{I} & \big((-.6,.3),(-.5,.8)\big) & \big((-.6,.3),(-.3,.7)\big) \\ \boldsymbol{A} & \big((-.1,.9),(-.5,.8)\big) & \big((-.1,.9),(-.3,.7)\big) \end{bmatrix}$$

For illustration, the relation between cities *I* and *K* (the element $(I,N) \in W \times V$) may be represented with membership value of $\big((-.6,.3),(-.5,.8)\big) \in K$, where the values "0.3", and "0.8" have pointed the opinion of tourists that the cities *I*, and *N*, respectively, are considered as modern city and the values "−0.6", and "−0.5" have pointed the opinion of tourists that the cities *I*, and *N*, respectively, are not considered as a modern city.

Then, the composition $\beta_1 \circ \beta_2$ presented the current approach may be running as follows:

$\beta_1 \circ \beta_2(W,V)=$

$$\begin{bmatrix} & \boldsymbol{N} & \boldsymbol{Z} \\ \boldsymbol{M} & \big((-.4,.6),(-.5,.8)\big) & \big((-.4,.6),(-.3,.7)\big) \\ \boldsymbol{Q} & \big((-.6,.9),(-.5,.8)\big) & \big((-.6,.9),(-.3,.7)\big) \end{bmatrix}$$

Clearly, Definition 4.4 has the same algebraic structure with ordinary composition relation and composition fuzzy relation. Therefore, no need to do an additional process to evaluate the membership values of the composition of two fuzzy relations as in, [1], [26].

**Theorem 4.1** For any fuzzy relations $\beta, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2$ defined on the appropriate sets, we have:

$$(\beta_1 \circ \beta_2) \circ \beta_3 = \beta_1 \circ (\beta_2 \circ \beta_3), \quad (6)$$

$$\beta_1 \subset \beta_2 \text{ and } \gamma_1 \subset \gamma_2 \Rightarrow \beta_1 \circ \gamma_1 \subset \beta_2 \circ \gamma_2, \quad (7)$$

$$\beta_1 \circ (\beta_2 \cup \beta_3) = \beta_1 \circ \beta_2 \cup \beta_1 \circ \beta_3, \quad (8)$$

$$\beta_1 \circ (\beta_2 \cap \beta_3) \subset \beta_1 \circ \beta_2 \cap \beta_1 \circ \beta_3, \quad (9)$$

$$\beta_1 \subset \beta_2 \Rightarrow \beta_1^{-1} \subset \beta_2^{-1}, \quad (10)$$

$$(\beta^{-1})^{-1} = \beta \text{and } (\beta_1 \circ \beta_2)^{-1} = \beta_2^{-1} \circ \beta_1^{-1}, \quad (11)$$

$$(\beta_1 \cup \beta_2)^{-1} = \beta_1^{-1} \cup \beta_2^{-1}, \quad (12)$$

$$(\beta_1 \cap \beta_2)^{-1} = \beta_1^{-1} \cap \beta_2^{-1}, \quad (13)$$

**Proof.** Straightforward from Definition 2.3, 4.2, 4.3, and 4.4.

**Definition 4.5** Let $\beta$ be a BVFR in $U$, i.e. $\beta \subset U \overline{\times} U$. Then

1. $\beta$ is called reflexive in $U$ if and only if $\forall u \in U$ and $\forall(\delta^-,\delta^+) \in W$, $\exists H \in \beta$ such that $((u,u),((\delta^-,\delta^+),(\delta^-,\delta^+))) \in H \in \beta$, that is if and only if $\Delta_U \subset \beta$.
2. $\beta$ is called symmetric if and only if whenever $((u,v),((\delta^-,\delta^+),(n^-,n^+))) \in H \in \beta, \exists H \in \rho$ such that $((v,u),((n^-,n^+),(\delta^-,\delta^+))) \in T \in \beta$, that is if and only if $\beta^{-1} = \beta$.
3. $\beta$ is called transitive if and only if whenever $((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+))) \in H \in \beta$ and $\Big((v,z),\big((\vartheta^-,\vartheta^+),(\alpha^-,\alpha^+)\big)\Big) \in T \in \beta,\ \exists C \in \beta$ such that $((u,z),((\delta^-,\delta^+),(\alpha^-,\alpha^+))) \in C \in \beta$, that is if and only if $\beta \circ \beta \subset \beta$.

A BVFR in $U$ is called a BVFER in $U$ if and only if it is satisfied all three axioms above.

**Example 4.1** For any set $U$, $U \overline{\times} U$ and $\Delta_U$ are BVFER in $U$.

**Theorem 4.2** Let $\beta$ and $\gamma$ be BVFRs of a nonempty set $U$. As a result,

**(i)** If $\beta$ is reflexive, then it follows that $\beta^{-1}$ and $\beta{\circ}\beta$ are also reflexive. (This applies to both symmetric and transitive cases).

**(iii)** If $\beta$ is reflexive, then $\beta$ is a subset of $\beta \circ \beta$.

**(iv)** If $\beta$ is symmetric, then both the union and intersection between $\beta$ and $\beta^{-1}$ are symmetric and their composition is commutative.

**(v)** If $\beta$ and $\gamma$ are reflexive, then their intersection is reflexive. (Holds for both symmetric and transitive)

**(vi)** If $\beta$ and $\gamma$ are symmetric, then their union is symmetric.

Regarding properties 5.1 part (i) and (v), we may deduce that $\beta^{-1}$, $\beta{\circ}\beta$ and $\beta \cap \gamma$ are BVFER in $U$, if $\beta$ and $\gamma$ are BVFR in $U$.

**Theorem 4.3** Let $\beta$ be a BVFER in $U$. Then,

**(i)** $\forall u_0 \in U$, $\beta$ induces a FER, $\beta_W(u_0)$, in $W$ defined by:

$\beta_W(u_0) = \{\big((|\delta^-|,\delta^+),(|\delta^-|,\delta^+)\big) \in [0,1] \times [0,1]: |\delta^-| = \delta^+, |\alpha^-| = \alpha^+,$ and $\big((u_0,u_0),((\delta^-,\delta^+),(\alpha^-,\alpha^+))\big) \in \text{H}$ for some $H \in \beta\}$.

**(ii)** $\forall\big({\delta_0}^-,{\delta_0}^+\big) \in W$, $\beta$ induces an equivalence relation, $\beta_U\Big(\big({\delta_0}^-,{\delta_0}^+\big)\Big)$, in the ordinary case, in $U$ defined by

$\beta_U\left(\left({\delta_0}^-, {\delta_0}^+\right)\right) = \{(u, v) \in U \times U$:
$\left((u, v), \left(\left({\delta_0}^-, {\delta_0}^+\right), \left({\delta_0}^-, {\delta_0}^+\right)\right)\right) \in H$
for some $H \in \beta\}$.

**Proof.** (i) We want to prove that $\beta_W(u_0)$ is,

- Reflexive: $\forall(\delta^-, \delta^+) \in W$, since $\beta$ is reflexive, we get $\left((u_0, u_0), ((\delta^-, \delta^+), (\delta^-, \delta^+))\right) \in H$ for some $H \in \beta$, therefore the elements on the form $((|\delta^-|, \delta^+), (|\delta^-|, \delta^+)) \in \beta_W(u_0)\ \ \forall(\delta^-, \delta^+) \in W$.
- Symmetric: if $((\delta^-, \delta^+), (|\vartheta^-|, \vartheta^+)) \in \beta_W(u_0)$, then $\left((u_0, u_0), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+))\right) \in H$ for some $H \in \beta$. But $\beta$ is symmetric, then $\left((u_0, u_0), ((\vartheta^-, \vartheta^+), (\delta^-, \delta^+))\right) \in T$ for some $T \in \beta$. Therefore the elements in the form $((|\vartheta^-|, \vartheta^+), (\delta^-, \delta^+)) \in \beta_W(u_0)$.
- Transitive: if $((\delta^-, \delta^+), (|\vartheta^-|, \vartheta^+)) \in \beta_W(u_0)$. and $((|\vartheta^-|, \vartheta^+), (|\alpha^-|, \alpha^+)) \in \beta_W(u_0)$, then $\left((u_0, u_0), ((\delta^-, \delta^+), (\vartheta^-, \vartheta^+))\right) \in H$ and $\left((u_0, u_0), ((\vartheta^-, \vartheta^+), (\alpha^-, \alpha^+))\right) \in H$ for some $H, T \in \beta$. However, since $\beta$ is transitive, then $\left((u_0, u_0), ((\delta^-, \delta^+), (\alpha^-, \alpha^+))\right) \in C$ for some $C \in \beta$. Therefore, the elements in the form $((\delta^-, \delta^+), (|\alpha^-|, \alpha^+)) \in \beta_W(u_0)$.

(ii) the proof of (ii) is like (i).

# 5 Bipolar Valued Fuzzy Function

To extend an ordinary theory of fuzzy relation, we mean to introduce the notion of BVF function, since functions are considered as a kind of relation in the ordinary set theory. In this section, we identify BVF functions similarly to a kind of BVFRs.

**Definition 5.1** Let $U$ and $V$ be nonempty sets. A BVF function from $U$ to $V$ can be described as a function **F** from $W^U$ to $W^V$ characterized by the ordered pair $(F, \{(f_u(\delta^-),\ f_u(\delta^+))\}_{u\in U})$, where $F: U \to V$ is a function from $U$ to $V$ and $\{(f_u(\delta^-),\ f_u(\delta^+))\}_{u\in U}$ is a family of functions $(f_u(\delta^-),\ f_u(\delta^+)): W \to W$ that satisfy the following conditions:

1. $f_u(\delta^-),\ f_u(\delta^+)$ are nondecreasing on $W$, and
2. $f_u(\delta^- = 0) = 0 = f_u(\delta^+ = 0)$, $f_u(\delta^- = -1) = -1$, and $f_u(\delta^+ = 1) = 1$

In such a way that the image of any bipolar valued fuzzy subset $M$ of $U$ under F results in the bipolar valued fuzzy subset F($M$) of $V$, defined as:

$$\mathbf{F}(M)v = \begin{cases} \begin{pmatrix} \bigwedge_{u\in F^{-1}(v)} f_u(\delta^-), \\ \bigvee_{u\in F^{-1}(v)} f_u(\delta^+) \end{pmatrix} & \text{if } F^{-1}(v) \neq \emptyset, \\ [0,0] = 0 & \text{if } F^{-1}(v) = \emptyset. \end{cases} \tag{14}$$

for every $v \in V$, We write $\mathbf{F} = (F, \{f_u(\delta^-),\ f_u(\delta^+)\}_{u\in U}); U \to V$ or, simply, $\mathbf{F} = (F, f_u(\delta^-),\ f_u(\delta^+)): U \to V$ to represent a BVF function from U to V, and we refer to the individual functions as follows: $f_u(\delta^-),\ f_u(\delta^+), u \in U$, the comembership functions associated to **F**.

Two BVF functions $\mathbf{F} = \left(F, (f_u(\delta^-),\ f_u(\delta^+))\right)$ and $\mathbf{G} = \left(G, (g_u(\delta^-),\ g_u(\delta^+))\right)$ from $U$ to $V$ are considered equal, denoted as $\mathbf{F} = \mathbf{G}$, iff $\mathbf{F}(M) = \mathbf{G}(M)$ for every $M \in W^U$. we have:

**Theorem 5.1** Two BVF functions $\mathbf{F} = \left(F, (f_u(\delta^-),\ f_u(\delta^+))\right)$ and $\mathbf{G} = \left(G, (g_u(\delta^-),\ g_u(\delta^+))\right)$ from $U$ to $V$ are equal iff $F = G$ and $f_u = (f_u(\delta^-),\ f_u(\delta^+)) = (g_u(\delta^-),\ g_u(\delta^+)) = g_u$, where $f_u(\delta^-) = g_u(\delta^-)$and $f_u(\delta^+) = g_u(\delta^+)$ for every $u \in U$.

**Proof:** It is evident that if $F = G$ and $f_u = g_u$, for every $u \in U$, then $\mathbf{F} = \mathbf{G}$.

Alternatively, assuming $\mathbf{F} = \mathbf{G}$. If $F \neq G$, then there exists an element $u_0 \in U$ such that $F(u_0) \neq G(u_0)$. Now, let us consider the bipolar-valued fuzzy subset M of U defined by:

$$M(u) = \begin{cases} [-1,1] & \text{if } u = u_0, \\ [0,0] = 0 & \text{if } u \neq u_0. \end{cases} \tag{15}$$

Then we have

$$\mathbf{F}(M)v = \begin{cases} [-1,1] & \text{if } v = F(u_0), \\ [0,0] = 0 & \text{if } v \neq F(u_0), \end{cases} \tag{16}$$

and

$$\mathbf{G}(M) = v \begin{cases} [-1,1] & \text{if } v = G(u_0) \\ [0,0] = 0 & \text{if } v \neq G(u_0) \end{cases} \tag{17}$$

Now, if $F(u_0) \neq G(u_0)$, then $\mathbf{F}(M) \neq \mathbf{G}(M)$, This Refutes the assertion that F = G.

Alternatively, if $f_u \neq g_u$ then there exist $u_0 \in U$ and $(\delta^-, \delta^+)_0 \in W$ such that $f_{u_0}\left(\left(\delta^{f-}, \delta^{f+}\right)_0\right) \neq g_{u_0}((\delta^{g-}, \delta^{g+})_0)$. In such a case, let us consider the bipolar- valued fuzzy subset of U.

$$N(u) = \begin{cases} (\delta^-, \delta^+)_0 & \text{if } u = u_0, \\ 0 & \text{if } u \neq u_0, \end{cases} \tag{18}$$

then $F = G$ and $f_{u_0}\left(\left(\delta^{f-}, \delta^{f+}\right)_0\right) \neq g_{u_0}((\delta^{g-}, \delta^{g+})_0)$ This implies that $\mathbf{F}(N) \neq \boldsymbol{G}(N)$. Hence, the theorem is proven.

Let $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ be a BVF function. The inverse image under $\mathbf{F}$ of a bipolar valued fuzzy subset $N$ of V, denoted by $\mathbf{F}^{-1}(N)$, is a bipolar valued fuzzy subset of $U$ defined by:

$$\mathbf{F}^{-1}(N) = \cup \ \{C \in W^u : \mathbf{F}(C) \in N\} \qquad (19)$$

If the comembership functions $(f_u(\delta^-), f_u(\delta^+))$, $u \in U$, are surjective, then, taking the properties of $(f_u(\delta^-), f_u(\delta^+))$ into account, we get

$$f_u\Big(\bigvee_{(\delta^-,\delta^+)\in\Delta} (\delta^-, \delta^+)\Big) = f_u\left(\bigwedge_{\delta^-\in\Delta} \delta^-, \bigvee_{\delta^+\in\Delta} \delta^+\right) = \left(\bigwedge_{\delta^-\in\Delta} f_u(\delta^-), \bigvee_{\delta^+\in\Delta} f_u(\delta^+)\right) \text{ and}$$

$$f_u\left(\bigwedge_{(\delta^-,\delta^+)\in\Delta} (\delta^-, \delta^+)\right) = f_u\left(\bigvee_{\delta^-\in\Delta} \delta^-, \bigwedge_{\delta^+\in\Delta} \delta^+\right) = \left(\bigvee_{\delta^-\in\Delta} f_u(\delta^-), \bigwedge_{\delta^+\in\Delta} f_u(\delta^+),\right)$$

where $\Delta$ is any W subset. In this instance, the preceding definition is equal to:

**Proposition 5.1** Assume that $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ be any BVF function whose comembership functions$(f_u(\delta^-), f_u(\delta^+))$ are onto. For every bipolar valued fuzzy subset $N$ of $V$ we have

$$\mathbf{F}^{-1}(N)u = \bigvee (f_u(\delta^-), f_u(\delta^+))^{-1}\left[N\big(F(u)\big)\right] = \begin{pmatrix} \bigwedge f_u^{-1}(\delta^-)\left[N\big(F(u)\big)\right], \\ \bigwedge f_u^{-1}(\delta^-)\left[N\big(F(u)\big)\right] \end{pmatrix} \qquad (20)$$

where the supremum is taken over the set of values $(f_u(\delta^-), f_u(\delta^+))^{-1}[N(F(u))] \subset W$.

**Proof.** Let $S = \cup\{C \in W^U; \mathbf{F}(C) \subset N\}$, and $T$ be the bipolar valued fuzzy subset of $U$ denoted by $T(u) = \vee f_u{}^{-1}\,[N(F(u))]$. We show that $T = S$.

For simplicity, let H= N(F(u)). For each $v \in V$, we have:

$$\mathbf{F}(T)v = \bigvee_{u\in \mathrm{F}^{-1}(v)} f_u\big(T(u)\big)$$

$$= \left(\bigvee_{u\in \mathrm{F}^{-1}(v)} f_u\left[\bigvee f_u^{-1}\left(N\big(F(u)\big)\right)\right]\right)$$

$$= \left(\bigvee_{u\in \mathrm{F}^{-1}(v)} f_u\left[\bigvee \begin{pmatrix} f_u(\delta^-), \\ f_u(\delta^+)\end{pmatrix}^{-1} (H)\right]\right)$$

$$= \left(\bigvee_{u\in F^{-1}(v)} f_u\begin{bmatrix} \bigwedge_{\delta^-\in\Delta} f_u^{-1}(\delta^-)[H], \\ \bigvee_{\delta^+\in\Delta} f_u^{-1}(\delta^+)[H]\end{bmatrix}\right)$$

$$= \left(\bigvee_{u\in F^{-1}(v)} f_u\begin{bmatrix} \bigwedge_{\delta^-\in f_u{}^{-1}(H)} \delta^-, \\ \bigvee_{\delta^+\in f_u{}^{-1}(H)} (\delta^+)\end{bmatrix}\right)$$

$$= \begin{pmatrix} \bigwedge_{u\in F^{-1}(v)} \bigwedge_{\delta^-\in f_u{}^{-1}(H)} f_u\big((\delta^-)\big), \\ \bigvee_{u\in F^{-1}(v)} \bigvee_{\delta^+\in f_u{}^{-1}(H)} f_u\big((\delta^+)\big)\end{pmatrix}$$

$$= \bigvee_{u\in \mathrm{F}^{-1}(v)} N(F(u)) = N(v).$$

Hence, $\mathbf{F}(T) = N$, Which implies that $T \subset S$.

Alternatively, assume there exists an element $C \in W^U$ such that $\mathbf{F}(\boldsymbol{C}) \subset N$.

$$\mathbf{F}(C) \subset N \Rightarrow \mathbf{F}(C)v \leq N(v) \text{ implies } \bigvee_{u\in \mathrm{F}^{-1}(v)} f_u\big(C(u)\big) \leq N(v)$$

$$= \bigvee (f_u(\delta^-), f_u(\delta^+))^{-1}\left[N\big(F(u)\big)\right]$$

$$= \left(\bigwedge f_u^{-1}(\delta^-)[H], \bigvee f_u^{-1}(\delta^+)[H]\right)$$

$$= \left(\bigwedge_{\delta^-\in f_u{}^{-1}(H)} (\delta^-), \bigvee_{\delta^+\in f_u{}^{-1}(H)} (\delta^+)\right) \leq N(v)$$

$\Rightarrow f_u(C(u)) \leq N(F(u))$

$\Rightarrow C(u) \leq (\delta^-, \delta^+)$, for every $(\delta^-, \delta^+) \in f_u^{-1}[N(F(u))$

$\Rightarrow C(u) \leq \vee f_u^{-1}[N(F(u))] = T(u) \Rightarrow C \subset T$.

We have thus showed that each $C \in W^U$ such that $\mathbf{F}(C) \subset N$ is a subset of $T$. This proves that $S \subset T$.

### Theorem 5.2

Let $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ be a bipolar valued fuzzy (BVF) function. For every bipolar valued fuzzy subsets $M, N, M_k$ of $U$ and for every l bipolar valued fuzzy subsets $C, D, C_k$ of $V$, the following holds:

a. $\mathbf{F}(\emptyset) = \emptyset$, (21)

b. $\mathbf{F}(U) = V$ if $F$ is onto, (22)

c. if $M \subset N$ then $\mathbf{F}(M) \subset \mathbf{F}(N)$, (23)

d. $\mathbf{F}(M \cup N) = \mathbf{F}(M) \cup \mathbf{F}(N)$, (24)

e. $\mathbf{F}(M \cap N) \subset \mathbf{F}(M) \cap \mathbf{F}(N)$
(equality holds if $F$ is $1-1$), (25)

f. $\mathbf{F}^{-1}(V) = U$ (26)

g. if $C \subset D$ then $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$, (27)

h. $\mathbf{F}^{-1}(\mathbf{F}(M)) \supset M$ (28)
(equality holds if $F$ is bijective).

i. If $(f_u(\delta^-), f_u(\delta^+))$ is onto for all $u \in U$, then:

j. $\mathbf{F}(\bigcup_{k\in K} M_k) = \bigcup_{k\in K} \mathbf{F}(M_k)$ ( $K$ is an index set), (29)

k. $\mathbf{F}(\bigcap_{k\in K} M_k) \subset \bigcap_{k\in K} \mathrm{F}(M_k)$ (equality holds if $F$ is $1-1$), (30)

l. $\mathbf{F}^{-1}(\bigcup_{k\in K} C_K) = \bigcup_{k\in K} \mathbf{F}^{-1}(C_k)$, (31)

m. $\mathbf{F}^{-1}(\bigcap_{k\in K} C_k) = \bigcap_{k\in K} \mathbf{F}^{-1}(C_k)$, (32)

n. $\mathbf{F}(\mathbf{F}^{-1}(C)) \subset C$ (equality holds if $F$ is onto). (33)

o. If $f_u(1-\delta^-) \geq 1 - f_u(\delta^-)$, $f_u(1-\delta^+) \geq 1 - f_u(\delta^+)$ for every $u \in U, (\delta^-, \delta^+) \in W$, then:
$\mathbf{F}(M^{\mathrm{C}}) \supset (\mathrm{F}(M))^{\mathrm{C}}$ if $F$ is onto. (34)

p. {Equality is achieved when F is a bijective function and if $f_u(1-\delta^-) = 1 - f_u(\delta^-)$, $f_u(1-\delta^+) = 1 - f_u(\delta^+)$.}
If $(f_u(\delta^-), f_u(\delta^+))$ is bijective and if $f_u(1-\delta^-) = 1 - f_u(\delta^-)$, $f_u(1-\delta^+) = 1 - f_u(\delta^+)$, then:
$$\mathbf{F}^{-1}(D^c) = (\mathbf{F}^{-1}(D))^c \quad (35)$$

**Proof.** To exemplify the employed technique, we shall solely demonstrate the proofs for parts (e), (l), and (p).

(e) $f_u(\vee_{(\delta^-,\delta^+)\in\Delta}(\delta^-,\delta^+)) = f_u(\wedge_{\delta^-\in\Delta}\delta^-, \vee_{\delta^+\in\Delta}\delta^+) = (\wedge_{\delta^-\in\Delta} f_u(\delta^-), \vee_{\delta^+\in\Delta} f_u(\delta^+))$

$$\mathbf{F}(M \cap N)y = \bigvee_{u\in F^{-1}(v)} (f_u(\delta^-), f_u(\delta^+))[(M \cap N)(u)]$$

$$= \bigvee_{u\in F^{-1}(v)} \begin{pmatrix} f_u(\delta^-), \\ f_u(\delta^+) \end{pmatrix} \begin{bmatrix} (M^-(u) \wedge N^-(u)), \\ (M^+(u) \vee N^+(u)) \end{bmatrix}$$

$$= \begin{pmatrix} \bigwedge_{\delta^-\in\Delta, u\in F^{-1}(v)} f_u(\delta^-)\,(M^-(u) \wedge N^-(u)), \\ \bigvee_{\delta^+\in\Delta, u\in F^{-1}(v)} f_u(\delta^+)(M^+(u) \vee N^+(u)) \end{pmatrix}$$

$$\leq \begin{pmatrix} \bigwedge_{\delta^-\in\Delta, u\in F^{-1}(v)} f_u(\delta^-)\,(M^-(u)) \wedge \\ \bigwedge_{\delta^-\in\Delta, u\in F^{-1}(v)} f_u(\delta^-)\,(N^-(u)), \\ \bigvee_{\delta^+\in\Delta, u\in F^{-1}(v)} f_u(\delta^+)(M^+(u)) \vee \\ \bigvee_{\delta^+\in\Delta, u\in F^{-1}(v)} f_u(\delta^+)(N^+(u)) \end{pmatrix}$$

$$= \begin{pmatrix} \bigwedge_{\delta^-\in\Delta, u\in F^{-1}(v)} f_u(\delta^-)\,(M^-(u)), \\ \bigvee_{\delta^+\in\Delta, u\in F^{-1}(v)} f_u(\delta^+)(M^+(u)) \end{pmatrix} \wedge \begin{pmatrix} \bigwedge_{\delta^-\in\Delta, u\in F^{-1}(v)} f_u(\delta^-)\,(N^-(u)), \\ \bigvee_{\delta^+\in\Delta, u\in F^{-1}(v)} f_u(\delta^+)(N^+(u)) \end{pmatrix}$$

$$= [\vee_{u\in F^{-1}(v)}(f_u(\delta^-), f_u(\delta^+))(M(u))] \wedge [\vee_{u\in F^{-1}(v)}(f_u(\delta^-), f_u(\delta^+))(N(u))]$$
$$= [\vee_{u\in F^{-1}(v)} f_u(M(u))] \wedge [\vee_{u\in F^{-1}(v)} f_u(N(u))]$$
$$= \mathbf{F}(M)v \wedge \mathbf{F}(N)v = [\mathbf{F}(M) \cap \mathbf{F}(N)](v).$$

If F is a one-to-one function, the operation ∨ is not used in the above steps, and the equality holds.

(l) It is evident, based on property (g), that
$$\bigcup_{k\in K} \mathbf{F}^{-1}(C_k) \subset \mathbf{F}^{-1}\left(\bigcup_{k\in K} C_k\right).$$
Now,
$$\mathbf{F}^{-1}\left(\bigcup_{k\in K} C_k\right)u = \bigvee f_u^{-1}\left[\left(\bigcup_{k\in K} C_k\right)(F(u))\right]$$
$$= \bigvee \begin{pmatrix} f_u(\delta^-), \\ f_u(\delta^+) \end{pmatrix}^{-1} \left[\left(\bigcup_{k\in K} C_k\right)(F(u))\right]$$
$$= \begin{pmatrix} \bigwedge f_u^{-1}(\delta^-)\left[\bigvee_{k\in K} C_k(F(u))\right], \\ \bigvee f_u^{-1}(\delta^+)\left[\bigvee_{k\in K} C_k(F(u))\right] \end{pmatrix}$$

$$= \begin{pmatrix} \bigwedge_{\delta^- \in f_u{}^{-1}[\vee_k C_k(F(u))]} (\delta^-)\ , \\ \bigvee_{\delta^+ \in f_u{}^{-1}[\vee_k C_k(F(u))]} (\delta^+) \end{pmatrix}$$

$$= \begin{pmatrix} \bigwedge_{f_u(\delta^-)=\vee_k C_k(F(u))} (\delta^-)\ , \\ \bigvee_{f_u(\delta^+)=[\vee_k C_k(F(u))]} (\delta^+) \end{pmatrix}$$

$$\le \bigvee_{k \in K} \left[ \bigvee_{f_u(\delta^-,\delta^+)=C_K(F(u))} (\delta^-, \delta^+) \right]$$

$$= \bigvee_{k \in K} \left[ \bigvee f_u^{-1}(C_k(F(u))) \right]$$

$$= \left[ \bigcup_{k \in K} \mathbf{F}^{-1}(C_k) \right] u.$$

(p) $\mathbf{F}^{-1}(D^c)u = f_u^{-1}\left[D^c\left(F(u)\right)\right]$
$= f_u^{-1}[1 - D(F(u))]$
$= 1 - f_u^{-1}[D(F(u))]$,
since $f_u(1-\delta^-) = 1 - f_u(\delta^-)$, $f_u(1-\delta^+) = 1 - f_u(\delta^+)$ and $(f_u(\delta^-), f_u(\delta^+))$ is bijective
$= 1 - \mathbf{F}^{-1}(D)u = [\mathbf{F}^{-1}(D)]^c u$.

The composition of two bipolar valued fuzzy functions $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ and $\mathbf{G} = \left(G, (g_v(\delta^-), g_v(\delta^+))\right): V \to Z$ is the bipolar valued fuzzy function $\mathbf{G} \circ \mathbf{F}: U \to Z$ defined by $(\mathbf{G} \circ \mathbf{F})(M) = \mathbf{G}(\mathbf{F}(M))$, for all $M \in W^U$.
Let $g_v$, be onto, for all $v \in V$. Then,
$(\mathrm{G} \circ \mathbf{F})(M)z = \mathrm{G}\left(\mathbf{F}(M)\right)$

$$= \bigvee_{v \in G^{-1}(z)} g_v(\ \mathrm{F}(M)v)$$

$$= \bigvee_{v \in G^{-1}(z)} g_v \left[ \bigvee_{u \in F^{-1}(v)} \begin{pmatrix} f_u(\delta^-), \\ f_u(\delta^+) \end{pmatrix} (M(u)) \right]$$

$$= \bigvee_{v \in G^{-1}(z)} g_v \begin{bmatrix} \bigwedge_{u \in F^{-1}(v)} f_u(\delta^-)[M(u)], \\ \bigvee_{u \in F^{-1}(v)} f_u(\delta^+)[M(u)] \end{bmatrix}$$

$$= \begin{pmatrix} \bigwedge_{v \in G^{-1}(z)} \bigwedge_{u \in F^{-1}(v)} g_v(f_u(\delta^-)[M(u)])\ , \\ \bigvee_{v \in G^{-1}(z)} \bigvee_{u \in F^{-1}(v)} g_v(\ f_u(\delta^+)[M(u)]) \end{pmatrix}$$

$$= \bigvee_{v \in G^{-1}(z)} \bigvee_{u \in \mathrm{F}^{-1}(v)} g_v \left[ \begin{matrix} (f_u(\delta^-), \\ f_u(\delta^+))[M(u)] \end{matrix} \right].$$

since the $(g_v(\delta^-), g_v(\delta^+))$ are onto

$$= \bigvee_{u \in (G \circ F)^{-1}(z)} \left(g_{F(u)} \circ f_u\right)(M(u)).$$

This means that

$$\mathbf{G} \circ \mathbf{F} = \left(G \circ F, g_{F(u)} \circ f_u\right),$$

where $g_{F(u)} \circ f_u, u \in U$, evidently satisfy the conditions (1) and (2) of comembership functions.

Let $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ be a bipolar valued fuzzy (BVF) function. F is considered injective or one-to-one if, for any bipolar valued fuzzy (BVF) subsets $M_1$ and $M_2$ of $U$, $\mathrm{F}(M_1) = \mathrm{F}(M_2)$ implies $M_1 = M_2$. The definitions of surjective and bijective bipolar valued fuzzy functions can be established in a comparable manner.

Establishing that F is one-to-one (respectively, onto) is not a challenging task, as it can be demonstrated that F and $(f_u(\delta^-), f_u(\delta^+)); u \in U$, are one-to-one (resp. onto).

A bipolar valued fuzzy function $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ is said to be invertible if there exists a bipolar valued fuzzy function $\mathbf{G} = \left(G, (g_v(\delta^-), g_v(\delta^+))\right): V \to U$ such that $\mathbf{G} \circ \mathbf{F} - \mathbf{id}_U$ and $\mathbf{F} \circ \mathbf{G} = \mathbf{id}_V$, where $\mathbf{id}_U = (\mathrm{id}_U, \mathrm{id}_W)$. The bipolar valued fuzzy function $\mathbf{G}$ is called the inverse of $\mathbf{F}$ and is denoted by $\mathbf{F}^{-1}$.

**Theorem 5.3**
Let $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right): U \to V$ and $\mathbf{G} = \left(G, (g_v(\delta^-), g_v(\delta^+))\right): V \to Z$ be bipolar valued fuzzy functions. Let $(g_v(\delta^-), g_v(\delta^+))$ be onto, for all $v \in V$. Then
(i) The composition $\mathbf{G} \circ \mathbf{F}: U \to Z$ of $\mathbf{F}$ and $\mathbf{G}$ is given by:

$$\mathbf{G} \circ \mathbf{F} = \left(G \circ F, g_{F(u)} \circ (f_u(\delta^-), f_u(\delta^+))\right).$$

(ii) $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right)$ is one-to-one (resp. onto) iff $\mathbf{F}$ and $(f_u(\delta^-), f_u(\delta^+)), u \in U$, are one-to-one (respectively, onto).
(iii) $\mathbf{F} = \left(F, (f_u(\delta^-), f_u(\delta^+))\right)$ is invertible iff $\mathbf{F}$ and $(f_u(\delta^-), f_u(\delta^+))$ are invertible. The inverse $\mathbf{F}^{-1}$ of $\mathbf{F}$ is given by $\mathbf{F}^{-1} = (F^{-1}, (f_u(\delta^-), f_u(\delta^+))^{-1})$.

# 6 Conclusion

A novel structure of bipolar-valued fuzzy Cartesian products was introduced, indicating all parts of its structure. So, analogously to the basic use of crisp Cartesian products, BVF-relation, BVF-equivalence relations and BVFFs were proposed.

Some Results and numerical examples of BVFR, BVFER and BVFF related to ordinary and fuzzy relations were studied, distinguished and proved. What distinguishes our research is the logical structure that coincides with the basic structure of algebra. (i.e. the BVFCP "A×B" can reclaim the subset A and B without losing or omitting some values as in ordinary algebraic structure), The limitation of this study appears in the disability of representing two-dimensional phenomena using the form of BVFCP, BVFR, and BVF function. Therefore, complex bipolar-valued fuzzy Cartesian products, relations, and functions can handle this limitation by extending the range of BVFCP from [-1, 1] to the complex form [-1,1] + i [-1, 1]. As future research, our results will be a cornerstone to build the BVF-equivalence class, BVF partial order. Also, the concept of bipolar valued fuzzy function can be utilized to start an attractive journey to introduce the BVF group and ring.

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- Fadi Al-Zubi prepared, created and presented this work, specifically writing the initial draft.
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