Moments of Powered Rayleigh Distribution by Generalized Order Statistics

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Abstract: - The power Rayleigh distribution is considered via the power transformation technique. The suggested model behaves better than the original distribution in dealing with complex data. The key focus of this paper is to establish some simple recurrence relations for single and product moments of generalized order statistics from the power Rayleigh distribution. Some numeric computations are carried out at the different parameters of the power Rayleigh distribution. The moments of order statistics and record values are deduced from the single and product moments. Further, characterization results are also derived for power Rayleigh distribution via recurrence relations and conditional expectations.

Key-Words: - Generalized order statistics, order statistics, record value, powered Rayleigh distribution, recurrence relations, conditional expectation, characterization.

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1 Introduction

A mechanism of studying random variables (RVs) arranged in ascending order is called generalized order statistics (GOS). [1], was pioneer of this theory. It contains like, order statistics, record, progressive censoring, etc., as a particular cases. The GOS has vast applications in reliability, extreme value, physical sciences, medical sciences, remote censoring, and its applicability over several field of studies are steadily growing.

Let $l \geq 1, i \in \mathbb{N}, u_1, u_2, \ldots\ldots, u_{i-1} \in \Re^{i-1}, 1 \leq r \leq i-1, U_r = \sum_{j=r}^{i-1} u_j$. Such that $\gamma_r = l + i - r + U_r \geq 1$ for all r based on $\tilde{u}$. Further, let $G(\varphi)$ and $g(\varphi)$ be the continuous distribution function (CDF) and probability density function (PDF) of a random variable $(RV)\, \varphi$. Then $\varphi(r, i, \tilde{u}, l)$ are named GOS and its joint PDF takes the following form.

$$l\left(\prod_{j=1}^{i-1}\gamma_j\right)\left(\prod_{k=1}^{i-1}[\bar{G}(\varphi_k)]^{u_k} g(\varphi_k)\right)[G(\varphi_i)]^{l-1} g(\varphi_i) \quad (1)$$

Here we consider type I of GOS:

Let $u_1 = u_2 = \cdots = u_{i-1} = u$. The PDF of $\varphi(r, i, \tilde{u}, l)$ is.

$$g_{(r,i,u,l)}(\varphi) = \frac{w_{r-1}}{(r-1)!}[\bar{G}(\varphi)]^{\gamma_r - 1} g(\varphi) h_u^{r-1} \times [G(\varphi)],\ -\infty < \varphi < \infty. \quad (2)$$

The joint PDF of r^{th} and s^{th} is given by,

$$g_{(r,s,i,u,l)}(\varphi, \rho) = \frac{w_{s-1}}{(r-1)!(s-r-1)!}[\bar{G}(\varphi)]^{u} g(\varphi) \square_u^{r-1} G(\varphi) \times [c_u(G(\rho)) - c_u(G(\phi))]^{s-r-1} \times [\bar{G}(\rho)]^{\gamma_s - 1} g(\rho),\ -\infty < \rho < \varphi < \infty. \quad (3)$$

The conditional PDF of $\varphi(s, i, u, l)$ given $\varphi(r, i, u, l) = \varphi$, $1 \leq r \leq s \leq i$, is.

$$t_{s|r}(\rho|\varphi) = \frac{w_{s-1}}{w_{r-1}(s-r-1)!} \frac{[(\bar{G}(\varphi))^{u+1} - (\bar{G}(\rho))^{u+1}]^{s-r-1}[\bar{G}(\rho)]^{\gamma_s - 1}}{(u+1)^{s-r-1}[\bar{G}(\varphi)]^{\gamma_{r+1}}} g(\rho),\quad \rho < \varphi \quad (4)$$

where,

$$w_{r-1} = \prod_{k=1}^{r}\gamma_k,\quad \gamma_k = k + (i-k)(u+1),$$

$$c_u(\varphi) = \begin{cases} -\frac{(1-\varphi)^{u+1}}{u+1}, & u \neq -1 \\ -\ln(1-\varphi), & u = -1 \end{cases}$$

and

$$\square_u(\varphi) = c_u(\varphi) - c_u(0).$$

If $u = 0$, $l = 1$, then $\varphi(r, i, u, l)$ reduces to order statistic and when $l \to -1$ then $\varphi(r, i, u, l)$ reduces to r^{th} upper record values.

In recent years, recurrence relations based on GOS have received considerable attention. Several authors have established the recurrence relations for different distributions. For example, [2], [3], [4], [5], [6], [7], [8] and there are significantly more.

The organization of the paper is as follows. Section 2 discusses powered Rayleigh distribution and statistical properties. Recurrence relations are addressed in Section 3. Characterization results are proved in Section 4. The concluding remarks ends in Section 5.

1.1 Powered Rayleigh Distribution

[9], introduced the Rayleigh distribution (RD). It has vast applications in reliability, clinical studies, wireless signal, oceanography, wind energy and lifetime equipment. Many generalizations of RD have been noted in the existing literature.

[10], proposed the powered Rayleigh distribution (PRD) via power transformation of RD. The PDF is given by,

$$g(\varphi,\alpha,\beta)=\frac{\alpha^2}{\beta}\varphi^{2\beta-1}e^{-\frac{\varphi^{2\beta}}{2\alpha^2}},\varphi>0,\alpha,\beta>0 \quad (5)$$

where α is the scale parameter and β is the shape parameter.
The CDF is given by.

$$\bar{G}(\varphi,\alpha,\beta)=e^{-\frac{\varphi^{2\beta}}{2\alpha^2}},\ \varphi>0,\alpha,\beta>0. \quad (6)$$

For extensive information on PRD see, [10].
From equation (5) and (6), we note that.

$$\bar{G}(\varphi)=\frac{\alpha^2}{\beta}\varphi^{1-2\beta}g(\varphi) \quad (7)$$

which will be utilized for deriving the recurrence relations.

1.2 Statistical Measures of PRD.

The r^{th} moments of PRD is

$$\mu_r'=2^{\frac{r}{2\beta}}\left(\frac{1}{\alpha^2}\right)^{-\frac{r}{2\beta}}\Gamma\left(1+\frac{r}{2\beta}\right).$$

The statistical properties are given in Table 1 and Table 2 in Appendix.

2 Recurrence Relations

This section addresses the recurrence relations for single and product moments of GOS from the PRD.

2.1 Single Moments

Theorem 1. Let φ be a non-negative continuous RV follows the PRD. Suppose that $p>0$ and $1\le r\le i$, then.

$$E[\varphi^p(r,i,u,l)]-E[\varphi^p(r-1,i,u,l)]=\frac{p\alpha^2}{\gamma_r\beta}E[\varphi^{p-2\beta}(r,i,u,l)] \quad (8)$$

Proof: We have from (2),

$$E[\varphi^p(r,i,u,l)]=\frac{w_{r-1}}{(r-1)!}\int_0^\infty\varphi^p[\bar{G}(\varphi)]^{\gamma_r-1}h_u^{r-1}[\mathrm{G}(\varphi)]g(\varphi)d\varphi. \quad (9)$$

Integrating by parts $[\bar{G}(\varphi)]^{\gamma_r-1}$, we get.

$$E[\varphi^p(r,i,u,l)]=\frac{pw_{r-1}}{(r-1)!\gamma_r}\int_0^\infty\varphi^{p-1}[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-1}[G(\varphi)]d\varphi+\frac{pw_{r-2}}{(r-2)!\gamma_r}\int_0^\infty\varphi^p[\bar{G}(\varphi)]^{\gamma_{r-1}-1}h_u^{r-2}[G(\varphi)]d\varphi,$$

which implies that,

$$E[\varphi^p(r,i,u,l)]-E[\varphi^p(r-1,i,u,l)]=\frac{pw_{r-1}}{(r-1)!\gamma_r}\int_0^\infty\varphi^{p-1}[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-1}[G(\varphi)]d\varphi.$$

On using (7), we have,

$$\mathrm{E}[\varphi^p(r,i,u,l)]-\mathrm{E}[\varphi^p(\mathrm{r}-1,\mathrm{i},\mathrm{u},\mathrm{l})]=\frac{\mathrm{pw_{r-1}}}{(\mathrm{r}-1)!\gamma_\mathrm{r}}\int_0^\infty\varphi^{\mathrm{p}-1}[\bar{G}(\varphi)]^{\gamma_\mathrm{r}-1}\mathrm{h}_\mathrm{u}^{\mathrm{r}-1}[\mathrm{G}(\varphi)]\times\left\{\frac{\alpha^2}{\beta}\varphi^{1-2\beta}\right\}g(\varphi)\mathrm{d}\varphi,$$

after simplification, Theorem 1 is proved.

Remark 2.1

(i) Setting $\beta=1$ in (8) result reduced for Rayleigh distribution as follows,

$$E[\varphi^p(r,i,u,l)]-E[\varphi^p(r-1,i,u,l)]=\frac{p\alpha^2}{\gamma_r}E[\varphi^{p-2}(r,i,u,l)]$$

agreed by [11].

(ii) Setting $u=0,\ l=1$ in (8) result reduced for order statistic for PR distribution as follows.

$$E[\varphi_{r:i}^p]-E[\varphi_{r-1:i}^p]=\frac{p\alpha^2}{(i-r+1)\beta}E[\varphi_{r:i}^{p-2\beta}].$$

(iii) Setting $u=-1,\ l\ge1$ in (8) result reduced for l^{th} record values for PR distribution as follows,

$$E[\varphi_{U(r)}^p]-E[\varphi_{U(r-1)}^p]=\frac{p\alpha^2}{\beta l}E[\varphi_{U(r)}^{p-2\beta}].$$

(iv) Setting $\beta=1$ in Remark (iii) result reduced for record values for RD as follows,

$$E[\varphi_{U(r)}^p]-E[\varphi_{U(r-1)}^p]=\frac{p\alpha^2}{\beta l}E[\varphi_{U(r)}^{p-2\beta}].$$

agreed by [12].

2.2 Product Moments

Theorem 2 Let φ be a non-negative continuous RV follows the PRD. Suppose that $p,q>0$ and $1\le r<s\le i$, then,

$$E[\varphi^{p,q}(r,s,i,u,l)]-E[\varphi^{p,q}(r,s-1,i,u,l)]=\frac{q\alpha^2}{\beta\gamma_s}\left\{E(\varphi^{p,q-2\beta}(r,s,i,u,l)\right\}. \quad (10)$$

Proof: From (3), we have

$$E[\varphi^{p,q}(r,s,i,u,l)] = \frac{w_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty \varphi^p[\bar{G}(\varphi)]^u h_u^{r-1}[G(\varphi)]g(\varphi)I(\varphi)d\varphi, \quad (11)$$

where

$$I(\varphi) = \int_0^\infty \rho^q[\bar{G}(\rho)]^{\gamma_s-1}g(\rho). \times [c_u(G(\rho)) - c_u(G(\varphi))]^{s-r-1}d\rho$$

Solving the integral $I(\varphi)$ by parts and substituting the resulting expression in (11), we get:

$$E[\varphi^{p,q}(r,s,i,u,l)] - E[\varphi^{p,q}(r,s-1,i,u,l)] = \frac{qw_{s-1}}{(r-1)!\,(s-r-1)!\,\gamma_s}\int_0^\infty\int_0^\varphi \varphi^p\rho^{q-1}[\bar{G}(\varphi)]^u h_u^{r-1} \times [G(\varphi)][c_u(G(\rho) - c_u(G(\varphi))]^{s-r-1} \times [\bar{G}(\rho)]^{\gamma_s}\left[\frac{\alpha^2\rho^{1-2\beta}}{\beta}\right]g(\varphi)g(\rho)d\rho\, d\varphi,$$

after simplification (10) yields.

3 Characterizations

Characterization of PRD via GOS are presented below.

Theorem 3 The necessary and sufficient condition for a RV φ to be distributed with PDF given in (5) is that,

$$E[\varphi^p(r,i,u,l)] - E[\varphi^p(r-1,i,u,l)] = \frac{p\alpha^2}{\gamma_r\beta}E[\varphi^{p-2\beta}(r,i,u,l)] \quad (12)$$

if and only if

$$\bar{G}(\varphi) = e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}, \varphi > 0, \alpha,\beta > 0. \quad (13)$$

Proof: From (8) necessary part proved. If the relation given (12) is satisfied, then on organizing the terms in (12) as given

$$\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^p[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-2}[G(\varphi)]g(\varphi)\left[\frac{h_u[G(\varphi)]}{[\bar{G}(\varphi)]} - \frac{(r-1)[\bar{G}(\varphi)]^u}{\gamma_r}\right]d\varphi = \frac{p\alpha^2}{\beta\gamma_r}\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^{p-2\beta}[\bar{G}(\varphi)]^{\gamma_r-1}h_u^{r-1}[G(\varphi)]g(\varphi)d\varphi \quad (14)$$

Let,

$$h(\varphi) = -\frac{[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-1}[G(\varphi)]}{\gamma_r}. \quad (15)$$

Differentiating of (15) both sides, we get:

$$h'(\varphi) = [\bar{G}(\varphi)]^{\gamma_r}h_u^{r-2}[G(\varphi)]g(\varphi)\left[\frac{h_u[G(\varphi)]}{[\bar{G}(\varphi)]} - \frac{(r-1)[\bar{G}(\varphi)]^u}{\gamma_r}\right].$$

Thus

$$\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^p h'(\varphi)d\varphi = \frac{p\alpha^2}{\beta\gamma_r}\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^{p-2\beta}[\bar{G}(\varphi)]^{\gamma_r-1}h_u^{r-1}[G(\varphi)]g(\varphi)d\varphi. \quad (16)$$

On integration left hand side in (16) by parts and putting the value of $h(\varphi)$,

$$\frac{w_{r-1}}{(r-1)!\,\gamma_r}\int_0^\infty p\varphi^{p-1}[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-1}[G(\varphi)]d\varphi - \left\{\frac{p\alpha^2}{\beta\gamma_r}\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^{p-2\beta}[\bar{G}(\varphi)]^{\gamma_r-1}h_u^{r-1}[G(\varphi)]g(\varphi)d\varphi\right.$$

which reduces to,

$$\frac{w_{r-1}}{(r-1)!}\int_0^\infty \varphi^{p-1}[\bar{G}(\varphi)]^{\gamma_r}h_u^{r-1}[G(\varphi)]g(\varphi)\left[\frac{\bar{G}(\varphi)}{g(\varphi)} - \frac{\alpha^2\varphi^{1-2\beta}}{\beta}\right]d\varphi = 0 \quad (17)$$

On using Müntz-Szász theorem [13] to (17), we get

$$\bar{G}(\varphi) = \frac{\alpha^2}{\beta}\varphi^{1-2\beta}g(\varphi)$$

which proves that $g(\varphi)$ has the form as in (5).

Theorem 4 Let φ be a non-negative RV having CDF $G(\varphi)$ with $G(0) = 0$ and $0 \le G(\varphi) \le 1$ for all $\varphi > 0$, then,

$$E\left[\xi\{\varphi(s,i,u,l)\}\middle|X_g(r,i,u,l) = \varphi\right] = e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\prod_{j=1}^{s-l}\left(\frac{\gamma_{l+j}}{\gamma_{l+j}+1}\right), l = r, r+1 \quad (18)$$

if and only if

$$\bar{G}(\varphi) = e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}, \varphi > 0, \alpha,\beta > 0. \quad (19)$$

where $\xi(\rho) = e^{-\frac{\rho^{2\beta}}{2\alpha^2}}$

Proof: Necessary part:
From (4) for $s > r+1$, we have

$$E[\xi\{\varphi(s,i,u,l)\}|\varphi(r,i,u,l) = \varphi] = D\int_o^\varphi e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\left(\frac{\bar{G}(\rho)}{\bar{G}(\varphi)}\right)^{\gamma_s-1} \times \left\{1 - \left(\frac{\bar{G}(\rho)}{\bar{G}(\varphi)}\right)^{u+1}\right\}^{s-r-1}\frac{g(\rho)}{\bar{G}(\varphi)}d\rho \quad (20)$$

where $D = \frac{w_{s-1}}{w_{r-1}(s-r-1)!(u+1)^{s-r-1}}$

Now setting $A = \frac{\bar{G}(\rho)}{\bar{G}(\varphi)} = \frac{e^{-\frac{y^{2\beta}}{2\alpha^2}}}{e^{-\frac{y^{2\beta}}{2\alpha^2}}}$ in (20), we have

$$E[\xi\{\varphi(s,i,u,l)\}|\varphi(r,i,u,l) = \varphi] = De^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\int_0^1 A^{\gamma_s}(1-A^{u+1})^{s-r-1}dA. \quad (21)$$

Again, by setting $B = A^{u+1}$ in (21), we find that.

$$E[\xi\{\varphi(s,i,u,l)\}|\varphi(r,i,u,l)=\varphi] =$$

$$De^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\int_0^1 B^{\frac{\gamma_s+1}{u+1}-1}(1-B)^{s-r-1}dB.$$

$$= De^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\frac{\Gamma\left(\frac{l+1}{u+1}+i-s\right)}{\Gamma\left(\frac{l+1}{u+1}+i-r\right)} = \frac{w_{s-1}}{w_{r-1}}\frac{e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}}{\prod_{j=1}^{s-r}(\gamma_{r+j}+1)},$$

where,

$$\frac{w_{s-1}}{w_{r-1}} = \prod_{j=1}^{s-r}\gamma_{r+j}.$$

The necessary part is verified.

Sufficiency part:
From (4) and (18), we have:

$$\int_0^{\varphi} e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}[(\bar{G}(\varphi))^{u+1} - (\bar{G}(\rho))^{u+1}]^{s-r-1}[\bar{G}(\rho)]^{\gamma_s-1}g(\rho)d\rho = t_{s|r}(\varphi)[\bar{G}(\varphi)]^{\gamma_{r+1}}, \quad (22)$$

where,

$$t_{s|r}(x) = e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right).$$

Performing differentiation of (22) both sides with respect to φ, we acquire

$$\frac{w_{s-1}[\bar{F}(x)]^m f(x)}{w_{r-1}(s-r-2)!\,(u+1)^{s-r-2}}\int_0^{\varphi} e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}[(\bar{G}(\varphi)))^{u+1} - (\bar{G}(\rho))^{u+1}]^{s-r-2}[\bar{G}(\rho)]^{\gamma_s-1}g(\rho)d\rho = t'_{s|r}(\varphi)[\bar{G}(\varphi)]^{\gamma_{r+1}} + \gamma_{r+1}t_{s|r}(\varphi)[\bar{G}(\varphi)]^{\gamma_{r+1}-1}g(\varphi),$$

where

$$t'_{s|r}(\varphi) = -\frac{\beta}{\alpha^2}\varphi^{2\beta-1}e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right)$$

$$t_{s|r+1}(\varphi) = e^{-\frac{\varphi^{2\beta}}{2\alpha^2}}\left(\frac{\gamma_{r+1}+1}{\gamma_{r+1}}\right)\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right).$$

Therefore, by [14]

$$\frac{g(\varphi)}{\bar{G}(\varphi)} = -\frac{1}{\gamma_{r+1}}\left[\frac{t'_{s|r}(\varphi)}{t_{s|r+1}(\varphi)-t_{s|r}(\varphi)}\right]$$

$$\frac{g(\varphi)}{\bar{G}(\varphi)} = \frac{\beta}{\alpha^2}\varphi^{2\beta-1}$$

which confirms that $g(\varphi)$ has the form as in (5).

4 Conclusion

The results provided in this article will be useful for researchers who are working in the domain of mathematical statistics. It helps to obtain exploratory analysis based on ordered random variables.

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Conflict of Interest

The author has no conflicts of interest to declare.

APPENDIX

Statistical measures of *PRD*.

Table 1. $\alpha \in [0.5 - 4.0]$ and $\beta = 0.5$.

alpha	1st	2nd	3rd	4th	mean	variance	skewness	Kurtosis
0.5	0.5	0.5	0.75	1.5	0.50	0.25	2.00	9.00
1.0	2	8	48	384	2.00	4.00	2.00	9.00
1.5	4.5	40.5	546.75	9841.5	4.50	20.25	2.00	9.00
2.0	8	128	3072	98304	8.00	64.00	2.00	9.00
2.5	12.5	312.5	11718.8	585937.5	12.50	156.25	2.00	9.00
3.0	18	648	34992	2519424	18.00	324.00	2.00	9.00
3.5	24.5	1200.5	88236.8	8647201.5	24.50	600.25	2.00	9.00
4.0	32	2048	196608	25165824	32.00	1024.00	2.00	9.00

Table 2. $\alpha \in [0.5 - 4.0]$ and $\beta = 1.0$

alpha	1st	2nd	3rd	4th	mean	variance	skewness	Kurtosis
0.5	0.627	0.5	0.47	0.5	0.63	0.11	0.64	3.24
1.0	1.253	2	3.76	8	1.25	0.43	0.63	3.25
1.5	1.88	4.5	12.69	40.5	1.88	0.97	0.63	3.24
2.0	2.507	8	30.08	128	2.51	1.71	0.63	3.24
2.5	3.133	12.5	58.749	312.5	3.13	2.68	0.63	3.25
3.0	3.76	18	101.518	648	3.76	3.86	0.63	3.25
3.5	4.387	24.5	161.208	1200.5	4.39	5.25	0.63	3.24
4.0	5.013	32	240.636	2048	5.01	6.87	0.63	3.25

From Table 1 and Table 2, we conclude that when alpha increases at fix value of beta, other characteristics are also increases except skewness and kurtosis.