The Truncated EM Method for Stochastic Differential Equations Driven by Fractional Brownian Motion

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Abstract: -We mainly focus on the numerical method of fractional Brownian motion in this paper. On the basis of the numerical method of general SDEs, an approximation scheme is obtained for the stochastic differential equations about fractional noise. And we get it by using the Lipschitz condition and combining with the truncation function $f_\triangle$ and $g_\triangle$. Furthermore, we also prove the moment boundedness and convergence of the solution by some lemma. At last, we apply this method to the Gilpin-Ayala model. The orbital image of the solution and the form of numerical solution are given. The error of solution also has been simulated by MATLAB.

Key-Words: -fractional Brownian motion, truncated EM method method, numerical method, convergence, moment boundedness, Gilpin-Ayala model

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1 Introduction

In recent years, SDEs driven by fractional Brownian motion have attracted more attention and widely applied in many fields. FBM has some better properties than general ones such as self-similarity and long-term memory, which can describe the random phenomena. Therefore, many scholars pay attention to it. We know that fractional Brownian motion does not meet the conditions of a semi-martingale, so the usual It$\hat{o}$ formula is not suitable, in [1], gave the It$\hat{o}$ formula for fractions, It$\hat{o}$ representation formula and Girsanov theorem. When the fractional noise instead of a general one, the classical theory of random integration is no longer applicable. Therefore, in [2], [3], the author comprehensively introduced the definition of FBM random integral, gave some theoretical applications, and focused on the relationship between different research approaches. For the Hearst index, the concept is not clear in some literature, and in [4] it is explained that fractional Brownian motion differs from the Gauss-Markov process($H \neq \frac{1}{2}$) in that the increment of one is stationary and correlated, while the other is nonstationary and uncorrelated. Moreover, in [5], the uniqueness and existence of the solution to the neutral pulse random delay equation driven by FBM has been given.

The problem of the numerical solution has also attracted the attention of many scholars (see, [6], [7], [8], [9], [10]), but the numerical methods of the fractional Brownian motion are relatively few. In [11], the author derived some approximation schemes of the scalar SDEs, and get the exact rate of convergence of it. He showed that the error of the Euler method converges to a random variable a.s. The EM method of backward SDEs had been discussed in [12]. It is focus on stochastic Markovian neural networks with jump. The general mean-square stability of it has been obtained and there are sufficient conditions which guarantee the stability of the method. In addition, an accurate calculation scheme for solving FBM-driven stochastic differential equations is proposed in [13], and this discretization method is based on the quadratic interpolation technique, its error and convergence are analyzed for better application. A class of stochastic fractional integro-differential equations has been concerned by [14]. This type of equation has weakly singular kernels. The author proposed a modified Euler−Maruyama (EM) method and then analysed the strong convergence of it. In [15], numerical schemes for multi-dimensional fBms with Hurst parameter movtivating stochastic differential equations are investigated. The author provide the order conditions of Runge-Kutta method to achieve the optimal rate of convergence, which based on the continuous dependency of numerical solutions with the driving noises and introducing an Runge-Kutta methods. Finally by applying this method, simpler step-Euler schemes with a strong convergence rate are developed, and the rate is confirmed by numerical experiment. Sometimes the drift term will be special and it satisfies locally Lipschitz but not bounded in neighborhood of the origin, therefore the author developed an implicit Euler schemen which can maintain positivity in [16], , then obtained rate of convergence.

In this paper, we give a new truncated EM method for nonliner SDEs which is explicit. Suppose the equation is of the form

$$dx_t = f(x_t)dt + g(x_t)dB_t^H. \tag{1}$$

The coefficients meets local Lipschitz condition but unfortunately, they don't grow linearly.

This paper is organized as follows. In the first section, the research background of the numerical solution and fractional Brownian motion are introduced. Section 2 provides some definitions and theorems required for the next proof. Section 3 obtains the specific form of the truncated EM method and proves the convergence of the numerical solution. And in Section 4, the error of the Gilpin Ayala model between truncation EM method and actual solution be simulated. The last section summarizes the paper.

2 Preliminary

This section we will give some definitions and theorems needed in this paper, which are important for the proof. This paper only pay attention to the situation of $\frac{1}{2} < H < 1$.

Defintion 2.1. *([17]) Let B_t^H is a continuous Gaussian process, H is a Hurst index, $0 < H < 1$. If B_t^H satisfies the following three conditions:*

1.$B_0^H = 0$.

2.$E(B_{t+\Delta t}^H - B_t^H) = 0$, for any $t > 0$ and $\Delta t > 0$

.

3.For different t and s, their covariance function is

$$\mathbb{E}[B_u^H B_v^H] = \frac{1}{2}(|u|^{2H} + |v|^{2H} - |u-v|^{2H}),\ t, s \geq 0,$$

then $B_t^H, t \geq 0$ is named fractional Brownian motion.

From the above definition, we can see the three facts:
1.When $H = \frac{1}{2}$, B_t^H is standard Brownian motion.
2.We can know that it has stationary increments, that is $\mathbb{E}(B_t^H - B_s^H)^2 = |t-s|^{2H}$.
3.FBM has the incremental autocorrelation. If $H < \frac{1}{2}$, there is a negative correlation between the increments of FBM; if $H > \frac{1}{2}$, there is a positive correlation between the increments of FBM.

We have learned that $B_t^H, t \geq 0$ $(H \neq \frac{1}{2})$ is not a semi-martingale, so the properties of Brownian motion are no longer valid. But we can establish the relationship between them.

We have

$$B_t^H = \int_0^t K_H(t,s) dB_s$$

where $K_H(t,s)$ is a square integrable kernel,

$$K_H(t,s) = c_H s^{H-\frac{1}{2}} \int_s^t (w-s)^{H-\frac{3}{2}} w^{H-\frac{1}{2}} dw,$$

$$c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{1/2}.$$

For more details, refer to [18]. Suppose $\frac{1}{2} < H < 1$, we denote $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\phi(u,v) = H(2H-1)|u-v|^{2H-2},\ \ s,t \in \mathbb{R}.$$

Then L_ϕ^2 is a Hilbert space, the inner product is denoted by

$$< f, g >_\phi = \int_0^\infty \int_0^\infty f(s)g(t)\phi(s,t) ds dt.$$

Then $f \in L_\phi^2(\mathbb{R}_+)$ if

$$||f||_\phi^2 := \int_0^\infty \int_0^\infty f(u)f(v)\phi(u,v) du dv < \infty.$$

Defintion 2.2. *([19]) For a random variable $F \in L^p$. We defined*

$$D_{\Phi g} G(\alpha) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ G(\alpha + \varepsilon \int_0^{\cdot} (\Phi g)(u) du) - G(\alpha) \right\}$$

as the ϕ-derivative in the orientation of Φg. if the limit exists in L^p.

Moreover, F is said to be ϕ-differentiable if there has a process$(D^\phi F_s, s \geq 0)$ makes

$$D_{\Phi g} F = \int_0^\infty D^\phi F_s g_s ds \quad a.s.$$

for all $g \in L_\phi^2$. If $f : \mathbb{R} \to \mathbb{R}$ is smooth and $F : \Omega \to \mathbb{R}$ is ϕ-differentiable, we can say that $f(F)$ is ϕ-differentiable, then we have

$$D_{\Phi g} f(F) = f'(F) D_{\Phi g} F$$

and

$$D_s^\phi f(F) = f'(F) D_s^\phi F.$$

The rules are as follows:

$$D_s^\phi \int_0^\infty f_u dB_u^H = \int_0^\infty \phi(u,v) f_u du = (\Phi f)(s);$$

$$D_s^\phi \delta(f) = \delta(f) \int_0^\infty \phi(u,s) f_u du = \delta(f)(\Phi f)(s).$$

Theorem 2.1. *([19]) Fractional Itô formula(H> $\frac{1}{2}$) Let $\mathcal{L}(0,T)$ be a family of stochastic process on $[0,T]$. If $\mathbb{E}|F|_\phi^2 < \infty$, then $F \in \mathcal{L}(0,T)$ and F is ϕ-differentiable. F_t, G_t are process that satisfy the following assumptions:*

1.There is an $\beta > 1 - H$ such that

$$\mathbb{E}|F_a - F_b|^2 \leq C|a-b|^{2\beta}$$

where $|a-b| \leq \varepsilon$, $\varepsilon > 0$.

2.

$$\lim_{0\le a,b\le t,|a-b|\to 0} \mathbb{E}|D_a^\phi(F_a-F_b)|^2=0.$$

3. $\mathbb{E}\int_0^T|F_tD_t^\phi\mu_t|ds<\infty$, $\mathbb{E}\sup_{0\le t\le T}|G_t|<\infty$. *Denote* $\mu_t=\zeta+\int_0^t G_udu+\int_0^t F_udB_u^H$, $\zeta\in\mathbb{R}$ *for* $t\in[0,T]$. *Let* $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ *be a function having the first continuous derivative in its first variable as well as its second one. Let* $\left(\frac{\partial f}{\partial x}(s,\mu_s)F_s,\ s\in[0,T]\right)\in\mathcal{L}(0,T)$. *For* $t\in[0,T]$, *we have*

$$\begin{aligned}f(t,\mu_t)=&f(0,\zeta)+\int_0^t\frac{\partial f}{\partial s}(s,\mu_s)ds\\&+\int_0^t\frac{\partial f}{\partial x}(s,\mu_s)G_sds+\int_0^t\frac{\partial f}{\partial x}(s,\mu_s)F_sdB_s^H\\&+\int_0^t\frac{\partial^2 f}{\partial x^2}(s,\mu_s)F_sD_s^\phi\mu_sds\end{aligned}\tag{2}$$

Theorem 2.2. *([20])Gronwall's inequality*

For $T>0$ *and* $c\ge 0$. *Suppose* $u(\cdot)$ *is function which is Borel measurable bounded and nonnegative on* $[0,T]$, *and* $v(\cdot)$ *is nonnegation integrable on* $[0,T]$. *We have*

$$u(t)\le c\exp\left(\int_0^t v(s)ds\right),t\in[0,T],$$

if

$$u(t)\le c+\int_0^t v(s)u(s)ds.$$

3 The truncated EM method about FBM and convergence

3.1 Description of the method

We will give the form of truncated EM method in this subsection.

First, let's make two assumptions for preparation.

Assumption 3.1. There is the local Lipschitz condition for coefficients: If $L>0$, then

$$|f(x_1)-f(x_2)|\vee|g(x_1)-g(x_2)|\le K_L|x_1-x_2|,\tag{3}$$

is hold. For $x,y\in\mathbb{R}$, $|x|\vee|y|\le L$, where $K_L>0$ is a constant and $|\cdot|$ is the Euclidean norm.

Assumption 3.2. The coefficients of equations satisfy the inequality below

$$x^Tf(x)+(m-1)g(x)D_s^\phi x(t)\le Q(1+|x|^2).\tag{4}$$

where $m>2$ and $Q>0$ are constants.

Now consider a SDE B_t^H.

$$dx_t=f(x_t)dt+g(x_t)dB_t^H\tag{5}$$

where $t\ge 0$, and $x(0)=x_0$. The condition

$$x^Tf(x)+|g(x)|D_s^\phi x(t)\le Q(1+|x|^2),\tag{6}$$

can guarantee the global solution. Following lemma proves the existence and uniqueness of it.

Lemma 3.1. *Suppose assumption 3.1 and Assumption 3.2 are satisfied.*

(i) The SDE (5) has an unique global solution $x(t)$.

(ii)

$$\sup_{0\le t\le T}\mathbb{E}|x_t|^m<\infty,\ \forall T>0.\tag{7}$$

Proof. First, since Assumption 3.1 hold. We know that coefficients satisfy the local Lipschitz condition, the equation has an unique local solution on $t\in[0,\mu_\infty]$, μ_∞ is an explosion time (see, [21], Theorem 3.1). We only need to proof that $\mu_\infty=\infty$ a.s.

τ_l is a stopping time for $l\ge 1$,

$$\tau_l=\mu_\infty\wedge inf\{t\in[0,\mu_\infty]:|x_t|\ge l\},$$

where $\inf\varnothing=\infty$. Clearly, τ_l's are increasing so $\tau_\infty=\lim_{k\to\infty}\tau_l$ and $\tau_\infty\le\mu_\infty$ a.s. By the Theorem 2.1 and the condition (6), we can claim that

$$\begin{aligned}&\mathbb{E}|x_{t\wedge\tau_l}|^2=|x_0|^2\\&+\mathbb{E}\int_0^{t\wedge\tau_l}2|x_s|f(x)ds+\mathbb{E}\int_0^{t\wedge\tau_l}2|x_s|g(x)dB_s^H\\&+\mathbb{E}\int_0^{t\wedge\tau_l}2g(x)D_s^\phi x_sds\\&\le|x_0|^2+2\mathbb{E}\int_0^{t\wedge\tau_l}Q(1+|x_s|^2)ds\\&\le C+2Qt+2Q\int_0^t\mathbb{E}|x_{t\wedge\tau_l}|^2ds.\end{aligned}$$

According to Theorem 2.2,

$$\mathbb{E}|x_{t\wedge\tau_l}|^2\le(C+2Qt)e^{2Qt}.$$

Where C is a constant. Define $\rho:\mathbb{R}_+\to\mathbb{R}_+$ by

$$\rho(r)=\inf_{|x|\ge r,0\le t\le\infty}|x_t|^2,\quad for\ r\ge 0.$$

Apparently,

$$\lim_{|x|\to\infty}\inf_{0\le t\le\infty}|x_t|^2=\infty.\tag{8}$$

We can see $\rho(|x_t|)\le|x_t|^2$ from the definition of ρ. And $\lim_{0\le t<\infty}|x_t|^2=\infty$ combine with condition (8), then

$$\lim_{r\to\infty}\rho(r)=\infty.$$

Based on the above analyses, we can obtain

$$\mathbb{E}\rho(|x_{t\wedge\tau_l}|) \leq \mathbb{E}|x_{t\wedge\tau_l}|^2 \leq (C+2Qt)e^{2Qt},$$

it follows that

$$(C+2Qt)e^{2Qt} \geq \mathbb{E}\rho(|x_{t\wedge\tau_l}|) \geq \rho(l)\mathbb{P}(\tau_l \leq t).$$

Setting l tends to infinity and when $t \to \infty$, there is

$$\mathbb{P}(\tau_\infty < \infty) = 0.$$

Immediately, there is $\tau_\infty = \infty$ a.s. Therefore, $\mu_\infty = \infty$ a.s.

Next, the certificate of (ii) is resemble to the theorem 5.1.1 in [22].

In order to get this method, we select a continuous $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ which is strictly increasing and when $r \to \infty$, we have $\eta(r) \to \infty$. In addition, it's also satisfied

$$sup_{|x|\leq r}(|f| \vee |g|) \leq \eta(r), \ \ \forall r \geq 0. \tag{9}$$

Then η^{-1} is also has the similar properties from $[\eta(0),\infty)$ to $\mathbb{R}_+$.

Select a $\Delta^* \in (0,1]$. There is a function $k : (0,\Delta^*] \to (0,\infty)$ which strictly decreasing and it meets

$$\begin{aligned} k(\Delta^*) \geq \eta(2), \lim_{\Delta\to 0} k(\Delta) = \infty, \\ \Delta^{H/2}k(\Delta) \leq 1, \ \ \forall \Delta \in (0,1). \end{aligned} \tag{10}$$

For any $\Delta \in (0,1)$, we define

$$\begin{aligned} f_\Delta(x) = f\left((|x| \wedge \eta^{-1}(k(\Delta)))\frac{x}{|x|}\right), \\ g_\Delta(x) = g\left((|x| \wedge \eta^{-1}(k(\Delta)))\frac{x}{|x|}\right) \end{aligned} \tag{11}$$

for $x \in \mathbb{R}$. They called truncated functions. When $x = 0$, we have $\frac{x}{|x|} = 0$. By (9), we get

$$|f_\Delta| \vee |g_\Delta| \leq \eta(\eta^{-1}(k(\Delta))) = k(\Delta), \ \ x \in \mathbb{R}. \tag{12}$$

It means that, even while f and g might not be bounded, both f_Δ and g_Δ are.

Moreover, as stated in the following lemma, we will demonstrate f_Δ and g_Δ also keep the condition (4) for $\Delta \in (0,\Delta^*]$.

Lemma 3.2. *We suppose that Assumption 3.2 is ture. For any $\Delta \in (0,\Delta^*]$, we gain that*

$$x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \leq 2Q(1+|x|^2), \quad \forall x \in \mathbb{R}^d. \tag{13}$$

Proof. Due to k is a strictly decreasing function, from (10), we deduced that

$$\eta^{-1}(k(\Delta)) \geq \eta^{-1}(k(\Delta^*)) \geq 2, \forall \Delta \in (0,\Delta^*]. \tag{14}$$

Choose a $\Delta \in (0,\Delta^*]$. For $x \in \mathbb{R}^d$,

(i) If $|x| \leq \eta^{-1}(k(\Delta))$, by (4), there have

$$\begin{aligned} &x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \\ =&x^T f(x) + (m-1)g(x)D_s^\phi x(s) \leq 2Q(1+|x|^2) \end{aligned}$$

so the (13) hold.

(ii) If $|x| > \eta^{-1}(k(\Delta))$, we can use the (4),

$$\begin{aligned} &x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \\ =&x^T f\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right) \\ &+(m-1)g\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right)D_s^\phi x(s). \end{aligned}$$

Insert an intermediate term to construct the following form,

$$\begin{aligned} &x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \\ =&\eta^{-1}(k(\Delta))\frac{x^T}{|x|}f\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right) \\ &+(m-1)g\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right)D_s^\phi x(s) \\ &+\left(\frac{|x|}{\eta^{-1}(k(\Delta))}-1\right)\eta^{-1}(k(\Delta))\frac{x^T}{|x|}f\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right) \\ \leq&Q(1+[\eta^{-1}(k(\Delta))]^2) \\ &+\left(\frac{|x|}{\eta^{-1}(k(\Delta))}-1\right)\eta^{-1}(k(\Delta))\frac{x^T}{|x|}f\left(\eta^{-1}(k(\Delta))\frac{x}{|x|}\right). \end{aligned}$$

In other hand, we note (4) $x^T f(x) \leq Q(1+|x|^2)$ by (4) for any $x \in \mathbb{R}$.

For convenience, let $\eta^{-1}(k(\Delta)) = M$. Then the above equation becomes

$$\begin{aligned} &x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \\ \leq&Q(1+M^2) + \left(\frac{|x|}{M}-1\right)Q(1+M^2) \\ =&Q(1+M^2)\frac{|x|}{M} \\ \leq&Q|x|\left(M+\frac{1}{M}\right). \end{aligned}$$

By (14), we know $M \geq 2$, therefore we can obtain

$$\begin{aligned} &x^T f_\Delta(x) + (m-1)g_\Delta(x)D_s^\phi x(s) \\ \leq&Q|x|(\frac{1}{2}+\eta^{-1}(k(\Delta))) \\ \leq&Q(1+|x|)^2 \leq 2Q(1+|x|^2). \end{aligned}$$

The proof is completed.

Through the above theoretical preparation, the form of the truncated EM $X_\Delta(t_l) \approx x(t_l)$ can now developed. By taking $X_\Delta(0) = x_0$, $t_l = l\Delta$, we can get that

$$X_\Delta(t_{l+1}) = X_\Delta(t_l) + f_\Delta(X_\Delta(t_l))\Delta + g_\Delta(X_\Delta(t_l))\Delta B_l^H \tag{15}$$

where $l = 0, 1, ...,$ where $\Delta B_l^H = B_{t_{l+1}}^H - B_{t_l}^H$ and in this form, time is discrete.

The truncated EM solutions with continuous time defined as

$$\hat{x}_\Delta(t) = \sum_{l=0}^{\infty} X_\Delta(t_l) I_{[t_l, t_{l+1})}(t), \;\; t \geq 0. \tag{16}$$

and

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\hat{x}_\Delta(s))ds + \int_0^t g_\Delta(\hat{x}_\Delta(s))dB_s^H \tag{17}$$

for $t \geq 0$.

By the definition above, $x_\Delta(t_l) = \hat{x}_\Delta(t_l) = X_\Delta(t_l)$ can be seen for all $l \geq 0$. Furthermore, $x_\Delta(t)$ has the form of Itô differential

$$dx_\Delta(t) = x_0 + f_\Delta(\hat{x}_\Delta(t))dt + g_\Delta(\hat{x}_\Delta(t))dB_t^H. \tag{18}$$

Following, we will demonstrate the convergence.

3.2 The moment bound of solution

We wii show the numerical solutions will converge in L^P.Via (12),

$$\sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^m < \infty, \;\; \forall T > 0.$$

can be seen easily. Nevertheless, obtaining the following inequality is difficult,

$$\sup_{0 \leq t \leq \Delta^*} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^m < \infty, \;\; \forall T > 0. \tag{19}$$

In this subsection, we will establish this. First, We will illustrates that $x_\Delta(t)$ close to $\hat{x}_\Delta(t)$.

Lemma 3.3. *Given a $\Delta \in (0, \Delta^*]$, $m \geq 2$, then*

$$\mathbb{E}|x_\Delta(t) - \hat{x}_\Delta(t)|^m \leq c_{m,\Delta,H}, \;\; \forall t \geq 0. \tag{20}$$

where $c_{m,\Delta,H} > 0$ is a constant which dependent on m, Δ and H. Thus,

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(t) - \hat{x}_\Delta(t)|^m = 0, \;\; \forall t \geq 0. \tag{21}$$

Proof. The generic positive real constants $c_{m,\Delta,H}$ which are only dependent on m, Δ, H and whose values might differ between occurrences, will be used in what follows. For $t \geq 0$, Fix a $\Delta \in (0, \Delta^*]$. there is a $l \geq 0$ enables $t_l \leq t \leq t_{l+1}$. By (12) and the fraction Itô integra (see,e.g., [23]). Afterward, we infer from (17),

$$\begin{aligned}
&\mathbb{E}|x_\Delta(t) - \hat{x}_\Delta(t)|^m = \mathbb{E}|x_\Delta(t) - x_\Delta(t_l)|^m \\
\leq &c_m \left(\mathbb{E}|\int_{t_l}^t f_\Delta(\hat{x}_\Delta(s))ds|^m + \mathbb{E}|\int_{t_l}^t g_\Delta(\hat{x}_\Delta(s))dB_s^H|^m \right) \\
\leq &c_m \left(\Delta^{m-1}\mathbb{E}\int_{t_l}^t |f_\Delta(\hat{x}_\Delta(s))|^m ds + \mathbb{E}|\int_{t_l}^t g_\Delta(\hat{x}_\Delta(s))dB_s^H|^m \right) \\
\leq &c_m \left(\Delta^m (k(\Delta))^m + \mathbb{E}|\int_{t_l}^t g_\Delta(\hat{x}_\Delta(s))dB_s^H|^m \right).
\end{aligned}$$

We also have (see, e.g., [18])

$$\begin{aligned}
&\mathbb{E}|\int_{t_l}^t g_\Delta(\hat{x}_\Delta(s))dB_s^H|^m \\
\leq &c(H,m)||g_\Delta(\hat{x}_\Delta(s))||_{L^{1/H}(t,t_l)}^m \\
= &c(H,m)\left(\int_t^{t_l} |g_\Delta(\hat{x}_\Delta(s))|^{\frac{1}{H}} dt\right)^{mH} \\
\leq &c(H,m)(k(\Delta))^m \left(\int_t^{t_l} 1dt\right)^{mH} \\
\leq &c(H,m)(k(\Delta))^m \Delta^{mH}.
\end{aligned}$$

Therefore,

$$\mathbb{E}|x_\Delta(t) - \hat{x}_\Delta(t)|^m \leq c_{m,\Delta,H}.$$

(20) and (21) are proofed immediately.

Next, we give proof of (19), which is also an important part.

Lemma 3.4. *If Assumptions 3.1 and 3.2 are ture. The inequality (19) is hold. Here, C might vary between occurrences, and represents generic positive real constants going forward, which are they are dependent on T, m, Q, x_0 but independent of Δ.*

Proof. For any $\Delta \in (0, \Delta^*)$ and $T \geq 0$. From (17), we can infer using the Theorem 2.1, for $0 \leq t \leq$

T.

$$\mathbb{E}|x_\Delta(t)|^m \leq |x_0|^m + \mathbb{E}|\int_0^t m|x_\Delta(t)|^{m-1} f_\Delta(\hat{x}_\Delta(s))|ds$$

$$+\mathbb{E}|\int_0^t m(m-1)|x_\Delta(t)|^{m-2} g_\Delta(\hat{x}_\Delta(s)) D_s^\phi x_\Delta(s)|ds$$

$$=|x_0|^m + \mathbb{E}|\int_0^t m|x_\Delta(s)|^{m-2} \left(\hat{x}_\Delta(s) f_\Delta(\hat{x}_\Delta(s))\right)|ds$$

$$+\mathbb{E}|\int_0^t m|x_\Delta(s)|^{m-2} \Big((m-1) g_\Delta(\hat{x}_\Delta(s)) D_s^\phi x_\Delta(s)\Big)|ds$$

$$+\mathbb{E}|\int_0^s m|x_\Delta(t)|^{m-2} (x_\Delta(s) - \hat{x}_\Delta(s))^T f_\Delta(\hat{x}_\Delta(s))|ds.$$

we can determined

$$\mathbb{E}|x_\Delta(t)|^m \leq |x_0|^m$$

$$+\mathbb{E}\int_0^t Qm|x_\Delta(s)|^{m-2}\left(1+|\hat{x}_\Delta(s)|^2\right)ds$$

$$+\mathbb{E}\int_0^t m|x_\Delta(s)|^{m-2}(x_\Delta(s)-\hat{x}_\Delta(s)) f_\Delta(\hat{x}_\Delta(s))ds$$

$$\leq |x_0|^m + \mathbb{E}\int_0^t Qm|x_\Delta(s)|^{m-2}\left(1+|\hat{x}_\Delta(s)|^2\right)ds$$

$$+(m-2)\mathbb{E}\int_0^t |x_\Delta(s)|^m ds$$

$$+2\mathbb{E}\int_0^t |x_\Delta(s)-\hat{x}_\Delta(s)|^{\frac{m}{2}} |f_\Delta(\hat{x}_\Delta(s))|^{\frac{m}{2}} ds$$

$$\leq P_1 + P_2\int_0^t \left(\mathbb{E}|x_\Delta(s)|^m + \mathbb{E}|\hat{x}_\Delta(s)|^m\right)ds$$

$$+2\mathbb{E}\int_0^t |x_\Delta(s)-\hat{x}_\Delta(s)|^{\frac{m}{2}} |f_\Delta(\hat{x}_\Delta(s))|^{\frac{m}{2}} ds$$

which based on the Young inequality and Lemma 3.2 and

$$a^{m-2}b \leq \frac{m-2}{m}a^m + \frac{2}{m}b^{\frac{m}{2}}, \;\; \forall a, b \geq 0,$$

Here, P_1 and P_2 can able to change along the progress of this proof. Lemma 3.3 and inequalities (12) and (10) provide us

$$\begin{aligned} &\mathbb{E}\int_0^t |x_\Delta(s)-\hat{x}_\Delta(s)|^{\frac{m}{2}} |f_\Delta(\hat{x}_\Delta(s))|^{\frac{m}{2}} ds \\ \leq& (k(\Delta))^{\frac{m}{2}} \int_0^T \mathbb{E}(|x_\Delta(s)-\hat{x}_\Delta(s)|^{\frac{m}{2}})ds \\ \leq& c_{m,\Delta,H} T. \end{aligned} \tag{22}$$

then

$$\begin{aligned} \mathbb{E}|x_\Delta(t)|^m &\leq P_1 + P_2\int_0^t \left(\mathbb{E}|x_\Delta(s)|^m + \mathbb{E}|\hat{x}_\Delta(s)|^m\right)ds \\ &\leq P_1 + P_2\int_0^t \left(\sup_{0\leq r\leq s} \mathbb{E}|x_\Delta(r)|^m\right)ds. \end{aligned}$$

If the right side is not decrease with t, the above formula holds for all $t \in [0, T]$.

We are able to notice

$$\sup_{0\leq r\leq t} \mathbb{E}|x_\Delta(r)|^m \leq P_1 + P_2\int_0^t \left(\sup_{0\leq r\leq s} \mathbb{E}|x_\Delta(r)|^m\right)ds.$$

Theorem 2.2 bring that

$$\sup_{0\leq r\leq t} \mathbb{E}|x_\Delta(r)|^m \leq P.$$

Here $\Delta \in (0, \Delta^*]$ and P is independent of Δ, then (19) can eb detect.

3.3 Strong convergence

Lemma 3.5. *We set Assumptions 3.1 and 3.2 are ture. Let $Z > |x_0|$ is a number with real value. We have the following conclusion:*

1.Define

$$\theta = \inf\{t \geq 0 : |x(t)| \geq Z\},$$

where $\inf \varnothing = \infty$. θ is a stopping time. We obtained that

$$\mathbb{P}(\theta_Z \leq T) \leq \frac{C}{Z^2}.$$

2.For any $\Delta \in (0, \Delta^*)$, Defined $\nu_{\Delta,Z}$ as

$$\nu_{\Delta,Z} = \inf\{t \geq 0 : |x_\Delta(t)| \geq Z\}.$$

then

$$\mathbb{P}(\nu_{\Delta,Z} \leq T) \leq \frac{C}{Z^2}.$$

3.For any $n \in (2, m]$,

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(T) - x(T)|^n = 0, \lim_{\Delta\to 0} \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^n = 0.$$

The proof of these lemmas is similar to the [24] (section 3.2), so we will not go into much detail here.

4 Simulation

Consider the Gilpin-Ayala model driven by fraction Brownian Motion,

$$dN_t = N_t\left[1 - \left(\frac{N_t}{K}\right)^\theta\right](rdt + \beta dB_t^H). \tag{23}$$

Let $f(N_t) = rN_t\left[1 - \left(\frac{N_t}{K}\right)^\theta\right]$ and $g(N_t) = N_t\left[1 - \left(\frac{N_t}{K}\right)^\theta\right]$. We can prove that there is a unique continuous solution N_t, $0 < N_t < K$. Obviously, the Assumption 3.1 and 3.2 are satisfied. Set $\theta = 1$ and choose $\eta(s) = s^2$, futhermore, (4) is hold. We also can select a strictly decreasing function $k(\Delta) = \Delta^{-\frac{1}{2}}$.

There is a truncated EM numerical solution $N_\Delta(t)$ of Eq.(23). The form is

$$N_\Delta^{(t_{k+1})} = N_\Delta^{(t_k)} + f_\Delta(N_\Delta^{(t_k)})\Delta + g_\Delta(N_\Delta^{(t_k)})\Delta B_k^H,$$

where the truncated functions are $f_\Delta(N_t) = f(N_t)$, $g_\Delta(N_t) = g(N_t)$.

The image of the orbit for this equation is shown as Figure 1.

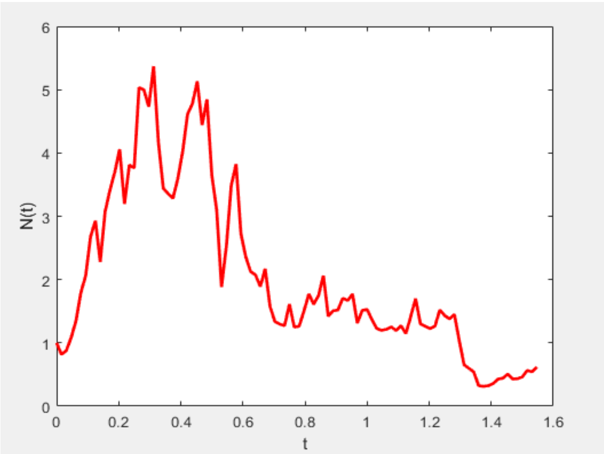

Fig. 1: Sample orbit for Equation (23)

And by Lemma 3.5, we can claim that $N_\Delta(t)$ is strongly convergent to $N(t)$.

We simulate the convergence rate of equation (23). And we set $N_0 = 1$, $T = 1$, $H = 0.6$, $\theta = 1$, $\beta = \frac{1}{2}$, $r = 1$, and simulate 1000 sample trajectories.

Next we will focus on the error at the endpoint $t = T$, and compute the average error $\delta = |N_\Delta(T) - N(T)|$. $N(T)$ represents real solution of equation (23). The result is shown in the Figure 2.

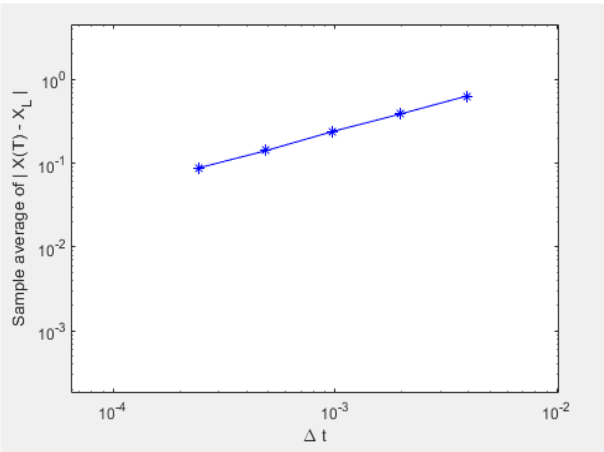

Fig. 2: The average error δ

In this graph, the solid blue line connected by asterisks represents the approximation to δ against Δt on log-log scale and that's implies the numerical solution of the equation (23) is convergent. And as Δt decreases, $\delta = |N_\Delta(T) - N(T)|$ also decreases accordingly.

5 Conclusion

This paper extend the method in [24] to SDEs driven by fractional noise. Regarding the research in this paper, we can draw the following conclusions:

1.This method will be applied to the nonlinear stochastic differential equations of fractional Brownian motion without linear growth condition.

2.The moment boundedness of the solution is guaranteed by using the stopping time and its strong convergence is proved.

3.We use the Gilpin-Ayala equation as an example to simulate the convergence rate of the numerical solution, and verify the error of this method.

In future works, this numerical method can be applied to some stochastic models of fractional Brownian motion. But it was limited to satisfying the local Lipschitz condition. On the basis of this research, we can continue to study some numerical methods of equations without Lipschitz condition. In addition, there will be some inspiration for the study of some other noise-driven stochastic differential equations.

References:

[1] Decreusefond L., Ustunel A S. Stochastic analysis of the fractional Brownian motion. Potential analysis, 1999, 10: 177-214.

[2] Biagini F., Hu Y., ksendal B., Zhang T., Stochastic calculus for fractional Brownian motion and applications, Springer-Verlag, London, 2008.

[3] Duncan T. E., Hu Y., Pasik-Duncan B. Stochastic calculus for fractional Brownian motion I. Theory. SIAM Journal on Control and Optimization, 2000, 38(2): 582-612.

[4] Millan G. On the Hurst exponent, Markov processes, and fractional Brownian motion. arXiv preprint arXiv, 2021, 2103.05019.

[5] Ahmed A M S. Existence and uniqueness of mild solution to neutral impulsive fractional stochastic delay differential equations driven by both Brownian motion and fractional Brownian motion. Differ. Appl,2022, 14:433-446.

[6] Higham D J., Mao X., Stuart A M.. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. SIAM Journal on Numerical Analysis, 2002, 40(3): 1041-1063.

[7] Hutzenthaler M., Jentzen A., Kloeden P E. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 2011, 467(2130): 1563-1576.

[8] Tretyakov M V, Zhang Z. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. SIAM Journal on Numerical Analysis, 2013, 51(6): 3135-3162.

[9] Saito Y., Mitsui T.. T-stability of numerial scheme for stochastic differential equations. Contributions in numerical mathematics. 1993: 333-344.

[10] Valinejad A., Hosseini S M. A variable step-size control algorithm for the weak approximation of stochastic differential equations. Numerical Algorithms, 2010, 55: 429-446.

[11] Neuenkirch A., Nourdin I. Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. Journal of Theoretical Probability, 2007, 20: 871-899.

[12] Yang H., Liu J., Liu Y., Yue X. Backward Euler-Maruyama method for a class of stochastic Markovian jump neural networks, MIPPR 2015: Pattern Recognition and Computer Vision. SPIE, 2015, 9813: 304-309.

[13] Shahnazi-Pour A., Moghaddam B. P.,Babaei A. Numerical simulation of the Hurst index of solutions of fractional stochastic dynamical systems driven by fractional Brownian motion. Journal of Computational and Applied Mathematics, 2021, 386:113210.

[14] Zheng Y., Qian S., Arshad S., Huang J. A modified Euler−Maruyama method for Riemann−Liouville stochastic fractional integro-differential equations. Journal of Statistical Computation and Simulation, 2023, 93(2): 249-265.

[15] Hong J., Huang C.,Wang X.. Optimal rate of convergence for two classes of schemes to stochastic differential equations driven by fractional Brownian motion. IMA Journal of Numerical Ayalysis, 2021, 41(2):1608-1638.

[16] Zhang S. Q., Yuan C. Stochastic differential equations driven by fractional Brownian motion with locally Lipschitz drift and their implicit Euler approximation. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 2021, 151(4): 1278-1304.

[17] Mishura I U S, Mishura Y. Stochastic calculus for fractional Brownian motion and related processes. Springer Science and Business Media, 2008.

[18] Wang S., Jiang Y.. Asymptotic analysis of a kernel estimator for parabolic stochastic partial differential equations driven by fractional noises. Frontiers of Mathematics in China, 2018, 13: 187-201.

[19] Duncan T E, Hu Y, Pasik-Duncan B. Stochastic calculus for fractional Brownian motion I. Theory. SIAM Journal on Control and Optimization, 2000, 38(2): 582-612.

[20] Mao X. Stochastic differential equations and applications. Elsevier, 2007.

[21] Jing G., Li B.. Comparison theorem and its applications for SDEs driven by fractional Brownian Motions. Journal of Mathematics, 2014, 34(5): 875-883.

[22] Kan X. Existence and uniqueness of solutions to Wick type integral Stochastic differential equation driven by fractional Brownian motion. Donghua University, 2009.

[23] Memin J., Mishura Y., Valkeila E. Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Statistics and Probability Letters, 2001, 51(2): 197-206.

[24] Mao X. The truncated Euler−Maruyama method for stochastic differential equations. Journal of Computational and Applied Mathematics, 2015, 290: 370-384.

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Suxin Wang, instructed and checked the reasonableness and correctness of the article.
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