Robust finite-time H_∞ control for discrete-time nonlinear uncertain singular systems with time-delay

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Abstract: This paper deals with the problems of finite-time H_∞ control for a class of discrete-time nonlinear singular systems subject to uncertainties and external disturbance. Firstly, the finite-time stability problem is discussed for discrete-time nonlinear uncertain singular systems with time-delay. The sufficient conditions of finite-time stability of discrete-time nonlinear singular systems are established. Then, by using the Lyapunov functional method, a criterion is established to ensure that the closed-loop system with external disturbance is H_∞ finite-time bounded. We design the controller gain matrix. Finally, we provide a numerical example to illustrate the effectiveness of the proposed results.

Key-Words: Finite-time stability; discrete-time singular systems; H_∞ finite-time boundedness; parametric uncertainty.

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1. Introduction

A singular system is a mixture of algebraic equations and differential equations, which can be regarded as a generalization of the standard state-space system. It not only describes the dynamics of the system, but also reveals algebraic constraints [1]. It is well known that singular systems have been widely used in many scientific fields, such as electrical network, circuit system, mechanical system, etc. Singular systems have also attracted considerable attention over the past thirty years [2-5]. In [6], Xia et al. considered the control problem of discrete singular hybrid systems. Feng et al. [7] studied singular linear quadratic optimal control for singular stochastic discrete-time system. In [8], Shuping et al. investigated the robust exponential stability and H_∞ control for uncertain discrete-time Markovian jump singular system. In [9], Ma et al. dealt with the finite-time H_∞ control for discrete-time switched singular time-delay systems.

It is widely accepted that most systems are inevitably nonlinear, and nonlinear dynamical systems have different characteristics and complex behaviors [10, 11]. In addition, time-delay and uncertainties often appear in various practical systems, such as chemical systems, biological systems and networked control systems. Time-delay and uncertainties may lead to system performance deterioration or system instability [12]. Some results on the stability and stabilization for uncertain discrete singular time-delay systems were reported in [13-16].

On the other hand, the behavior of a practical system in a finite time interval is often concerned. Finite-time stability and Lyapunov stability are two different concepts. Finite time stability considers the boundedness of a state system within a fixed interval. A system is called finite time stable if the system state does not exceed a certain domain during a fixed time interval with a given bound on the initial condition. Many valuable results have been obtained for finite-time stability and finite-time boundedness [16-20]. In [16], finite-time stability and stabilization for singular discrete-time linear positive systems were considered. In [18], the finite-time stability of discrete-time singular systems with nonlinear perturbations was studied. In [19], based on finite-time disturbance observer, the robust adaptive finite-time stabilization control for a class of nonlinear switched systems was investigated.

However, to the best of our knowledge, there is little research on the finite-time H_∞ control for uncertain discrete-time nonlinear singular systems with time-delay. This motivates us to study the finite-time H_∞ control for uncertain discrete-time nonlinear singular systems.

This paper investigates the finite-time H_∞ control problem for discrete-time nonlinear singular systems with uncertainties and time-delay via state feedback control. We present sufficient conditions which guarantee that a class of discrete-time nonlinear singular systems is finite-time stable. Then, we establish a new criterion to ensure that the closed-loop system with external disturbance is H_∞ finite-time bounded. Lastly, a numerical example is given to show the validity of the proposed method.

Notations. The superscript "T" denotes the transpose. $M<0(M>0)$ denotes the matrix M is a negative-definite (positive-definite) symmetric matrix. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denotes the maximum and minimum eigenvalue of the real symmetric matrix. $\mathbb{N}$ denotes the non-negative integer set. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry.

2. Problem Formulation

Consider the following discrete-time nonlinear singular system with state delay and parametric uncertainty

$$\begin{cases} Ex(k+1)=(A+\Delta A(k))x(k)+Bu(k) \\ \qquad +(A_d+\Delta A_d(k))x(k-d)+C_1 f(x(k)) \\ \qquad +Dw(k)+C_2 G(x(k-d)), \\ w(k+1)=Fw(k), \\ z(k)=\Lambda B^T x(k), \\ x(\theta)=\varphi(\theta), \forall\theta\in\{-d,-d+1,\cdots,0\}, \end{cases} \tag{1}$$

where $x(k)\in R^n$ is the n-dimensional state vector, $u(k)\in R^m$ is the control input. $z(k)\in R^q$ is penalty signal. The matrix $E\in R^{n\times n}$ may be singular and we assume that rank$(E)=r\le n.$ d is a positive integer representing the time delay. $\varphi(k)$ is an initial condition defined on the interval $[-d,0]$. The external disturbance $w(k)\in R^s$ satisfies

$$w^T(k)w(k)\le b, \quad b\ge 0. \tag{2}$$

The matrices $A, A_d, B, C_1, C_2, D, F$ and Λ are system matrices of corresponding dimensions. $\Delta A(k)$, $\Delta A_d(k)$ are unknown matrices representing time-varying parameter uncertainties and are assumed to be of the following form:

$$\Delta A(k)=M_1\Delta(k)N_1, \quad \Delta A_d(k)=M_2\Delta(k)N_2, \tag{3}$$

where $M_i(i=1,2)$ and $N_i(i=1,2)$ are known real constant matrices and $\Delta(k)$ is the unknown time-varying matrix-valued function subject to the following condition:

$$\Delta^T(k)\Delta(k)\le I, \quad \forall k\in\mathbb{N}. \tag{4}$$

$f(x(k))$ and $G(x(k-d))$ are unknown and represent the nonlinear perturbations with $f(0)=0$, $G(0)=0$, and for any $x,\hat{x}\in R^n$, the following Lipschitz condition is satisfied:

$$\begin{aligned} &\|f(x(k))-f(\hat{x}(k))\|\le\beta_1\|T(x(k)-\hat{x}(k))\|, \\ &\|G(x(k-d))-G(\hat{x}(k-d))\|\le\beta_2\|U(x(k-d)-\hat{x}(k-d))\|, \end{aligned} \tag{5}$$

where β_1,β_2 are positive scalars, T, U are known constant matrices.

Throughout the paper, some useful definitions and lemmas are given.

Definition 1 ([18]) **.** The matrix pair (E,A) is said to be causal if $\deg(\det(zE-A))=\text{rank}(E)$.

Definition 2. The system (1) with $u(k)=0$ and $w(k)=0$ is said to be finite-time stable (FTS) with respect to (c_1,c_2,N,R), where $0<c_1<c_2$, $R>0$, if

$$\sup_{k\in\{-d,-d+1,\cdots,0\}}\varphi^T(k)R\varphi(k)\le c_1$$
$$\Rightarrow x^T(k)E^TREx(k)<c_2, \forall k\in\{1,2,\cdots,N\}.$$

Definition 3. The system (1) is said to be finite-time bounded (FTB) with respect to (c_1,c_2,N,R,b), where $0<c_1<c_2$, $R>0$, if

$$\sup_{k\in\{-d,-d+1,\cdots,0\}}\varphi^T(k)R\varphi(k)\le c_1$$
$$\Rightarrow x^T(k)E^TREx(k)<c_2, \forall k\in\{1,2,\cdots,N\}.$$

Lemma 1 (Young inequality [21]) **.** Given matrices of appropriate dimensions X , Y and P >0, the following inequality hold

$$\mathrm{XY}+\mathrm{Y}^T\mathrm{X}^T\le\varepsilon\mathrm{XP}^{-1}\mathrm{X}^T+\varepsilon^{-1}\mathrm{Y}^T\mathrm{P}\,\mathrm{Y}, \forall\varepsilon\in\mathbb{R}^+. \tag{6}$$

Lemma 2 (Schur complement lemma [22]) **.** Given constant matrices X_1, X_2, X_3, where $X_1=X_1^T>0$ and $X_2^T=X_2>0$, then

$$X_1+X_3^TX_2^{-1}X_3<0,$$

if and only if

$$\begin{bmatrix} X_1 & X_3^T \\ X_3 & -X_2 \end{bmatrix}<0. \tag{7}$$

This study aims to derive new conditions that guarantee FTS of discrete-time nonlinear singular system (1) with $u(k)=0$ and $w(k)=0$, and to design a state-feedback controller such that the resulting closed-loop system is robust H_∞ FTB.

3. Main Results

In this section, we study the FTS and H_∞ FTB for discrete-time nonlinear singular system.

3.1. Finite-time stability

From system (1) with $u(k)=0$ and $w(k)=0$, we have following system

$$\begin{cases} Ex(k+1)=(A+\Delta A(k))x(k)+(A_d+\Delta A_d(k))x(k-d) \\ \qquad +C_1 f(x(k))+C_2 G(x(k-d)), \\ x(\theta)=\varphi(\theta), \forall\theta\in\{-d,-d+1,\cdots,0\}, \end{cases} \tag{8}$$

Theorem 1. Assume that system (8) is causal. The nonlinear singular system (8) is FTS with respect to (c_1,c_2,N,R) if there exist positive scalars μ_1,μ_2, $\eta\geq 1$, and symmetric positive-definite matrices P, Q such that the following inequalities hold:

$$\eta^N[\lambda_1 c_1+\lambda_2\eta^{d-1}dc_1]<c_2\lambda_4, \tag{9}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix}<0, \tag{10}$$

where

$\Pi_{11}=(A+\Delta A)^T P(A+\Delta A)-\eta E^T PE+\mu_1\beta_1^2 T^T T+Q,$

$\Pi_{12}=(A+\Delta A)^T P(A_d+\Delta A_d),\ \Pi_{13}=(A+\Delta A)^T PC_1,$

$\Pi_{14}=(A+\Delta A)^T PC_2,$

$\Pi_{22}=(A_d+\Delta A_d)^T P(A_d+\Delta A_d)-\eta^d Q+\mu_2\beta_2^2 U^T U,$

$\Pi_{23}=(A_d+\Delta A_d)^T PC_1,\ \Pi_{24}=(A_d+\Delta A_d)^T PC_2,$

$\Pi_{33}=C_1^T PC_1-\mu_1 I,\ \Pi_{34}=C_1^T PC_2,\ \Pi_{44}=C_2^T PC_2-\mu_2 I,$

$\hat{P}=R^{-\frac{1}{2}}PR^{-\frac{1}{2}},\quad \hat{Q}=R^{-\frac{1}{2}}QR^{-\frac{1}{2}},\quad \lambda_1=\lambda_{\max}(E^T\hat{P}E),$

$\lambda_2=\lambda_{\max}(\hat{Q}),\quad \lambda_4=\lambda_{\min}(\hat{P}),\quad ER^{\frac{1}{2}}=R^{\frac{1}{2}}E.$

Proof. We construct the Lyapunov function

$$V(k)=V_1(k)+V_2(k), \tag{11}$$

where

$$V_1(k)=x^T(k)E^T PEx(k),$$

$$V_2(k)=\sum_{i=k-d}^{k-1}\eta^{k-i-1}x^T(i)Qx(i),$$

Along the trajectory of (8), we have

$$\begin{aligned} &V_1(k+1)-\eta V_1(k) \\ &=x^T(k+1)E^T PEx(k+1)-\eta x^T(k)E^T PEx(k) \\ &=[(A+\Delta A)x(k)+(A_d+\Delta A_d)x(k-d) \\ &+C_1 f(x(k))+C_2 G(x(k-d))]^T P[(A+\Delta A)x(k) \\ &+(A_d+\Delta A_d)x(k-d)+C_1(k)f(x(k)) \\ &+C_2(k)G(x(k-d))]-\eta x^T(k)E^T PEx(k), \end{aligned} \tag{12}$$

$$\begin{aligned} &V_2(k+1)-\eta V_2(k) \\ &=\sum_{i=k+1-d}^{k}\eta^{k-i}x^T(i)Qx(i)-\eta\sum_{i=k-d}^{k-1}\eta^{k-i-1}x^T(i)Qx(i) \\ &=x^T(k)Qx(k)-\eta^d x(k-d)^T Qx(k-d). \end{aligned} \tag{13}$$

From (12) and (13), it follows that

$$\Delta V(k)=\varsigma(k)^T\Phi\varsigma(k), \tag{14}$$

where

$\varsigma(k)=\begin{bmatrix} x^T(k) & x^T(k-d) & f^T(x(k)) & G^T(x(k-d)) \end{bmatrix}^T,$

$$\Phi=\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} \\ * & * & \Phi_{33} & \Phi_{34} \\ * & * & * & \Phi_{44} \end{bmatrix},$$

$\Phi_{11}=(A+\Delta A)^T P(A+\Delta A)-\eta E^T PE+Q,$

$\Phi_{12}=(A+\Delta A)^T P(A_d+\Delta A_d),$

$\Phi_{13}=(A+\Delta A)^T PC_1, \Phi_{14}=(A+\Delta A)^T PC_2,$

$\Phi_{22}=(A_d+\Delta A_d)^T P(A_d+\Delta A_d)-\eta^d Q,$

$\Phi_{23}=(A_d+\Delta A_d)^T PC_1, \Phi_{24}=(A_d+\Delta A_d)^T PC_2,$

$\Phi_{33}=C_1^T PC_1,\quad \Phi_{34}=C_1^T PC_2,\ \Phi_{44}=C_2^T PC_2.$

From the Lipschitz conditions (5), we obtain the following inequalities for any scalars $\mu_1>0, \mu_2>0$:

$$\mu_1(\beta_1^2 x^T(k)T^T Tx(k)-f^T(x(k))f(x(k)))\geq 0, \tag{15}$$

$$\begin{aligned} &\mu_2(\beta_2^2 x^T(k-d)U^T Ux(k-d) \\ &-G^T(x(k-d))G(x(k-d)))\geq 0, \end{aligned} \tag{16}$$

From (15)-(16) , the following inequality is obtained:

$$V(k+1)-\eta V(k)\leq\varsigma(k)^T\Omega\varsigma(k), \tag{17}$$

where

$$\Omega=\begin{bmatrix} \Omega_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ * & \Omega_{22} & \Phi_{23} & \Phi_{24} \\ * & * & \Omega_{33} & \Phi_{34} \\ * & * & * & \Omega_{44} \end{bmatrix},$$

$\Omega_{11}=\Phi_{11}+\mu_1\beta_1^2 T^T T,\quad \Omega_{22}=\Phi_{22}+\mu_2\beta_2^2 U^T U,$

$\Omega_{33}=\Phi_{33}-\mu_1 I,\quad \Omega_{44}=\Phi_{44}-\mu_2 I.$

From inequality (10), it can be seen that

$$\Omega<0.$$

From (17), it follows that

$$V(k+1)-\eta V(k)\leq 0, \tag{18}$$

which implies that

$$V(k+1) \le \eta V(k). \tag{19}$$

From (19), we get that

$$V(k) \le \eta^k V(0) \le \eta^N V(0). \tag{20}$$

Then, from the Lyapunov function (11), we can get

$$\begin{aligned} V(0) &= x^T(0)E^T PEx(0) + \sum_{i=-d}^{-1} \eta^{-1-i} x^T(i)Qx(i) \\ &\le \lambda_{\max}(E^T \hat{P} E)x^T(0)Rx(0) \\ &\quad + \eta^{d-1}\lambda_{\max}(\hat{Q})\sum_{i=-d}^{-1} x^T(i)Rx(i) \\ &\le \lambda_1 c_1 + \lambda_2 \eta^{d-1} dc_1, \end{aligned} \tag{21}$$

and

$$\begin{aligned} V(k) &\ge x^T(k)E^T PEx(k) \\ &\ge \lambda_{\min}(\hat{P})x^T(k)E^T REx(k) \\ &= \lambda_4 x^T(k)E^T REx(k). \end{aligned} \tag{22}$$

So, from (20)-(22), one get

$$\lambda_4 x^T(k)E^T REx(k) < \eta^N(\lambda_1 c_1 + \lambda_2 \eta^{d-1} dc_1),$$

Using (9), we can deduce that

$$x^T(k)E^T REx(k) < c_2, \quad \forall k \in \{1,2,\cdots,N\}.$$

According to Definition 2, nonlinear singular system (8) is finite-time stable. This completes the proof.

3.2. Finite-time H_∞ control

Next, we investigate the finite-time control problem for nonlinear singular system (1).

We choose the control law $u(k) = -Lx(k)$, and the system (1) can be written as follows:

$$\begin{cases} Ex(k+1) = (A + \Delta A - BL)x(k) \\ \qquad +(A_d + \Delta A_d)x(k-d) + C_1 f(x(k)) \\ \qquad +C_2 G(x(k-d)) + Dw(k), \\ w(k+1) = Fw(k), \\ z(k) = \Lambda B^T x(k), \\ x(\theta) = \varphi(\theta), \forall \theta \in [-d, -d+1, \cdots, 0], \end{cases} \tag{23}$$

Definition 4. The closed-loop system (23) is said to be H_∞ finite-time bounded (H_∞ FTB) with respect to $(c_1, c_2, N, R, b, \gamma)$, where $0 < c_1 < c_2$, $R > 0$, if the system (23) is FTB with respect to (c_1, c_2, N, R, b), and under the zero-initial condition the following condition is satisfied

$$\sum_{k=0}^{N} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{N} w^T(k)w(k), \tag{24}$$

where $\gamma > 0$ is an H_∞ performance bound.

We are now in the position to give a criteria for finite-time boundedness of system (23).

Theorem 2. Assume that system (23) is causal. The system (23) is finite-time bounded with respect to $(c_1, c_2, N, R, b, \gamma)$ if there exist positive scalars $\mu_1, \mu_2, \varepsilon_1, \varepsilon_2, \tilde{\gamma}$, $\eta \ge 1$, three symmetric positive-definite matrices P, Q, S and a matrix Y such that the following inequalities hold:

$$\eta^N(\lambda_1 c_1 + \lambda_2 \eta^{d-1} dc_1 + \lambda_3 b) + \frac{1}{2}\eta^{N-1}\tilde{\gamma}^2 b < \lambda_4 c_2, \tag{25}$$

$$\begin{bmatrix} \Gamma_{11} & 0 & 0 & 0 & 0 & \Gamma_{16} & 0 & 0 \\ * & \Gamma_{22} & 0 & 0 & 0 & \Gamma_{26} & 0 & 0 \\ * & * & \Gamma_{33} & 0 & 0 & \Gamma_{36} & 0 & 0 \\ * & * & * & \Gamma_{44} & 0 & \Gamma_{46} & 0 & 0 \\ * & * & * & * & \Gamma_{55} & \Gamma_{56} & 0 & 0 \\ * & * & * & * & * & \Gamma_{66} & \Gamma_{67} & \Gamma_{68} \\ * & * & * & * & * & * & \Gamma_{77} & 0 \\ * & * & * & * & * & * & * & \Gamma_{88} \end{bmatrix} < 0, \tag{26}$$

where

$$\Gamma_{11} = Q - \eta E^T PE + \frac{1}{2} B\Lambda^T \Lambda B^T + \mu_1 \beta_1^2 T^T T + \varepsilon_1 N_1^T N_1,$$
$$\Gamma_{16} = A^T P - YB^T, \Gamma_{22} = -\eta^d Q + \mu_2 \beta_2^2 U^T U + \varepsilon_2 N_2^T N_2,$$
$$\Gamma_{26} = A_d^T P, \quad \Gamma_{33} = -\mu_1 I, \quad \Gamma_{36} = C_1^T P, \quad \Gamma_{44} = -\mu_2 I,$$
$$\Gamma_{46} = C_2^T P, \quad \Gamma_{55} = F^T SF - \eta S - \frac{1}{2}\tilde{\gamma}^2 I, \quad \Gamma_{56} = D^T P,$$
$$\Gamma_{66} = -P, \Gamma_{67} = PM_1, \Gamma_{68} = PM_2, \Gamma_{77} = -\varepsilon_1 I,$$
$$\Gamma_{88} = -\varepsilon_2 I, \quad \hat{P} = R^{-\frac{1}{2}} P R^{-\frac{1}{2}}, \quad \hat{Q} = R^{-\frac{1}{2}} Q R^{-\frac{1}{2}},$$
$$\lambda_1 = \lambda_{\max}(E^T \hat{P} E), \quad \lambda_2 = \lambda_{\max}(\hat{Q}), \quad \lambda_3 = \lambda_{\max}(S),$$
$$\lambda_4 = \lambda_{\min}(\hat{P}), \quad \gamma = \tilde{\gamma}\sqrt{\eta^N}, \quad B^T P = \tilde{P} B^T, ER^{\frac{1}{2}} = R^{\frac{1}{2}} E.$$

Furthermore, the controller gain is given by

$$L = \tilde{P}^{-T} Y^T. \tag{27}$$

Proof . We construct the Lyapunov function

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{28}$$

where

$$V_1(k) = x^T(k)E^T PEx(k),$$
$$V_2(k) = \sum_{i=k-d}^{k-1} \eta^{k-i-1} x^T(i)Qx(i),$$
$$V_3(k) = w^T(k)Sw(k).$$

Then

$$
\begin{aligned}
&V(k+1)-\eta V(k)\\
&= x^T(k+1)E^TPEx(k+1)+\sum_{i=k+1-d}^{k}\eta^{k-i}x^T(i)Qx(i)\\
&+w^T(k+1)Sw(k+1)-\eta x^T(k)E^TPEx(k)\\
&-\eta\sum_{i=k-d}^{k-1}\eta^{k-i-1}x^T(i)Qx(i)-\eta w^T(k)Sw(k)\\
&=[(A+\Delta A-BL)x(k)+(A_d+\Delta A_d)x(k-d)\\
&+C_1f(x(k))+C_2G(x(k-d))+Dw(k)]^TP\\
&\times[(A+\Delta A(k)-BL)x(k)+(A_d+\Delta A_d)x(k-d)\\
&+C_1f(x(k))+C_2G(x(k-d))+Dw(k)]\\
&+(Fw(k))^TPFw(k)-\eta x^T(k)E^TPEx(k)\\
&-\eta w^T(k)Sw(k)+x^T(k)Qx(k)\\
&-\eta^d x^T(k-d)Qx(k-d)\\
&=\xi^T(k)\Sigma\xi(k),
\end{aligned}
\tag{29}
$$

where

$$\xi(k)=\begin{bmatrix}x^T(k) & x^T(k-d) & f^T(x(k)) & G^T(x(k-d)) & w^T(k)\end{bmatrix}^T,$$

$$\Sigma=\begin{bmatrix}\Sigma_{11}&\Sigma_{12}&\Sigma_{13}&\Sigma_{14}&\Sigma_{15}\\ *&\Sigma_{22}&\Sigma_{23}&\Sigma_{24}&\Sigma_{25}\\ *&*&\Sigma_{33}&\Sigma_{34}&\Sigma_{35}\\ *&*&*&\Sigma_{44}&\Sigma_{45}\\ *&*&*&*&\Sigma_{55}\end{bmatrix},$$

$$
\begin{aligned}
&\Sigma_{11}=(A+\Delta A-BL)^TP(A+\Delta A-BL)+Q-\eta E^TPE,\\
&\Sigma_{12}=(A+\Delta A-BL)^TP(A_d+\Delta A_d),\\
&\Sigma_{13}=(A+\Delta A-BL)^TPC_1,\\
&\Sigma_{14}=(A+\Delta A-BL)^TPC_2,\\
&\Sigma_{15}=(A+\Delta A-BL)^TPD,\\
&\Sigma_{22}=(A_d+\Delta A_d)^TP(A_d+\Delta A_d)-\eta^dQ,\\
&\Sigma_{23}=(A_d+\Delta A_d)^TPC_1,\Sigma_{24}=(A_d+\Delta A_d)^TPC_2,\\
&\Sigma_{25}=(A_d+\Delta A_d)^TPD,\quad \Sigma_{33}=C_1^TPC_1,\\
&\Sigma_{34}=C_1^TPC_2,\ \Sigma_{35}=C_1^TPD,\quad \Sigma_{44}=C_2^TPC_2,\\
&\Sigma_{45}=C_2^TPD,\quad \Sigma_{55}=D^TPD+F^TSF-\eta S.
\end{aligned}
$$

It is obvious from (29)

$$
\begin{aligned}
&V(k+1)-\eta V(k)\\
&=\xi^T(k)\Sigma\xi(k)-\frac{1}{2}\tilde{\gamma}^2w^T(k)w(k)-\frac{1}{2}\|z(k)\|^2\\
&+\frac{1}{2}\tilde{\gamma}^2\|w(k)\|^2+\frac{1}{2}x^T(k)B\Lambda^T\Lambda B^Tx(k)\\
&=\xi^T(k)\tilde{\Sigma}\xi(k)+\frac{1}{2}\tilde{\gamma}^2\|w(k)\|^2-\frac{1}{2}\|z(k)\|^2,
\end{aligned}
\tag{30}
$$

where

$$\tilde{\Sigma}=\begin{bmatrix}\tilde{\Sigma}_{11}&\Sigma_{12}&\Sigma_{13}&\Sigma_{14}&\Sigma_{15}\\ *&\Sigma_{22}&\Sigma_{23}&\Sigma_{24}&\Sigma_{25}\\ *&*&\Sigma_{33}&\Sigma_{34}&\Sigma_{35}\\ *&*&*&\Sigma_{44}&\Sigma_{45}\\ *&*&*&*&\tilde{\Sigma}_{55}\end{bmatrix},$$

$$\tilde{\Sigma}_{11}=\Sigma_{11}+\frac{1}{2}B\Lambda^T\Lambda B^T,\quad \tilde{\Sigma}_{55}=\Sigma_{55}-\frac{1}{2}\tilde{\gamma}^2I.$$

From inequality (15)-(16), we can obtain that

$$\xi^T(k)\begin{bmatrix}\mu_1\beta_1^2T^TT&0&0&0&0\\ *&\mu_2\beta_2^2U^TU&0&0&0\\ *&*&-\mu_1I&0&0\\ *&*&*&-\mu_2I&0\\ *&*&*&*&0\end{bmatrix}\xi(k)\ge 0, \tag{31}$$

Then adding the left hand side of (31) to (30), we can obtain that

$$
\begin{aligned}
&V(k+1)-\eta V(k)\\
&\le\xi^T(k)\hat{\Sigma}\xi(k)+\frac{1}{2}\tilde{\gamma}^2\|w(k)\|^2-\frac{1}{2}\|z(k)\|^2,
\end{aligned}
\tag{32}
$$

where

$$\hat{\Sigma}=\begin{bmatrix}\hat{\Sigma}_{11}&\Sigma_{12}&\Sigma_{13}&\Sigma_{14}&\Sigma_{15}\\ *&\hat{\Sigma}_{22}&\Sigma_{23}&\Sigma_{24}&\Sigma_{25}\\ *&*&\hat{\Sigma}_{33}&\Sigma_{34}&\Sigma_{35}\\ *&*&*&\hat{\Sigma}_{44}&\Sigma_{45}\\ *&*&*&*&\tilde{\Sigma}_{55}\end{bmatrix},$$

$$\hat{\Sigma}_{11}=\tilde{\Sigma}_{11}+\mu_1\beta_1^2T^TT,\quad \hat{\Sigma}_{22}=\Sigma_{22}+\mu_2\beta_2^2U^TU,$$
$$\hat{\Sigma}_{33}=\Sigma_{33}-\mu_1I,\ \hat{\Sigma}_{44}=\Sigma_{44}-\mu_2I.$$

By Schur complement $\hat{\Sigma}<0$ is equivalent to

$$\Theta=\begin{bmatrix}\Theta_{11}&0&0&0&0&\Theta_{16}\\ *&\Theta_{22}&0&0&0&\Theta_{26}\\ *&*&-\mu_1I&0&0&C_1^TP\\ *&*&*&-\mu_2I&0&C_2^TP\\ *&*&*&*&\Theta_{55}&D^TP\\ *&*&*&*&*&-P\end{bmatrix}<0, \tag{33}$$

where

$$\Theta_{11}=Q-\eta E^TPE+\frac{1}{2}B\Lambda^T\Lambda B^T+\mu_1\beta_1^2T^TT,$$
$$\Theta_{16}=(A+\Delta A-BL)^TP,\quad \Theta_{22}=\mu_2\beta_2^2U^TU-\eta^dQ,$$
$$\Theta_{26}=(A_d+\Delta A_d)^TP,\quad \Theta_{55}=F^TSF-\eta S-\frac{1}{2}\tilde{\gamma}^2I.$$

By segregating the matrix (33) for known and uncertain parts, yield

$$\Theta=\tilde{\Theta}+\Delta\Theta, \tag{34}$$

where

$$\tilde{\Theta}=\begin{bmatrix}\Theta_{11} & 0 & 0 & 0 & 0 & (A-BL)^T P\\ * & \Theta_{22} & 0 & 0 & 0 & A_d^T P\\ * & * & -\mu_1 I & 0 & 0 & C_1^T P\\ * & * & * & -\mu_2 I & 0 & C_2^T P\\ * & * & * & * & \Theta_{55} & D^T P\\ * & * & * & * & * & -P\end{bmatrix},$$

$$\Delta\Theta=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & \Delta A^T P\\ * & 0 & 0 & 0 & 0 & \Delta A_d^T P\\ * & * & 0 & 0 & 0 & 0\\ * & * & * & 0 & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & * & 0\end{bmatrix}$$

$$=\mathrm{X}_1\Delta(k)\mathrm{Y}_1+(\mathrm{X}_1\Delta(k)\mathrm{Y}_1)^T$$
$$+\mathrm{X}_2\Delta(k)\mathrm{Y}_2+(\mathrm{X}_2\Delta(k)\mathrm{Y}_2)^T,$$
$$\mathrm{X}_1=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & M_1^T P\end{bmatrix}^T,$$
$$\mathrm{X}_2=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & M_2^T P\end{bmatrix}^T,$$
$$\mathrm{Y}_1=[N_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$
$$\mathrm{Y}_2=[0 \quad N_2 \quad 0 \quad 0 \quad 0 \quad 0].$$

For positive scalars ε_1 and ε_2, we have

$$\begin{aligned}&\mathrm{X}_1\Delta(k)\mathrm{Y}_1+(\mathrm{X}_1\Delta(k)\mathrm{Y}_1)^T\\ &\le \varepsilon_1^{-1}\mathrm{X}_1\mathrm{X}_1^T+\varepsilon_1\mathrm{Y}_1^T\Delta^T(k)\Delta(k)\mathrm{Y}_1\\ &\le \varepsilon_1^{-1}\mathrm{X}_1\mathrm{X}_1^T+\varepsilon_1\mathrm{Y}_1^T\mathrm{Y}_1,\end{aligned} \tag{35}$$

$$\begin{aligned}&\mathrm{X}_2\Delta(k)\mathrm{Y}_2+(\mathrm{X}_2\Delta(k)\mathrm{Y}_2)^T\\ &\le \varepsilon_2^{-1}\mathrm{X}_2\mathrm{X}_2^T+\varepsilon_2\mathrm{Y}_2^T\Delta^T(k)\Delta(k)\mathrm{Y}_2\\ &\le \varepsilon_2^{-1}\mathrm{X}_2\mathrm{X}_2^T+\varepsilon_2\mathrm{Y}_2^T\mathrm{Y}_2.\end{aligned} \tag{36}$$

From (34-36), we have

$$\Theta\le\tilde{\Theta}+\varepsilon_1^{-1}\mathrm{X}_1\mathrm{X}_1^T+\varepsilon_1\mathrm{Y}_1^T\mathrm{Y}_1+\varepsilon_2^{-1}\mathrm{X}_2\mathrm{X}_2^T+\varepsilon_2\mathrm{Y}_2^T\mathrm{Y}_2. \tag{37}$$

Let $Y=L^T P$. From $B^T P=\tilde{P}B^T$, Lemma 1 and (26), it can be seen that $\Theta<0$, which implies that

$$\hat{\Sigma}<0. \tag{38}$$

From (32) and (38), it is obvious that

$$V(k+1)-\eta V(k)\le\frac{1}{2}\tilde{\gamma}^2\|w(k)\|^2-\frac{1}{2}\|z(k)\|^2. \tag{39}$$

Thus, from the inequalities (39) we have

$$V(N+1)<\eta^{N+1}V(0)+\frac{1}{2}\sum_{i=0}^{N}\eta^{N-i}\left\{\tilde{\gamma}^2\|w(i)\|^2-\|z(i)\|^2\right\}. \tag{40}$$

Using the facts that $V(N+1)>0$ and $V(0)=0$, it follows from (40) that

$$\sum_{i=0}^{N}\|z(i)\|^2<\sum_{i=0}^{N}\eta^{N-i}\tilde{\gamma}^2\|w(i)\|^2.$$

It is deduced that

$$\sum_{i=0}^{N}z^T(i)z(i)<\sum_{i=0}^{N}\tilde{\gamma}^2\eta^N w^T(i)w(i).$$

Therefore

$$\sum_{i=0}^{N}z^T(i)z(i)\le\gamma^2\sum_{i=0}^{N}w^T(i)w(i).$$

where $\gamma=\tilde{\gamma}\sqrt{\eta^N}$. Thus, the system (23) satisfies the performance (24).

Moreover, from (29), (31) and (38), we have

$$\begin{aligned}&V(k+1)-\eta V(k)=\xi^T(k)\Sigma\xi(k)\\ &\le\xi^T(k)\Sigma\xi(k)+\frac{1}{2}x^T(k)B\Lambda^T\Lambda B^T x(k)+\xi^T(k)\\ &\cdot\begin{bmatrix}\mu_1\beta_1^2T^TT & 0 & 0 & 0 & 0\\ * & \mu_2\beta_2^2U^TU & 0 & 0 & 0\\ * & * & -\mu_1 I & 0 & 0\\ * & * & * & -\mu_2 I & 0\\ * & * & * & * & 0\end{bmatrix}\xi(k)\\ &=\xi^T(k)\hat{\Sigma}\xi(k)+\frac{1}{2}\tilde{\gamma}^2w^T(k)w(k),\end{aligned} \tag{41}$$

From (38), we have

$$V(k+1)<\eta V(k)+\frac{1}{2}\tilde{\gamma}^2w^T(k)w(k),$$

which implies that

$$\begin{aligned}V(k)&<\eta^kV(0)+\frac{1}{2}\sum_{i=0}^{k-1}\eta^{k-1-i}\left\{\tilde{\gamma}^2\|w(i)\|^2\right\}\\ &\le\eta^NV(0)+\frac{1}{2}\eta^{N-1}\tilde{\gamma}^2b.\end{aligned} \tag{42}$$

From the Lyapunov function (28), we can get

$$\begin{aligned}V(0)&=x^T(0)E^TPEx(0)+\sum_{i=-d}^{-1}\eta^{-i-1}x^T(i)Qx(i)\\ &+w^T(0)Sw(0)\\ &\le\lambda_{\max}(E^T\hat{P}E)x^T(0)Rx(0)\\ &+\lambda_{\max}(\hat{Q})\sum_{i=-d}^{-1}\eta^{-i-1}x^T(i)Rx(i)+\lambda_{\max}(S)b\\ &\le\lambda_1c_1+\lambda_2\eta^{d-1}dc_1+\lambda_3b,\end{aligned} \tag{43}$$

and

$$\begin{aligned}V(k)&>x^T(k)E^TPEx(k)\\ &>\lambda_4x^T(k)E^TREx(k).\end{aligned} \tag{44}$$

Using (42-44) and (25), it can be induced that

$$x^T(k)E^TPEx(k)<c_2,\quad\forall k\in\{1,\cdots,N\}.$$

According to Definition 4, the system (24) is finite-time boundness. This completes the proof.

4. Numerical example

In this section, a numerical example is presented to show the application of the developed theory.

Example 1. Consider the uncertain singular system (1) with the following parameters:

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0.4 & -0.02\\ 0.01 & 0.1\end{bmatrix},$$

$$A_d=\begin{bmatrix}0.3 & -0.05\\ 0.1 & 0.2\end{bmatrix}, B=\begin{bmatrix}0.2 & -0.1\\ 0.1 & 0.2\end{bmatrix},$$

$$C_1=\begin{bmatrix}0.2 & 0.1\\ 0.5 & 0.3\end{bmatrix},\quad C_2=\begin{bmatrix}0.1 & 0.2\\ 0.2 & 0.1\end{bmatrix},$$

$$D=\begin{bmatrix}-0.5 & 0.2\\ 0.1 & 0.2\end{bmatrix},\quad F=\begin{bmatrix}1.001 & 0\\ 0 & 1.001\end{bmatrix},$$

$$\Lambda=\begin{bmatrix}0.5 & 0.2\\ 0.2 & 0.5\end{bmatrix},\quad T=\begin{bmatrix}0.9 & 0\\ 0 & 0.9\end{bmatrix},$$

$$U=\begin{bmatrix}0.95 & 0\\ 0 & 0.95\end{bmatrix},\quad R=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$N_1=\begin{bmatrix}0.01 & 0.02\\ 0.01 & 0.01\end{bmatrix},\quad N_2=\begin{bmatrix}0.01 & 0.02\\ 0.01 & 0.01\end{bmatrix},$$

$$M_1=\begin{bmatrix}0.02 & 0.01\\ 0.02 & 0.02\end{bmatrix},\quad M_2=\begin{bmatrix}0.02 & 0.01\\ 0.02 & 0.02\end{bmatrix},$$

$$f(x(k))=\begin{bmatrix}0.01\sin(x_1(k))\\ 0\end{bmatrix},\quad d=2,$$

$$G(x(k-d))=\begin{bmatrix}0.02\sin(x_1(k-d))\\ 0\end{bmatrix}.$$

Take

$$c_1=0.2, \beta_1=\beta_2=0.8, \eta=1.001, \mu_1=\mu_2=0.5,$$
$$\varepsilon_1=\varepsilon_2=0.2,\ \gamma=2.02,\ b=1.2, N=20.$$

Solving the inequalities (25) and (26), we get the following feasible solution:

$$P=\begin{bmatrix}2.4646 & -0.7126\\ -0.7126 & 1.0453\end{bmatrix}, Q=\begin{bmatrix}1.3254 & -0.0251\\ -0.0251 & 0.0296\end{bmatrix},$$

$$S=\begin{bmatrix}3.8794 & 0.0014\\ 0.0014 & 3.8838\end{bmatrix}, Y=\begin{bmatrix}1.4628 & -1.8267\\ -1.8267 & 2.2031\end{bmatrix},$$

$c_2=9.0292.$

The controller gain matrice is

$$L=\begin{bmatrix}1.4055 & -1.7195\\ -2.4707 & 2.9833\end{bmatrix}.$$

According Theorem 2, the system (23) is finite-time bounded with respect to $(0.2, 9.0292, 20, I, 1.2, 2)$.

To demonstrate the effectiveness of controller, we present the trajectory of state in the Figure 1. The trajectory of $x^T(k)E^T REx(k)$ is shown in Figure 2. It can be seen from Figure 2 that the value of $x^T(k)E^T REx(k)$ is less than c_2.

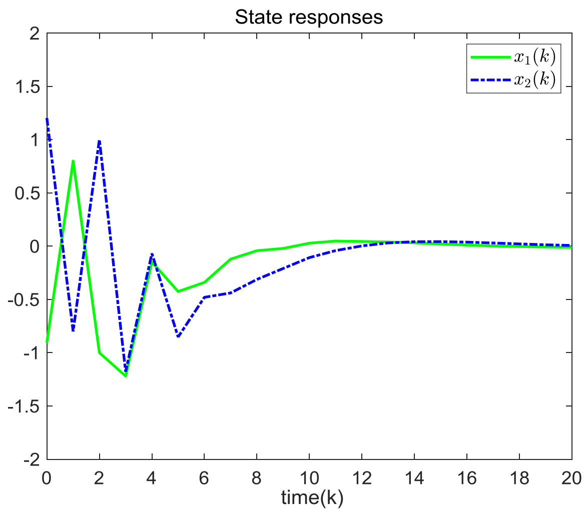

Fig. 1. The trajectories of $x_1(k)$ and $x_2(k)$.

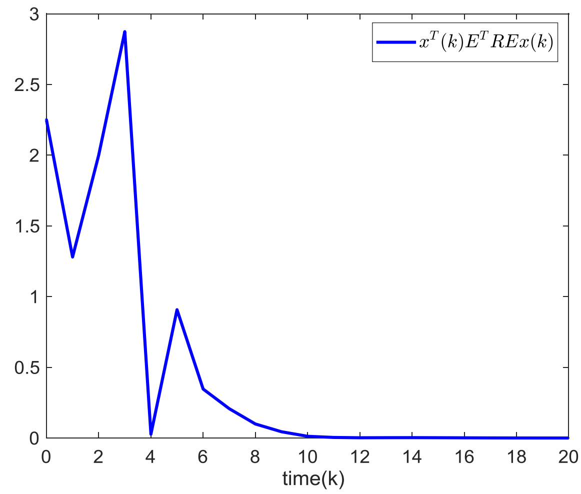

Fig. 2. The trajectory of $x^T(k)E^T REx(k)$.

5. Conclusions

In this paper, the problem of finite-time H_∞ control for discrete-time nonlinear singular systems with uncertainties and time-delay is investigated. By constructing an appropriate Lyapunov function, sufficient conditions are developed to ensure that the discrete-time nonlinear singular systems with uncertainties and a time-delay are finite-time stable. Then, the sufficient conditions are derived to ensure that the resulting closed-loop system is finite-time bounded via state feedback control. The controller gain matrix is given. Finally, a numerical example is presented to show the effective the presented results.

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