Puiseux and Taylor Series of the Einstein Functions and Their Role in the Solution of Inhomogeneous Airy's Equation

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Abstract: - The Einstein functions in generalized Puiseux and Taylor series are used as forcing functions in Airy's inhomogeneous equation, and particular and general solutions are obtained. Comparison are made with solutions obtained using the Nield-Kuznetsov functions' approach. For each of the Einstein's functions, the standard Nield-Kuznetsov function of the second kind is expressed in terms of Bessel functions. Computations and graphs in this work were produced using *Wolfram Alpha.*

Key-Words: - Puiseux series, Einstein and Nield-Kuznetsov functions.

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1 Introduction

In a recent article, Roach and Hamdan, [1], investigated solutions to Airy's inhomogeneous equation when the inhomogeneity (the forcing function in Airy's equation) is due to Einstein's functions. Einstein functions are combinations of logarithmic and exponential that arise in the study of distributions, and the determination of physical and chemical material constants arising in the study of Einstein's field equations. For these and many other applications of Einstein functions, one is referred to the elegant works of Abramowitz and Stegun, [2], Hilsenrath and Ziegler, [3], Cezairliyan, [4], and the references therein.

The main objective of the work of Roach and Hamdan, [1], was to find a connection between Airy's functions, [5], of the first and second kind, and the four Einstein functions, $Ej(x), j = 1,2,3,4,$ are given by:

$$\begin{cases} E1(x) = \dfrac{x}{e^x - 1} \\ E2(x) = \log(1 - e^{-x}) \\ E3(x) = \dfrac{x}{e^x - 1} - \log(1 - e^{-x}) \\ E4(x) = \dfrac{x^2 e^x}{(e^x - 1)^2} \end{cases} \tag{1}$$

where "log" is the natural logarithm.

In order to accomplish their objective, Roach and Hamdan, [1], provided particular and general solutions to Airy's inhomogeneous ordinary differential equation (ODE) of the form

$$y'' - xy = Ej(x) \quad (2)$$

wherein "prime" notation denotes ordinary differentiation with respect to the independent variable, and $Ej(x)$ is one of the four functions in (1).

The particular solution to (2) is given by, [1]:

$$y_p = \int_0^x \left\{ \int_0^\tau Ej(t)dt \right\} d\tau \quad (3)$$

and the general solution is given by

$$y = c_1 Ai(x) + c_2 Bi(x) + \int_0^x \left\{ \int_0^\tau Ej(t)dt \right\} d\tau \quad (4)$$

where c_1 and c_2 are arbitrary constants, and $Ai(x)$ and $Bi(x)$ are the linearly independent Airy's functions of the first and second kind, respectively, [6], with a non-zero Wronskian given by, [2,6]:

$$W\big(A_i(x), B_i(x)\big) = A_i(x)\frac{dB_i(x)}{dx} - B_i(x)\frac{dA_i(x)}{dx} = \frac{1}{\pi} \quad (5)$$

Roach and Hamdan, [1], obtained the following particular solutions corresponding to (1), respectively, by evaluating (3):

$$y_p = \begin{cases} xL_{i2}e^{-x} + 2L_{i3}e^{-x} - \frac{\pi^2 x}{6} - 2\zeta(3). \\ -\left(L_{i3}(e^{-x}) + \frac{x\pi^2}{6}\right) - \zeta(3). \\ -\left\{x[L_{i2}(e^{-x})] + 3L_{i3}(e^{-x}) + \frac{\pi^3}{3}x\right\} \\ -6\,\zeta(3). \\ -x^2 log(1-e^x) - 4xL_{i2}(e^x) + \\ 6L_{i3}(e^x) - \frac{x\pi^2}{3} - 6\,\zeta(3). \end{cases} \quad (6)$$

where $\zeta(x)$ is the zeta function and $\zeta(3) = 1.2020569$, and $L_{i2}(x)$ and $L_{i3}(x)$ are polylogarithmic functions, [7,8].

In this work, an alternative method is offered in which the series form of Einstein functions is used in the evaluation of particular solution (3). The use of series and tis approach might offer an easier alternative to computing particular and general solutions when dealing with initial and boundary value problems. In the process of this work, the existing relationships between Einstein's functions and the standard Nield-Kuznetsov functions, [1,9,10], will be utilized to express the standard Nield-Kuznetsov function of the second kind in terms of Bessel functions, [11].

2 Einstein and Bessel Functions

Roach and Hamdan, [1], obtained particular solution (3) through the following alternate form, introduced by Hamdan and Kamel, (2011):

$$y_p = \pi Ki(x) - \pi f(x) Ni(x) \quad (7)$$

where $Ni(x)$ and $Ki(x)$ are the standard Nield-Kuznetsov functions of the first and second kind, respectively, defined as, (Nield and Kuznetsov, 2009, Hamdan and Kamel, [9]:

$$Ni(x) = Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt \quad (8)$$

$$Ki(x) = f(x)Ni(x) - \left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\} \quad (9)$$

where $f(x) = Ej(x)$ in the work of Roach and Hamdan, [1], and in the current work.

The following relationships between Einstein functions, $Ei(x)$, Airy's functions, $Ai(x)$ and $Bi(x)$, and the standard Nield-Kuznetsov functions $Ni(x)$ and $Ki(x)$ were established in a theorem introduced by Roach and Hamdan, [1]:

$$Ki(x) = Ei(x)Ni(x) + \frac{1}{\pi}\int_0^x \left\{\int_0^\tau Ej(t)dt\right\} d\tau \quad (10)$$

$$Ki(x) = Ej(x)\left\{Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt\right\} + \frac{1}{\pi}\int_0^x \left\{\int_0^x Ej(t)dt\right\} dt \quad (11)$$

Using (6) and (7), the following relationships involving the polylogarithm functions are developed:

$$Ki(x) = E1(x)Ni(x) + \frac{x}{\pi}L_{i2}e^{-x} + \frac{2}{\pi}L_{i3}e^{-x} - \frac{\pi x}{6} - \frac{2}{\pi}\zeta(3) \quad (12)$$

$$Ki(x) = E2(x)Ni(x) - \left[\frac{1}{\pi}L_{i3}(e^{-x}) + \frac{x\pi}{6}\right] - \frac{1}{\pi}\zeta(3) \quad (13)$$

$$Ki(x) = E3(x)Ni(x) - \left\{\frac{x}{\pi}L_{i2}(e^{-x}) + \frac{3}{\pi}L_{i3}(e^{-x}) + \frac{\pi^2}{3}x\right\} - \frac{6}{\pi}\zeta(3) \quad (14)$$

$$Ki(x) = E4(x)Ni(x) - \frac{x^2}{\pi}log(1-e^x) - \frac{4x}{\pi}L_{i2}(e^x) + \frac{6}{\pi}L_{i3}(e^x) - \frac{x\pi}{3} - \frac{6}{\pi}\zeta(3) \quad (15)$$

With the knowledge of the expressions of $Ai(x)$ and $Bi(x)$, and their integrals in terms of Bessel's function of the first kind as, [11]:

$$Ai(x) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}(\mu) - I_{\frac{1}{3}}(\mu)\right] \quad (16)$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}(\mu) + I_{\frac{1}{3}}(\mu)\right] \quad (17)$$

$$\int_0^x Ai(t)dt = \frac{1}{3}\int_0^\mu \left[I_{-\frac{1}{3}}(t) - I_{\frac{1}{3}}(t)\right]dt \quad (18)$$

$$\int_0^x Bi(t)dt = \frac{1}{\sqrt{3}}\int_0^\mu \left[I_{-\frac{1}{3}}(t) + I_{\frac{1}{3}}(t)\right]dt \quad (19)$$

wherein $\mu = \frac{2}{3}x^{3/2}$, then using (16)-(19) in (8), the function $Ni(x)$ can be expressed in terms of Bessel's function, as, [11]:

$$Ni(x) = \frac{2\sqrt{x}}{3\sqrt{3}}[I_{-\frac{1}{3}}(\mu).\int_0^\mu I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\mu)\int_0^\mu I_{-\frac{1}{3}}(t)dt] \quad (20)$$

Using (16)-(19) in (12)-(15), the function $Ki(x)$ in (12)-(15) can be expressed, respectively, in terms of Bessel's function, as

$$Ki(x) = \frac{2x\sqrt{x}}{3\sqrt{3}(e^x-1)}\left\{I_{-\frac{1}{3}}(\mu).\int_0^\mu I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\mu)\int_0^\mu I_{-\frac{1}{3}}(t)dt\right\} + \frac{x}{\pi}L_{i2}e^{-x} + \frac{2}{\pi}L_{i3}e^{-x} - \frac{\pi x}{6} - \frac{2}{\pi}\zeta(3) \quad (21)$$

$$Ki(x) = \frac{2\sqrt{x}}{3\sqrt{3}}log(1-e^{-x})\left\{I_{-\frac{1}{3}}(\mu).\int_0^\mu I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\mu)\int_0^\mu I_{-\frac{1}{3}}(t)dt\right\} - \left[\frac{1}{\pi}L_{i3}(e^{-x}) + \frac{x\pi}{6}\right] - \frac{1}{\pi}\zeta(3) \quad (22)$$

$$Ki(x) = \left[\frac{2x\sqrt{x}}{3\sqrt{3}(e^x-1)} - \frac{2\sqrt{x}}{3\sqrt{3}}\log(1-e^{-x})\right]\left\{I_{-\frac{1}{3}}(\mu).\int_0^\mu I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\mu)\int_0^\mu I_{-\frac{1}{3}}(t)dt\right\} - \left\{\frac{x}{\pi}L_{i2}(e^{-x}) + \frac{3}{\pi}L_{i3}(e^{-x}) + \frac{\pi^2}{3}x\right\} - \frac{6}{\pi}\zeta(3) \quad (23)$$

$$Ki(x) = \frac{2\sqrt{x}x^2e^x}{3\sqrt{3}(e^x-1)^2}\left\{I_{-\frac{1}{3}}(\mu).\int_0^\mu I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\mu)\int_0^\mu I_{-\frac{1}{3}}(t)dt\right\} - \frac{x^2}{\pi}log(1-e^x)$$

$$-\frac{4x}{\pi}L_{i2}(e^x) + \frac{6}{\pi}L_{i3}(e^x) - \frac{x\pi}{3} - \frac{6}{\pi}\zeta(3) \quad (24)$$

Although (21)-(24) do not have direct implications on solving Airy's inhomogeneous equation at present, they are of theoretical value and are presented in this work for the sake of completeness and to provide a connection of Bessel's function and Einstein's functions.

3 Series Expressions of Einstein Functions

Some of the elementary properties of the Einstein functions, their domains, ranges, graphs and series representations are summarised in what follows, [12].

Case 1: $\boldsymbol{E_1(x) = \frac{x}{e^x - 1}}$

Domain of $E_1(x)$ is the set of real numbers except $x = 0$, and its range is the set of values $0 < E_1(x) < 1$ or $E_1(x) > 1$. Its graph is given in **Fig. 1**

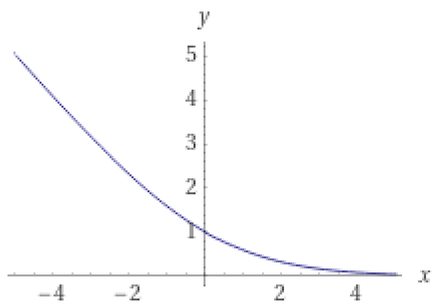

Fig. 1. Graph of $\boldsymbol{E_1(x)}$

The following improper integral of $E_1(x)$ converges:

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \frac{\pi^2}{6} \qquad (25)$$

and the function has a horizontal asymptote at 0, namely:

$$\lim_{x \to \infty} \frac{x}{e^x - 1} = 0 \qquad (26)$$

First derivative of $E_1(x)$ and its indefinite integral are given by:

$$E'_1(x) = \frac{e^x(x-1)+1}{(e^x-1)^2} \qquad (27)$$

$$\int \frac{x}{e^x - 1} dx = x\log(1 - e^{-x}) - Li_2(e^{-x}) + C \quad (28)$$

$E_1(x)$ can be approximated by the following Maclaurin series:

$$E_1(x) = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} - \frac{691x^{12}}{1307674368\,000} + O(x^{13}) \qquad (29)$$

Using (29) in (3), the following particular solution is obtained for (2) when its forcing function is $E_1(x)$:

$$y_p = -\frac{x^3}{12} + \frac{x^4}{144} - \frac{x^6}{21\,600} + \frac{x^8}{1\,693\,440} - \frac{x^{10}}{108\,864\,000} + \frac{x^{12}}{632\,2821\,120} - \frac{691x^{14}}{237\,996\,734\,976\,000} \qquad (30)$$

and the general solution can be obtained by substituting (30) in (4).

Case 2: $\boldsymbol{E_2(x) = \log(1 - e^{-x})}$

Domain of $E_2(x)$ is the set of positive real numbers and its range is the set of negative real numbers. Its graph is shown in **Fig. 2.**

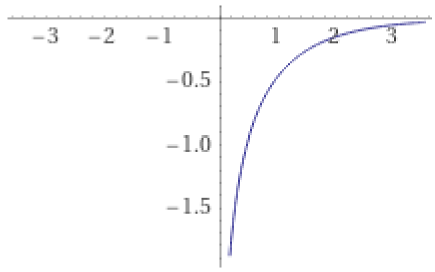

Fig. 2. Graph of $\boldsymbol{E_2(x)}$

The following improper integral of $E_2(x)$ converges:

$$\int_0^\infty \log(1 - e^{-x})\, dx = -\frac{\pi^2}{6} \qquad (31)$$

and the function has a horizontal asymptote at 0, namely:

$$\lim_{x \to \infty} \log(1 - e^{-x}) = 0 \qquad (32)$$

First derivative of $E_1(x)$ and its indefinite integral are given by:

$$E'_2 = \frac{1}{e^x - 1} \qquad (33)$$

$$\int \log(1-e^{-x})\,dx = Li_2(e^{-x}) + C \qquad (34)$$

$E_2(x)$ can be approximated by the following Puiseux series:

$$E_2(x) = logx - \frac{x}{2} + \frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181440} - \frac{x^8}{9676800} + \frac{x^{10}}{4790011600} \qquad (35)$$

Using (35) in (3), the following particular solution is obtained for (2) when its forcing function is $E_2(x)$:

$$y_p = \frac{x^2}{2}logx - \frac{3x^2}{4} - \frac{x^3}{12} + \frac{x^4}{288} - \frac{x^6}{86\,400} + \frac{x^8}{10\,160\,640} - \frac{x^{10}}{870\,912\,000} + \frac{x^{12}}{632\,281\,531\,200} \qquad (36)$$

and the general solution can be obtained by substituting (36) in (4).

Case 3: $E_3(x) = \frac{x}{e^x-1} - \log(1-e^{-x})$

Domain of $E_3(x)$ is the set of positive real numbers and its range is the set of positive real numbers. Its graph is shown in **Fig. 3.**

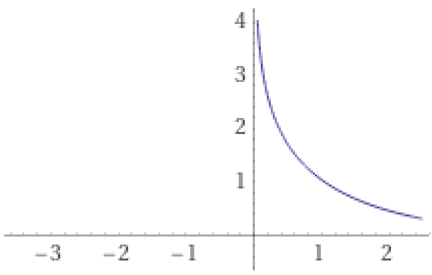

Fig. 3. Graph of $E_3(x)$

The following improper integral of $E_3(x)$ converges:

$$\int_0^\infty \left[\frac{x}{e^x-1} - \log(1-e^{-x})\right]dx = \frac{\pi^2}{3} \qquad (37)$$

and the function has a horizontal asymptote at 0, namely:

$$\lim_{x\to\infty}\left[\frac{x}{e^x-1} - \log(1-e^{-x})\right] = 0 \qquad (38)$$

First derivative of $E_3(x)$ and its indefinite integral are given by:

$$E'_3 = -\frac{xe^x}{(e^x-1)^2} \qquad (39)$$

$$\int \frac{x}{e^x-1} - \log(1-e^{-x})\,dx = xlog(1-e^{-x}) - 2\,Li_2(e^{-x}) + C \qquad (40)$$

$E_3(x)$ can be approximated by the following Puiseux series:

$$E_3(x) = 1 - logx + O(x^2) \qquad (41)$$

Using (41) in (3), the following particular solution is obtained for (2) when its forcing function is $E_3(x)$:

$$y_p = \frac{x^2}{4}(5 - 2logx) \qquad (42)$$

and the general solution can be obtained by substituting (42) in (4).

Case 4: $E_4(x) = \frac{x^2e^x}{(e^x-1)^2}$

Domain of $E_4(x)$ is the set of real numbers except $x = 0$, and its range is the set of values $0 < E_4(x) < 1$. Its graph is given in **Fig.**

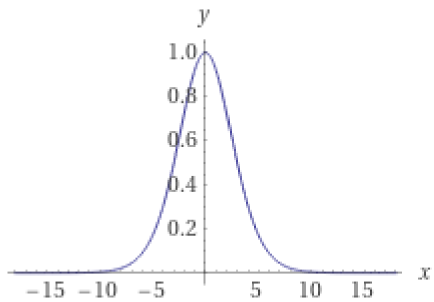

Fig. 4. Graph of $E_4(x)$

The following improper integral of $E_4(x)$ converges:

$$\int_0^\infty \frac{x^2e^x}{(e^x-1)^2}dx = \frac{\pi^2}{3} \qquad (43)$$

and the function has a horizontal asymptote at 0, namely:

$$\lim_{x\to\infty}\frac{x^2e^x}{(e^x-1)^2}=0 \tag{44}$$

First derivative of $E_4(x)$ and its indefinite integral are given by:

$$E'_4=\frac{xe^x[e^x(x-2)+(x+2)]}{(e^x-1)^3} \tag{45}$$

$$\int\frac{x^2e^x}{(e^x-1)^2}dx=2Li_2(e^x)+2xlog(1-e^x)+\frac{x^2}{1-e^x}+C \tag{46}$$

$E_4(x)$ can be approximated by the following Maclaurin series:

$$E_4(x)=1-\frac{x^2}{12}+\frac{x^4}{240}-\frac{x^6}{6048}+\frac{x^8}{172800}-\frac{x^{10}}{5322240}+\frac{691x^{12}}{118879488000} \tag{47}$$

Using (47) in (3), the following particular solution is obtained for (2) when its forcing function is $E_4(x)$:

$$y_p=\frac{x^2}{2}-\frac{x^4}{144}+\frac{x^6}{7\,200}-\frac{x^8}{338\,688}+\frac{x^{10}}{15\,552\,000}-\frac{x^{12}}{702\,535\,680}+\frac{691x^{14}}{21\,636\,066\,816\,000} \tag{48}$$

and the general solution can be obtained by substituting (48) in (4).

4 Results and Discussion

Polylogarithmic expressions for the particular solutions of Airy's inhomogeneous equation with Einstein's functions as its right-hand-side, as given by equation (6), are graphed in **Figs.5(a), 6(a), 7(a) and 8(a).** Corresponding particular solutions obtained from Taylor and Puiseux series, as given by equations (30), (36), (42) and (48), are plotted in **Figs.5(b), 6(b), 7(b) and 8(b).**

For particular solutions obtained using Taylor series expressions of the Einstein function, **Fig. 5(a)** and **5(b)** show similar trends in the graphs, although their numerical values are different. Similar behavior is observed in **Fig. 8(a)** and **8(b)**. Although no solution to initial or boundary value problems has been obtained in this work, hence solutions based on the general solutions have not been computed, it is expected that numerical values of the general solutions in both approaches should be close.

For particular solutions in **Fig. 6(a)** and **6(b)**, and in **Fig. 8(a)** and **8(b)**, differences occur in both the graphical trends and in the numerical values. This might be attributed to the use of Puiseux series in these cases, or it might be possible that solutions to initial and boundary value problems based on the general solutions would adjust themselves numerically. While this is inconclusive at present, graphs of the particular solutions are meant to provide information with regard to how close the polylogarithm expressions are to series expressions.

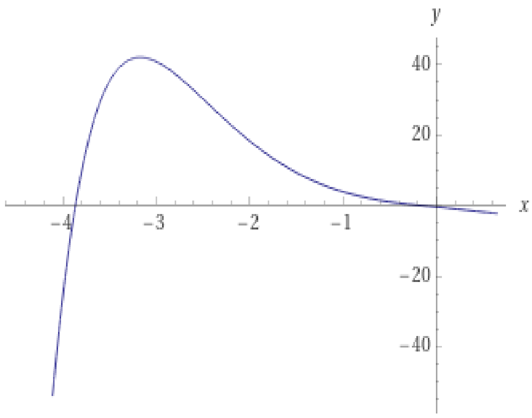

Fig. 5(a)
y_p Corresponding to $E_1(x)$ in Equation (6)

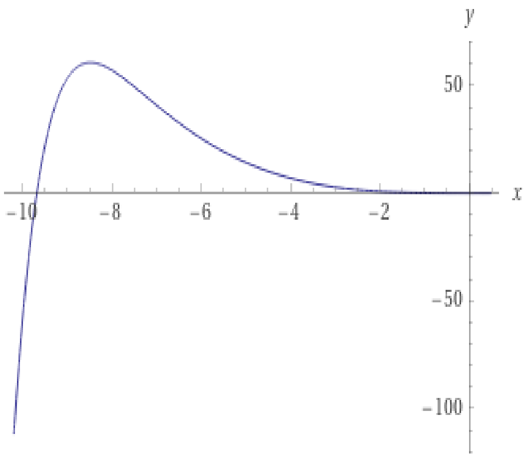

Fig. 5(b) y_p of Equation (30)

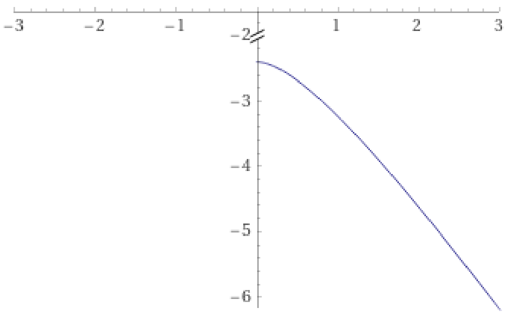

Fig. 6(a)
y_p Corresponding to $E_2(x)$ in Equation (6)

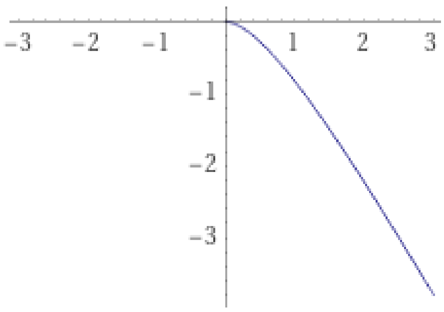

Fig. 6(b) y_p of Equation (36)

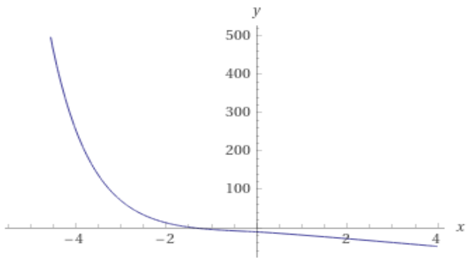

Fig. 7(a)
y_p Corresponding to $E_1(x)$ in Equation (6)

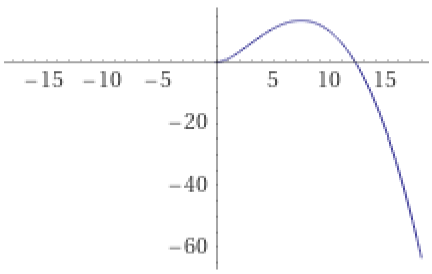

Fig. 7(b) y_p of Equation (42)

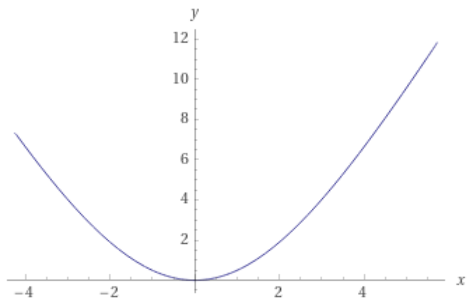

Fig. 8(a)
y_p Corresponding to $E_1(x)$ in Equation (6)

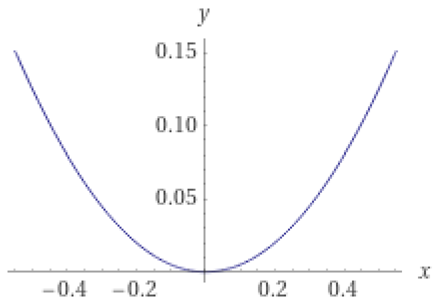

Fig. 8(b) y_p of Equation (48)

5 Conclusion

In this work, a method of solution of the inhomogeneous Airy's equation when the right-hand-side is one of Einstein's functions is investigated. The method is based on expressing the Einstein functions in series form followed by obtaining particular solutions. The main conclusion that can be drawn from this work is that Taylor series expansions of Einstein's functions produce particular solutions with a trend that is similar to that of particular solutions expressed in polylogarithmic functions.

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Contribution of individual authors

Both authors reviewed the literature, formulated the problem, provided independent analysis, and jointly wrote and revised the manuscript.

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