Dynamical Behavior of High-order Rational Difference System

QIANHONG ZHANG
Guizhou University of Finance
and Economics
School of Mathematics and Statistics
Guiyang, Guizhou
CHINA
zqianhong68@163.com

WENZHUAN ZHANG
Guizhou University of Finance
and Economics
Guizhou Key Laboratory of
Economics System Simulation
Guiyang, Guizhou
CHINA
zhwenzh97@hotmail.com

Abstract: This paper is concerned with the boundedness, persistence and global asymptotic behavior of positive solution for a system of two high-order rational difference equations. Moreover, some numerical examples are given to illustrate results obtained.

Key–Words: difference equations, boundedness, persistence, global asymptotic behavior

1 Introduction

Difference equation or discrete dynamical system is diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n, x_{n-2}, \cdots, x_{n-k})$ determines a difference equation and vise versa. Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models [1] describing real life situations in population biology [2], economic, probability theory, genetics, psychology, etc.

The study of properties of rational difference equations [3] and systems of rational difference equations has been an area of interest in recent years. There are many papers in which systems of difference equations have studied.

Cinar et al. [4] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{m}{y_n},\ y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}.$$

Cinar [5] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

Also, Cinar [6] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{z_n},\ y_{n+1} = \frac{x_n}{x_{n-1}},\ z_{n+1} = \frac{1}{x_{n-1}}.$$

Ozban [7] has investigated the positive solutions of the system of rational difference equtions

$$x_{n+1} = \frac{1}{y_{n-k}},\ \ y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m+k}}.$$

Papaschinopoulos et al. [8] investigated the global behavior for a system of the following two nonlinear difference equations.

$$x_{n+1} = A + \frac{y_n}{x_{n-p}},\ y_{n+1} = A + \frac{x_n}{y_{n-q}},\ n = 0, 1, \cdots,$$

where A is a positive real number, p, q are positive integers, and $x_{-p}, \cdots, x_0, x_{-q}, \cdots, x_0$ are positive real numbers.

In 2012, Zhang, Yang and Liu [9] investigated the global behavior for a system of the following third order nonlinear difference equations.

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n},\ y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n},$$

where $A, B \in (0, \infty)$, and the initial values $x_{-i}, y_{-i} \in (0, \infty), i = 0, 1, 2.$

Ibrahim [10] has obtained the positive solution of the difference equation system in the modeling competitive populations.

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n + \alpha},\ y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n + \beta}.$$

Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. In book [11], Kocic and Ladas have studied global behavior of nonlinear difference equations of higher order. Similar nonlinear systems of rational difference equations were investigated (see [12],[13]). Other related results reader can refer ([14], [15], [16], [17], [18],[19],[20],[21],[22],[23],[24],[25]).

Motivated by above discussion, our goal, in this paper is to investigate the solutions of the two-dimensional system of rational nonlinear difference equations in the form

$$\begin{cases} x_{n+1} = \frac{x_n}{B+y_{n-r}y_{n-s}}, \\ \\ y_{n+1} = \frac{y_n}{A+x_{n-p}x_{n-q}}, \ n = 0, 1, \cdots. \end{cases} \quad (1)$$

where $A, B \in (0, \infty), p, q, r, s \in N^+$, and the initial values $x_{-\max\{p,q\}}, x_{1-\max\{p,q\}}, \cdots, x_0 \in (0, \infty)$; $y_{-\max\{r,s\}}, y_{1-\max\{r,s\}}, \cdots, y_0 \in (0, \infty)$. Moreover, we have studied the local stability, global stability, boundedness of solutions. We will consider some special cases of (1) as applications. Finally, we give some numerical examples.

2 Preliminaries

Let I_x, I_y be some intervals of real number and $f : I_x^m \times I_y^t \to I_x, g : I_x^m \times I_y^t \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_j) \in I_x \times I_y (i = -m, -m+1, \cdots, -1, 0; j = -t, -t+1, \cdots, 0)$, the system of difference equations, for $n = 0, 1, 2, \cdots$,

$$\begin{cases} x_{n+1} = f(x_n, \cdots, x_{n-m}, y_n, \cdots, y_{n-t}), \\ \\ y_{n+1} = g(x_n, \cdots, x_{n-m}, y_n, \cdots, y_{n-t}), \end{cases} \quad (2)$$

has a unique solution $\{(x_n, y_n)\}_{n=-\max\{m,t\}}^{\infty}$. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of (2) if $\bar{x} = f(\bar{x}, \cdots, \bar{x}, \bar{y}, \cdots, \bar{y}), \bar{y} = g(\bar{x}, \cdots, \bar{x}, \bar{y}, \cdots, \bar{y})$, namely, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$.

Let I_x, I_y be some intervals of real numbers, interval $I_x \times I_y$ is called invariant for system (1) if, for all $n > 0$,

$$x_{-m}, x_{-1}, \cdots, x_0 \in I_x, y_{-t}, y_{-1}, \cdots, y_0 \in I_y$$

$$\Rightarrow x_n \in I_x, \ \ y_n \in I_y.$$

Definition 1 *Assume that $(\bar{x}, \bar{y})$ be a fixed point of (2). Then*
(i) $(\bar{x}, \bar{y})$ is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_i, y_j) \in I_x \times I_y (i = -m, -m+1, \cdots, -1, 0; j = -t, -t+1, \cdots, 0)$, with $\sum_{i=-m}^{0} |x_i - \bar{x}| < \delta, \sum_{j=-t}^{0} |y_j - \bar{y}| < \delta$, implies $|x_n - \bar{x}| < \varepsilon, |y_n - \bar{y}| < \varepsilon$.
(ii) $(\bar{x}, \bar{y})$ is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_j) \in I_x \times I_y (i = -m, -m+1, \cdots, -1, 0; j = -t, -t+1, \cdots, 0)$, $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$.
(iii) $(\bar{x}, \bar{y})$ is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.
(iv) Unstable if it is not stable.

Theorem 2 *[11] Assume that $X(n+1) = F(X(n)), n = 0, 1, \cdots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system i.e., $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix J_F, evaluated at $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has modulus greater than one then $\overline{X}$ is unstable.*

Theorem 3 *[12] Assume that $X(n+1) = F(X(n)), n = 0, 1, \cdots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\overline{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $p(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if*

$$\Delta_k > 0 \ for \ k = 1, 2, \cdots, n, \quad (3)$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

3 Main results

Consider the system (1), if $A < 1, B < 1$, system (1) has equilibrium $(0, 0)$ and $(\sqrt{1-A}, \sqrt{1-B})$. In addition, if $A < 1, B = 1$, then system (1) has infinite equilibrium points $(\bar{x}, 0)$, where $\bar{x} \geq 0$, and if $A = 1, B < 1$, then system (1) has infinite equilibrium points $(0, \bar{y})$, where $\bar{y} \geq 0$. Finally, if $A > 1$ and $B > 1$, $(0, 0)$ is the unique equilibrium point.

Theorem 4 *Assume that $A < 1, B < 1$. Then the following statements are true.*
(i) The equilibrium $(0, 0)$ is locally unstable.
(ii) The unique positive equilibrium $(\sqrt{1-A}, \sqrt{1-B})$ is locally unstable.

Proof: (i) Let $M = \max\{p, q, r, s\}$. We can easily obtain that the linearized system of (1) about the equilibrium $(0, 0)$ is

$$\Phi_{n+1} = D\Phi_n, \tag{4}$$

where $\Phi_n = (x_n, x_{n-1}, \cdots, x_{n-M}, y_n, y_{n-1}, \cdots, y_{n-M})^T$, $D = (d_{ij})_{(2M+2)\times(2M+2)} =$

$$\begin{pmatrix} \frac{1}{B} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{A} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{5}$$

The characteristic equation of (4) is

$$f(\lambda) = \lambda^{2M}\left(\lambda - \frac{1}{A}\right)\left(\lambda - \frac{1}{B}\right) = 0. \tag{6}$$

This shows that the roots of characteristic equation $\lambda = \frac{1}{A}$ and $\lambda = \frac{1}{B}$ lie outside unit disk. So the unique equilibrium $(0, 0)$ is locally unstable.

(ii) We can easily obtain that the linearized system of (1) about the equilibrium $(\sqrt{1-A}, \sqrt{1-B})$ is

$$\Phi_{n+1} = G\Phi_n, \tag{7}$$

where $\Phi_n = (x_n, x_{n-1}, \cdots, x_{n-M}, y_n, y_{n-1}, \cdots, y_{n-M})^T$, $G =$

$$\left(\begin{array}{cccccccc} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & \cdots & -\sqrt{1-A} & \cdots & -\sqrt{1-A} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & & & & & & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{cccccccc} 0 & \cdots & -\sqrt{1-B} & \cdots & -\sqrt{1-B} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & & & & & & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 \end{array}\right)$$

in which $-\sqrt{1-B}$ are in column $M + r + 2$ and $M + s + 2$, respectively. $-\sqrt{1-A}$ are in column $p + 1$ and $q + 1$, respectively.

Let $\lambda_1, \lambda_2, \cdots, \lambda_{2M+2}$ denote the $2M + 2$ eigenvalues of Matrix G. Let $D = \text{diag}(d_1, d_2, \cdots, d_{2M+2}), d_i \neq 0 (i = 1, 2, \cdots, 2M + 2)$ be a diagonal matrix,

Clearly D is invertible. Computing DGD^{-1}, we obtained $DGD^{-1} =$

$$\left(\begin{array}{cccccccc} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \frac{d_2}{d_1} & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ & & \frac{d_{p+1}}{d_p} & & & & & \\ & & & \ddots & & & & \\ & & & & \frac{d_{q+1}}{d_q} & & & \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{d_{M+1}}{d_M} & 0 \\ 0 & \cdots & -\sqrt{1-A} & \cdots & -\sqrt{1-A} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & & & & & & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \end{array}\right.$$

$$\begin{array}{cccccccc} 0 & \cdots & -\sqrt{1-B} & \cdots & -\sqrt{1-B} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \frac{d_{M+3}}{d_{M+2}} & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ & & \frac{d_{M+r+2}}{d_{M+r+1}} & & & & & \\ & & & \ddots & & & & \\ & & & & \frac{d_{M+s+2}}{d_{M+s+1}} & & & \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{d_{2M+2}}{d_{2M+1}} & 0 \end{array}\right)$$

It is well known that G has the same eigenvalues as DGD^{-1}, we obtain that
$\max_{1\le k\le 2M+2}|\lambda_k|$

$$\begin{aligned} &= \|DED^{-1}\| \\ &= \max\Big\{d_2d_1^{-1},\cdots,d_{M+1}d_M^{-1},d_{M+3}d_{M+2}^{-1},\cdots, \\ &\quad d_{2M+2}d_{2M+1}^{-1},1+2\sqrt{1-A},1+2\sqrt{1-B}\Big\} \\ &> 1 \end{aligned}$$

It follows from Theorem 3 that equilibrium $(\sqrt{1-A},\sqrt{1-B})$ is locally unstable. □

Theorem 5 *Assume that $A>1, B>1$. Then the equilibrium $(0,0)$ is globally asymptotically stable.*

Proof: For $A>1, B>1$, from Theorem 4 $(0,0)$ is locally asymptotically stable. From (1), it is easy to see that every positive (x_n,y_n) is bounded, i. e., $0\le x_n\le x_0, 0\le y_n\le y_0$. Now, it is sufficient to prove that(x_n,y_n) is decreasing. From (1), we have

$$\begin{cases} x_{n+1} = \frac{x_n}{B+y_{n-r}y_{n-s}} \le \frac{x_n}{B} < x_n, \\ y_{n+1} = \frac{y_n}{A+x_{n-p}x_{n-q}} \le \frac{y_n}{A} < y_n. \end{cases}$$

This implies that the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence, $\lim_{n\to\infty}x_n=0, \lim_{n\to\infty}y_n=0$. Therefore, the equilibrium $(0,0)$ is globally asymptotically stable. □

Theorem 6 *Let $A<1$ and $B<1$. Then, for solution (x_n,y_n) of system (1) following statements are true.*
(i) If $x_n\to 0$, then $y_n\to\infty$.
(ii) If $y_n\to 0$, then $x_n\to\infty$.

4 Rate of convergence

In order to study the rate of convergence of positive solutions of (1) which converge to equilibrium point $(0,0)$ of this system, first we consider the following results that gives the rate of convergence of solution of a system of difference equations.

$$X_{n+1}=[A+B(n)]X_n \tag{8}$$

where X_n be m dimensional vector,$A\in C^{m\times m}$ is a constant matrix. $B: Z^+\to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\|\to 0 \tag{9}$$

as $n\to\infty$, where $\|\cdot\|$ be any matrix norm which is associated with the vector norm

$$\|(x,y)\|=\sqrt{x^2+y^2}.$$

Proposition 7 *(Perrons Theorem)[26] Suppose that condition (9) holds. If X_n is any solution of (8), then $X_n=0$ for all large n or*

$$\rho=\lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|}. \tag{10}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 8 *[26] Suppose that condition (9) holds. If X_n is any solution of (8), then $X_n=0$ for all large n or*

$$\rho=\lim_{n\to\infty}\sqrt[n]{\|X_{n+1}\|}. \tag{11}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Let(x_n,Y_n) be an arbitrary positive solution of system (1) such that $\lim_{n\to\infty}x_n=0, \lim_{n\to\infty}y_n=0$. It follows from (1) that

$$x_{n+1}-0=\frac{x_n}{B+y_{n-r}y_{n-s}}=\frac{1}{B+y_{n-r}y_{n-s}}x_n$$

and

$$y_{n+1}-0=\frac{y_n}{A+x_{n-p}y_{n-q}}=\frac{1}{A+x_{n-p}x_{n-q}}y_n$$

Let $E_n^1=x_n-0, E_n^2=y_n-0$, then we have

$$E_{n+1}^1=A_nE_n^1+B_nE_n^2,\quad E_{n+1}^2=C_nE_n^1+D_nE_n^2.$$

where

$$A_n=\frac{1}{B+y_{n-r}y_{n-s}}, B_n=0,$$

$$C_n = 0, D_n = \frac{1}{A + x_{n-p}x_{n-q}}.$$

Moreover

$$\lim_{n\to\infty} A_n = \frac{1}{B}, \quad \lim_{n\to\infty} D_n = \frac{1}{A}.$$

Now the limiting system of error terms can be written as

$$\begin{pmatrix} E^1_{n+1} \\ E^2_{n+1} \end{pmatrix} = \begin{pmatrix} 1/B & 0 \\ 0 & 1/A \end{pmatrix} \begin{pmatrix} E^1_n \\ E^2_n \end{pmatrix},$$

which is similar to linearized system of (1) about the equilibrium point $(0, 0)$.

Using Proposition 7 and Proposition 8, we have following result.

Theorem 9 *Assume that* (x_n, y_n) *be a positive solution of (1) such that* $\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} y_n = 0$*, then the error vector* $E_n = (E^1_n, E^2_n)^T$ *of every solution of (1) satisfies the following asymptotic relations*

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|} = |\lambda_{1,2}F_j(0, 0)|,$$

$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_{1,2}F_j(0, 0)|,$$

where $\lambda_{1,2}F_j(0, 0) = \frac{1}{A}$ *or* $\frac{1}{B}$ *are the characteristic of Jacobian matrix* $F_J(0, 0)$.

5 Numerical examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations and system of nonlinear difference equations.

Example 1. If the initial conditions $x_0 = 0.6, x_{-1} = 0.2, x_{-2} = 0.8, x_{-3} = 0.3, x_{-4} = 0.4, y_0 = 0.6, y_{-1} = 0.3, y_{-2} = 0.5, y_{-3} = 0.2, y_{-4} = 0.8$ and $A = 1.2, B = 1.3, r = 1, s = 3, p = 2, q = 4$, we have the following system

$$x_{n+1} = \frac{x_n}{1.3 + y_{n-1}y_{n-3}},$$

$$y_{n+1} = \frac{y_n}{1.2 + x_{n-2}x_{n-4}}.$$

It is clear that $A > 1, B > 1$. Then the equilibrium $(0, 0)$ is globally asymptotically stable.(See Theorem 3.2 , Fig. 1)

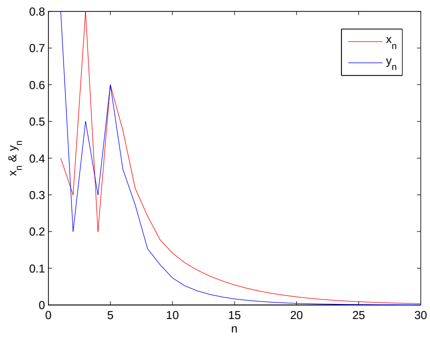

Figure 1: The fixed point (0,0) is globally asymptotically stable

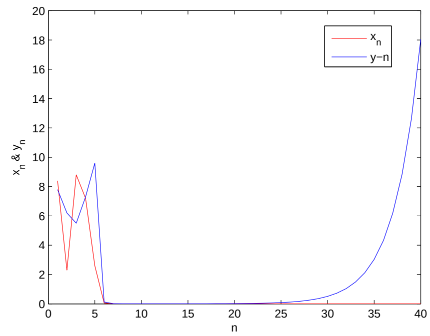

Figure 2: The fixed point (0,0) and $(\sqrt{1-A}, \sqrt{1-B})$ is unstable

Example 2. If If the initial conditions $x_0 = 2.6, x_{-1} = 7.2, x_{-2} = 8.8, x_{-3} = 2.3, x_{-4} = 8.4, y_0 = 9.6, y_{-1} = 7.3, y_{-2} = 5.5, y_{-3} = 6.2, y_{-4} = 7.8$ and $A = 0.9, B = 0.7, r = 1, s = 3, p = 2, q = 4$, we have the following system

$$x_{n+1} = \frac{x_n}{0.7 + y_{n-1}y_{n-3}}, \ y_{n+1} = \frac{y_n}{0.9 + x_{n-2}x_{n-4}}.$$

It is clear that $A < 1, B < 1$. Then equilibrium $(0, 0)$ and $(\sqrt{1-A}, \sqrt{1-B})$ are unstable.(See Theorem 4, Theorem 6, Fig 2)

6 Conclusion and future work

In this paper, we have studied the behavior of positive solution to system (1) under some conditions. If $A > 1$ and $B > 1$, the system (1) has an unique equilibrium $(0, 0)$ which is globally asymptotically stable. If $A < 1$ and $B < 1$, the system (1) has equilibrium $(0, 0)$ and $(\sqrt{1-A}, \sqrt{1-B})$, and these equilibriums are unstable. We will study the behavior of positive solution to system under the conditions $A > 1, B < 1$ or $A = B = 1$ in the future.

Acknowledgements: This work was financially supported by the National Natural Science Foundation of China (Grant No. 11361012), and the Scientific Research Foundation of Guizhou Provincial Science and Technology Department([2013]J2083, [2009]J2061), and the Natural Science Foundation of Guizhou Provincial Educational Department (No.2008040).

References:

[1] M.P. Hassell and H.N. Comins, *Discrete time models for two-species competition*, Theoretical Population Biology, Vol. 9, no. 2,1976, pp. 202–221.

[2] J.E. Franke and A.A. Yakubu, *Mutual exclusion versus coexistence for discrete competitive Systems*, Journal of Mathematical Biology, Vol.30, no. 2,1991, pp. 161–168.

[3] B. Iricanin and S. Stevic, *Some Systems of Nonlinear Difference Equations of Higher Order with Periodic Solutions*, Dynamics of Continuous, Discrete and Impulsive Systems. Series A Mathematical Analysis, Vol. 13, No. 3-4, 2006, pp. 499–507.

[4] C. Cinar, I. Yalcinkaya and R. Karatas, *On the positive solutions of the difference equation system* $x_{n+1} = \frac{m}{y_n}, y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}$, J. Inst. Math. Comp. Sci., Vol.18, 2005, pp.135–136.

[5] C. Cinar, *On the positive solutions of the difference equation system* $x_{n+1} = \frac{1}{y_n}, y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$, Applied Mathematics and Computation, Vol.158, 2004, pp. 303–305.

[6] C. Cinar and I. Yalcinkaya, *On the positive solutions of the difference equation system* $x_{n+1} = \frac{1}{z_n}, y_{n+1} = \frac{x_n}{x_{n-1}}, z_{n+1} = \frac{1}{x_{n-1}}$, International Mathematical Journal, Vol.5, 2004, pp. 525–527.

[7] A.Y. Ozban, *On the positive solutions of the system of rational difference equations* $x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$, J. Math. Anal. Appl., Vol.323, 2006, pp. 26–32.

[8] G. Papaschinopoulos and C.J. Schinas, *On a system of two nonlinear difference equations*, J. Math. Anal. Appl. Vol. 219, 1998, pp. 415–426.

[9] Q. Zhang,L. Yang and J. Liu, *Dynamics of a system of rational third-order difference equation*, Advances in Difference Equations 2012, 2012: 136, pp. 1–8.

[10] T.F. Ibrahim, *Two-dimensional fractional system of nonlinear difference equations in the modeling competitive populations*, International Journal of Basic & Applied Sciences, Vol.12, no. 05, 2012, pp. 103–121.

[11] V.L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with application* , Kluwer Academic Publishers, Dordrecht, 1993.

[12] M.R.S. Kulenovic and O. Merino, *Discrete dynamical systems and difference equations with mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[13] K. Liu , Z. Zhao, X. Li and P. Li, *More on three-dimensional systems of rational difference equations*, Discrete Dynamics in Nature and Society, Vol. 2011, Article ID 178483, 2011.

[14] Q.Zhang, W.Zhang, *On a system of two high-order nonlinear difference equations*, Advances in Mathematical Physics, Vol. 2014 (2014), Article ID 729273, 8 pages.

[15] Q. Zhang, W.Zhang, Y,Shao, J.Liu, *On the system of high order rational difference equations*, International Scholarly Research Notices Volume 2014 (2014), Article ID 760502, 5 pages

[16] Q. Zhang, J. Liu, and Z. Luo, *Dynamical behavior of a system of third-order rational difference equation*, Discrete Dynamics in Nature and Society, vol. 2015, pp. 1C6, 2015.

[17] T.F. Ibrahim and Q. Zhang, *Stability of an anti-competitive system of rational difference equations* , Archives Des Sciences, Vol. 66, no. 5, 2013, pp. 44–58.

[18] E.M.E. Zayed and M.A. El-Moneam, *On the global attractivity of two nonlinear difference equations*, J. Math. Sci., Vol. 177, 2011, pp. 487–499.

[19] N. Touafek and E.M. Elsayed, *On the periodicity of some systems of nonlinear difference equations*, Bull. Math. Soc. Sci. Math. Roumanie, Vol. 2, 2012, pp. 217–224.

[20] N. Touafek and E.M. Elsayed, *On the solutions of systems of rational difference equations*, Mathematical and Computer Modelling, Vol. 55, 2012, pp. 1987–1997.

[21] S. Kalabusic, M.R.S. Kulenovic and E. Pilav, *Dynamics of a two-dimensional system of rational difference equations of Leslie–Gower*

type, Advances in Difference Equations, 2011, doi:10.1186/1687-1847-2011-29.

[22] Q. Din, *Global behavior of a plant-herbivore model*, Advance sin Difference Equations, 1(2015), 1-12.

[23] Q. Din, K. A. Khan, A. Nosheen, *Stability analysis of a system of exponential difference equations*, Discrete Dyn. Nat. Soc., Volume 2014, Article ID 375890, 11 pages.

[24] Q. Din, *Global stability of a population model*, Chaos Soliton Fract., Vol. 59, 2014, pp. 119-128.

[25] Q. Din, T. Donchev, *Global character of a host-parasite model*, Chaos Soliton Fract., Vol. 54, 2013, pp. 1-7.

[26] M. Pituk, *More on Poincares and Perrons theorems for difference equations*, J. Diff. Eq. App., Vol. 8, 2002, pp. 201-216.