Positive Periodic Solutions in Shifts $\delta_{\pm}$ for a Class of Dynamic Equations with Feedback Control on Time Scales

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Abstract: Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. By using a multiple fixed point theorem in cones, some criteria are established for the existence and multiplicity of positive solutions in shifts $\delta_{\pm}$ for a class of higher-dimensional functional dynamic equations with feedback control on time scales of the following form:

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + b(t)f(t, x(\tau(t)), u(t)), \\ u^{\Delta}(t) = -r(t)u(t) + g(t)x(t), \ t \in \mathbb{T}, \end{cases}$$

where $A(t) = (a_{ij}(t))_{n \times n}$ is a nonsingular matrix with continuous real-valued functions as its elements. Finally, numerical examples are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Periodic solution; Functional dynamic equation; feedback control; Shift operator; Time scale.

1 Introduction

In the past few years, different types of ecosystems with feedback controls have been studied extensively both in theory and applications [1-6]. The reasons for introducing control variables are based on main two points. On one hand, ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables (for more details, one can see [7]). On the other hand, in the literature, it has been proved that, under certain conditions, some species are permanent, but some may possibly be extinct in the competitive system, for example, see [8]. In order to search for certain schemes to ensure all the species coexist, feedback control variables should be introduced to ecosystem.

In fact, both continuous and discrete systems are very important in implementation and application. The study of dynamic equations on time scales, which unifies differential, difference, h-difference, q-differences equations and more, has received much attention; see, for example, [9-12] and the references therein.

The theory of dynamic equations on time scales was introduced by Hilger in his Ph.D. thesis in 1988 [13]. The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are only a few results concerning periodic solutions of dynamic equations on time scales; see, for example, [14-17]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy this condition. Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. They defined a new periodicity concept with the aid of shift operators $\delta_{\pm}$ which are first defined in [18] and then generalized in [19].

Recently, based on a fixed-point theorem in cones, several researchers studied the existence of positive periodic solutions in shifts $\delta_{\pm}$ for some nonlinear first-order functional dynamic equation on time scales; see [20-23]. However, to the best of our knowledge, there are few papers published on the existence of positive periodic solutions in shifts $\delta_\pm$ for higher-dimensional functional dynamic equations with feedback control on time scales, especially systems with the coefficient matrix being an arbitrary nonsingular $n \times n$ matrix.

Motivated by the above, in the present paper, we consider the following system:

$$\begin{cases} x^\Delta(t) = A(t)x(t) + b(t)f(t, x(\tau(t)), u(t)), \\ u^\Delta(t) = -r(t)u(t) + g(t)x(t), \ t \in \mathbb{T}, \end{cases} \tag{1}$$

where $\mathbb{T} \subset \mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_\mathbb{T}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $A = (a_{ij})_{n\times n}$ is a nonsingular matrix with continuous real-valued functions as its elements, $A \in \mathcal{R}$, and $a_{ij} \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω; $b = \mathrm{diag}(b_1, b_2, \cdots, b_n)$, and $b_i \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω; $f = (f_1, f_2, \cdots, f_n)^T$, and $f_i \in C(\mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ is periodic in shifts $\delta_\pm$ with period ω with respect to the first variable; $\tau \in C(\mathbb{T}, \mathbb{T})$ is periodic in shifts $\delta_\pm$ with period ω; $r = \mathrm{diag}(r_1, r_2, \cdots, r_n)$, $-r \in \mathcal{R}$ and $r_i \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω; $g = \mathrm{diag}(g_1, g_2, \cdots, g_n)$, and $g_i \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω.

In [24], Hu and Xie studied the existence of positive periodic solutions of system (1) on a periodic time scale $\mathbb{T}$ with $b(t) = 1$. The time scale $\mathbb{T}$ considered in [24] is unbounded above and below. Moreover, the condition (P_4) in [24] is too strict so that it cannot be satisfied even if the coefficient matrix A is a diagonal matrix. Therefore, the results in [24] are less applicable.

The main purpose of this paper is to study the existence and multiplicity of positive periodic solutions in shifts $\delta_\pm$ of system (1) under more general assumptions. By using Leggett-Williams fixed point theorem, sufficient conditions for the existence of at least three positive periodic solutions in shifts $\delta_\pm$ of system (1) will be established. The results presented in this paper improve and generalize the results in [24].

In this paper, for each $x = (x_1, x_2, \cdots, x_n)^T \in C(\mathbb{T}, \mathbb{R}^n)$, the norm of x is defined as $\|x\| = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_\mathbb{T}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^n |x_i(t)|$, and when it comes to that x is continuous, delta derivative, delta integrable, and so forth; we mean that each element x_i is continuous, delta derivative, delta integrable, and so forth.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. In Section 3, we establish our main results for positive periodic solutions in shifts $\delta_\pm$ by applying Leggett-Williams fixed point theorem. In Section 4, numerical examples are presented to illustrate that our results are feasible and more general.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}^n$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R}^n)$.

For the basic theories of calculus on time scales, see [25].

Definition 1. ([25]) *An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided*

$$I + \mu(t)A(t)$$

is invertible for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Definition 2. ([25]) *Let $t_0 \in \mathbb{T}$ and assume that A is a regressive $n\times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$Y^\Delta = A(t)Y, \ Y(t_0) = I,$$

where I denotes as usual the $n \times n$-identity matrix, is called the matrix exponential function(at t_0), and is denoted by $e_A(\cdot, t_0)$.

Lemma 3. ([25]) *If A is a regressive $n \times n$-matrix-valued functions on $\mathbb{T}$, then*
(i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;
(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
(iii) $e_A(t, s) = e_A^{-1}(s, t)$;
(iv) $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Lemma 4. ([25]) *Let A be a regressive $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and*

$$y^\Delta = A(t)y + f(t), \;\; y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, the solution is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

The following definitions, lemmas about the shift operators and the new periodicity concept for time scales which can be found in [26].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$(called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_\mathbb{T}$ in $\delta_\pm(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_\pm := \{(s, t) \in [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 5. ([26]) *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 6. ([26]) *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 7. ([26]) *(Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta_\pm^\omega$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Lemma 8. ([26]) *$\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ and $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 9. ([26]) *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\Delta s.$$

Lemma 10. ([23]) *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

$$e_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_A(t, t_0) \; for \; t, t_0 \in \mathbb{T}^*.$$

Lemma 11. ([23]) *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

$$e_A(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_A(t, \sigma(s)) \; for \; t, s \in \mathbb{T}^*.$$

Set

$$X = \big\{x(t) : x(t) \in C(\mathbb{T}, \mathbb{R}^n), x(\delta_+^\omega(t)) = x(t)\big\}$$

with the norm defined by $\|x\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_\mathbb{T}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^{n} |x_i(t)|$, then X is a Banach space.

Lemma 12. *The function $x \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1) if and only if x is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_{t}^{\delta_+^\omega(t)} G(t, s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s, \tag{2}$$

where

$$\begin{aligned} G(t, s) &= \big[e_A(t_0, \delta_+^\omega(t_0)) - I\big]^{-1} e_A(t, \sigma(s)) \\ &:= (G_{ik})_{n\times n}, \end{aligned} \tag{3}$$

and $\Phi(x)$ is defined in (4).

Proof. If $u(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of the second equation of system (1). By using Lemma 4, for $s \in [t, \delta_+^\omega(t)]_\mathbb{T}$, we have

$$\begin{aligned} u(s) = {} & e_{-r}(s, t)u(t) \\ & + \int_{t}^{s} e_{-r}(s, \sigma(\theta))g(\theta)x(\theta)\Delta\theta. \end{aligned}$$

Let $s = \delta_+^\omega(t)$ in the above equality, we have

$$u(\delta_+^\omega(t)) = e_{-r}(\delta_+^\omega(t), t)u(t) + \int_t^{\delta_+^\omega(t)} e_{-r}(\delta_+^\omega(t), \sigma(\theta))g(\theta)x(\theta)\Delta\theta.$$

Noting that $u(\delta_+^\omega(t)) = u(t)$ and $e_{-r}(t, \delta_+^\omega(t)) = e_{-r}(t_0, \delta_+^\omega(t_0))$, then

$$u(t) = \int_t^{\delta_+^\omega(t)} \bar{G}(t,s)g(s)x(s)\Delta s := (\Phi x)(t), \quad (4)$$

where

$$\bar{G}(t,s) = \left[e_{-r}(t_0, \delta_+^\omega(t_0)) - I\right]^{-1} e_{-r}(t, \sigma(s)).$$

Let $u(t)$ be an ω-periodic solution in shifts $\delta_\pm$ of (4). By (4) and Lemma 8, we have

$$\begin{aligned} & u^\Delta(t) \\ = \; & -r(t)u(t) \\ & +\bar{G}(\sigma(t), \delta_+^\omega(t))g(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t)x(\delta_+^\omega(t)) \\ & -\bar{G}(\sigma(t), t)g(t)x(t) \\ = \; & -r(t)u(t) + g(t)x(t). \end{aligned}$$

Denote $(\Phi x) = ((\Phi_1 x), (\Phi_2 x), \cdots, (\Phi_n x))^T$, then any ω-periodic solutions in shifts $\delta_\pm$ of system (1) is equivalent to that of the following equation

$$x^\Delta(t) = A(t)x(t) + b(t)f(t, x(\tau(t)), (\Phi x)(t)).$$

Again using Lemma 4, repeating the above process, we have

$$x(t) = \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s,$$

where

$$G(t,s) = \left[e_A(t_0, \delta_+^\omega(t_0)) - I\right]^{-1} e_A(t, \sigma(s)).$$

This completes the proof. □

By using Lemma 10 and Lemma 11, it is easy to verify that the Green's function $G(t,s)$ satisfies

$$G(\delta_+^\omega(t), \delta_+^\omega(s)) = G(t,s), \; \forall t \in \mathbb{T}^*, s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}. \quad (5)$$

For convenience, we introduce the following notations:

$$A_1 := \min_{1\le k\le n} \inf_{s,t\in[t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n G_{ik}(t,s)\right|,$$

$$B_1 := \max_{1\le k\le n} \sup_{s,t\in[t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n G_{ik}(t,s)\right|.$$

Hereafter, we assume that

(P_1) $A_1 > 0, \; B_1 > 0$;

(P_2) $G_{ik}b_k f_k \ge 0, \; \forall \, i, k = 1, 2, \cdots, n$.

Let

$$K = \left\{x \in X : |x(t)|_0 \ge \xi\|x\|, t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\right\}, \quad (6)$$

where $\xi = \frac{A_1}{B_1} \in (0,1)$. Obviously, K is a cone in X.

Define an operator H by

$$\begin{aligned} & (Hx)(t) \\ = & \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s, \quad (7) \end{aligned}$$

for all $x \in K, \; t \in \mathbb{T}$, where $G(t,s)$ is defined by (3) and

$$\begin{aligned} & (Hx)(t) \\ = \; & ((H_1 x)(t), (H_2 x)(t), \cdots, (H_n x)(t))^T, \quad (8) \end{aligned}$$

where

$$\begin{aligned} & (H_i x)(t) \\ = & \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik} b_k(s) f_k(s, x(\tau(s)), (\Phi x)(s))\Delta s, \\ & i = 1, 2, \cdots, n. \end{aligned}$$

In the following, we shall give some lemmas concerning K and H defined by (6) and (7), respectively.

Lemma 13. *Assume that $(P_1) - (P_2)$ hold, then $H : K \to K$ is well defined.*

Proof. For any $x \in K$. In view of (7), by Lemma 9 and (5), for $t \in \mathbb{T}$, we obtain

$$\begin{aligned} & (Hx)(\delta_+^\omega(t)) \\ = \; & \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t), s)b(s) \\ & \times f(s, x(\tau(s)), (\Phi x)(s))\Delta s \\ = \; & \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t), \delta_+^\omega(s))b(\delta_+^\omega(s))\delta_+^{\Delta\omega}(s) \\ & \times f(\delta_+^\omega(s), x(\tau(\delta_+^\omega(s))), (\Phi x)(\delta_+^\omega(s)))\Delta s \\ = \; & \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s \\ = \; & (Hx)(t), \end{aligned}$$

that is, $Hx \in X$.

Furthermore, for any $x \in K, \forall\, t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, by (P_2), we have

$$\begin{aligned}
&|(Hx)(t)|_0\\
&= \Big|\int_t^{\delta_+^\omega(t)} G(t,s)b(s)f(s,x(\tau(s)),(\Phi x)(s))\Delta s\Big|_0\\
&= \sum_{i=1}^n \Big|\int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}b_k(s)\\
&\quad \times f_k(s,x(\tau(s)),(\Phi x)(s))\Delta s\Big|\\
&\geq A_1 \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n |b_k(s)f_k(s,x(\tau(s)),(\Phi x)(s))|\Delta s\\
&= A_1 \int_{t_0}^{\delta_+^\omega(t_0)} |b(s)f(s,x(\tau(s)),(\Phi x)(s))|_0 \Delta s\\
&= \frac{A_1}{B_1} B_1 \int_{t_0}^{\delta_+^\omega(t_0)} |b(s)f(s,x(\tau(s)),(\Phi x)(s))|_0 \Delta s\\
&\geq \xi\|Hx\|,
\end{aligned}$$

that is, $Hx \in K$. This completes the proof. □

Define

$$B^m := \min_{1\leq i\leq n} \int_{t_0}^{\delta_+^\omega(t_0)} |b_i(s)|\Delta s,$$
$$B^M := \max_{1\leq i\leq n} \int_{t_0}^{\delta_+^\omega(t_0)} |b_i(s)|\Delta s.$$

Lemma 14. *Assume that $(P_1)-(P_2)$ hold, then $H : K \to K$ is completely continuous.*

Proof. We first show that H is continuous. Because of the continuity of f, for any $\nu > 0$ and $\varepsilon > 0$, there exists a $\eta > 0$ such that

$$\big\{\phi,\psi \in C(\mathbb{T},\mathbb{R}^n), \|\phi\| \leq \nu, \|\psi\| \leq \nu, \|\phi-\psi\| < \eta\big\}$$

implying

$$\begin{aligned}
&|f(s,\phi(\tau(s)),(\Phi\phi)(s)) - f(s,\psi(\tau(s)),(\Phi\psi)(s))|_0\\
&< \frac{\varepsilon}{B_1 B^M}.
\end{aligned}$$

Therefore, if $x,y,\in K$ with $\|x\|\leq\nu, \|y\|\leq\nu, \|x-y\|<\eta$, then

$$\begin{aligned}
&|(Hx)(t)-(Hy)(t)|_0\\
\leq\ & \sum_{i=1}^n \Big|\int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}b_k(s)\\
&\times f_k(s,x(\tau(s)),(\Phi x)(s))\Delta s\\
&-\int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik}b_k(s)\\
&\times f_k(s,y(\tau(s)),(\Phi y)(s))\Delta s\Big|\\
\leq\ & \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n |\sum_{i=1}^n G_{ik}||b_k(s)\\
&\times f_k(s,x(\tau(s)),(\Phi x)(s))\\
&-b_k(s)f_k(s,y(\tau(s)),(\Phi y)(s))|\Delta s\\
<\ & B_1\Big(\int_t^{\delta_+^\omega(t)} |b(s)f(s,x(\tau(s)),(\Phi x)(s))\\
&-b(s)f(s,y(\tau(s)),(\Phi y)(s))|_0\Delta s\Big)\\
<\ & B_1 B^M \frac{\varepsilon}{B_1 B^M} = \varepsilon,
\end{aligned}$$

for all $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, which yields

$$\begin{aligned}
\|Hx-Hy\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t)-(Hy)(t)|_0\\
&\leq \varepsilon,
\end{aligned}$$

that is, H is continuous.

Next, we show that H maps any bounded sets in K into relatively compact sets. We first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$, for any $\nu > 0$, there exists $\eta > 0$ such that $\big\{x,y \in K, \|x\| \leq \nu, \|y\| \leq \nu, \|x-y\| < \eta, s \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}\big\}$ implying

$$|f(s,x(\tau(s)),(\Phi x)(s)) - f(s,y(\tau(s)),(\Phi y)(s))|_0 < 1.$$

Choose a positive integer N such that $\frac{\nu}{N} < \eta$. Let $x \in K$ and define $x^k(\cdot) = \frac{x(\cdot)k}{N}$, $k = 0,1,2,\cdots,N$. If $\|x\| < \nu$, then

$$\begin{aligned}
\|x^k - x^{k-1}\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \Big|\frac{x(\cdot)k}{N} - \frac{x(\cdot)(k-1)}{N}\Big|_0\\
&\leq \|x\|\frac{1}{N} \leq \frac{\nu}{N} < \eta.
\end{aligned}$$

So

$$\begin{aligned}
&|f(s,x^k(\tau(s)),(\Phi x^k)(s))\\
&-f(s,x^{k-1}(\tau(s)),(\Phi x^{k-1})(s))|_0 < 1,
\end{aligned}$$

for all $s \in [t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}$, and this yields

$$\begin{aligned}
&|f(s,x(\tau(s)),(\Phi x)(s))|_0\\
=\ & |f(s,x^N(\tau(s)),(\Phi x^N)(s))|_0\\
\leq\ & \sum_{k=1}^N |f(s,x^k(\tau(s)),(\Phi x^k)(s))\\
&-f(s,x^{k-1}(\tau(s)),(\Phi x^{k-1})(s))|_0 + |f(s,0,0)|_0\\
<\ & N + \sup_{s\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |f(s,0,0)|_0 =: W. \qquad (9)
\end{aligned}$$

It follows from (8) and (9) that for $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$,

$$\begin{aligned}\|Hx\| &= \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \sum_{i=1}^{n} |(H_ix)(t)| \le B_1WB^M \\ &:= D.\end{aligned}$$

Finally, for $t \in \mathbb{T}$, we have

$$\begin{aligned}(Hx)^\Delta(t) &= A(t)(Hx)(t) \\ &\quad +b(t)f(t, x(\tau(t)), (\Phi x)(t)).\end{aligned}$$

So

$$\begin{aligned}&|(Hx)^\Delta(t)|_0 \\ &= |A(t)(Hx)(t) + b(t)f(t, x(\tau(t)), (\Phi x)(t))|_0 \\ &\le A^uD + B^uW,\end{aligned}$$

where $A^u := \max\limits_{1\le j\le n} \sup\limits_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \sum\limits_{i=1}^{n} |a_{ij}(t)|$, $B^u := \max\limits_{1\le i\le n} \sup\limits_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |b_i(t)|$.

To sum up, $\{Hx : x \in K, \|x\| \le \nu\}$ is a family of uniformly bounded and equiv-continuous functionals on $[t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, we know that the functional H is completely continuous. This completes the proof. □

3 Main results

In this section, we shall state and prove our main results about the existence of at least three positive periodic solutions of system (1) via the Leggett-Williams fixed point theorem.

Let X be a Banach space with cone K. A map α is said to be a nonnegative continuous concave functional on K if $\alpha : K \to [0, +\infty)$ is continuous and

$$\alpha(\lambda x + (1 - \lambda)y) \ge \lambda\alpha(x) + (1 - \lambda)\alpha(y)$$

for all $x, y \in K, 0 < \lambda < 1$.

Let a, b be two numbers such that $0 < a < b$ and α be a nonnegative continuous concave functional on K. We define the following convex sets:

$$\begin{aligned}&K_a = \{x \in K : \|x\| < a\}, \\ &K(\alpha, a, b) = \{x \in K : a \le \alpha(x), \|x\| \le b\}.\end{aligned}$$

Lemma 15. ([27]) *(Leggett-Williams fixed point theorem) Let $H : \overline{K}_c \to \overline{K}_c$ be completely continuous and α be a nonnegative continuous concave functional on K such that $\alpha(x) \le \|x\|$ for all $x \in \overline{K}_c$. Suppose that there exist $0 < d < a < b \le c$ such that*

(1) *$\{x \in K(\alpha, a, b) : \alpha(x) > a\} \ne \emptyset$ and $\alpha(Hx) > a$ for $x \in K(\alpha, a, b)$;*

(2) *$\|Hx\| < d$ for all $\|x\| \le d$;*

(3) *$\alpha(Hx) > a$ for all $x \in K(\alpha, a, c)$ with $\|H(x)\| > b$.*

Then H has at least three fixed points $x_1, x_2, x_3 \in \overline{K}_c$ satisfying $\|x_1\| < d, a < \alpha(x_2), \|x_3\| > d$ and $\alpha(x_3) < a$.

For convenience, we introduce the following notations:

$$\begin{aligned}f^\vartheta &:= \limsup_{\|u\|\to\vartheta} \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \frac{|f(t, u)|_0}{\|u\|}, \\ f_b &:= \min_{\xi b\le|u|_0\le b}\ \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |f(t, u)|_0.\end{aligned}$$

Theorem 16. *Assume that $(P_1)-(P_2)$ hold, and there exist a number $b > 0$ such that the following conditions:*

(i) $f^0 < \frac{1}{B_1}$, $f^\infty < \frac{1}{B_1}$;

(ii) $B^mf_b > \frac{\xi b}{A_1}$ *for* $\xi b \le |u|_0 \le b, t \in \mathbb{T}$;

hold. Then system (1) has at least three positive ω-periodic solutions in shifts $\delta_\pm$.

Proof. By the condition $f^\infty < \frac{1}{B_1}$ of (i), one can find that for

$$0 < \varepsilon < \frac{1}{B_1} - f^\infty,$$

there exists a $c_0 > b$ such that

$$|f(s, u, \Phi u)|_0 \le \frac{f^\infty + \varepsilon}{B^M}\|u\|,$$

where $\|u\| > c_0$.

Let $c_1 = \frac{c_0}{\xi}$, if $x \in K$, $\|x\| > c_1$, then $\|x\| > c_0$, and we have

$$\begin{aligned}&|(Hx)(t)|_0 \\ =& \left|\int_t^{\delta_+^\omega(t)} G(t, s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s\right|_0 \\ \le& B_1\sum_{k=1}^{n}\int_t^{\delta_+^\omega(t)} |b_k(s)f_k(s, x(\tau(s)), (\Phi x)(s))|\Delta s \\ \le& B_1B^M|f(s, x(\tau(s)), (\Phi x)(s))|_0 \\ \le& B_1(f^\infty + \varepsilon)\|x\| \\ <& \|x\|. \qquad (10)\end{aligned}$$

Take $k_{c_1} = \{x | x \in K, \|x\| \leq c_1\}$, then the set k_{c_1} is a bounded set. According to that H is completely continuous, then H maps bounded sets into bounded sets and there exists a number c_2 such that

$$\|Hx\| \leq c_2, \ \forall x \in k_{c_1}.$$

If $c_2 \leq c_1$, we deduce that $H : k_{c_1} \to k_{c_1}$ is completely continuous. If $c_2 < c_1$, then from (10), we know that for any $x \in k_{c_2} \backslash k_{c_1}$ and $\|Hx\| < \|x\| < c_2$ hold. Thus we have $H : k_{c_2} \to k_{c_2}$ is completely continuous. Now, take $c = \max\{c_1, c_2\}$, then $c > b$, so $H : k_c \to k_c$ is completely continuous.

Denote the positive continuous concave functional $\alpha(x)$ as $\alpha(x) = \inf\limits_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|_0$. Firstly, let $a = \xi b$ and take $x \equiv \frac{a+b}{2}, x \in K(\alpha, a, b), \alpha(x) > a$, then the set $\{x \in K(\alpha, a, b)\} \neq \emptyset$. By (ii), if $x \in K(\alpha, a, b)$, then $\alpha(x) \geq a$, and we have

$$\begin{aligned}
&\alpha(Hx)\\
&= \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t)|_0\\
&= \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \bigg| \int_t^{\delta_+^\omega(t)} G(t,s)b(s)\\
&\quad \times f(s, x(\tau(s)), (\Phi x)(s))\Delta s \bigg|\\
&\geq A_1 \sum_{k=1}^{n} \int_t^{\delta_+^\omega(t)} |b_k(s) f_k(s, x(\tau(s)), (\Phi x)(s))| \Delta s\\
&\geq A_1 B^m |f(s, x(\tau(s)), (\Phi x)(s))|_0\\
&\geq A_1 B^m f_b\\
&> A_1 \frac{\xi b}{A_1} = a.
\end{aligned}$$

Hence condition (1) of Lemma 15 holds.

Secondly, by the condition $f^0 < \frac{1}{B_1}$ of (i), one can find that for

$$0 < \varepsilon < \frac{1}{B_1} - f^0,$$

there exists a d $(0 < d < a)$ such that

$$|f(s, u, \Phi u)|_0 \leq \frac{f^0 + \varepsilon}{B^M} \|u\|,$$

where $0 \leq \|u\| \leq d$. If $x \in K_d = \{x | \|x\| \leq d\}$, we have

$$\begin{aligned}
&|(Hx)(t)|_0\\
= & \left| \int_t^{\delta_+^\omega(t)} G(t,s)b(s)f(s, x(\tau(s)), (\Phi x)(s))\Delta s \right|_0\\
\leq & B_1 \sum_{k=1}^{n} \int_t^{\delta_+^\omega(t)} |b_k(s) f_k(s, x(\tau(s)), (\Phi x)(s))| \Delta s\\
\leq & B_1 B^M |f(s, x(\tau(s)), (\Phi x)(s))|_0\\
\leq & B_1 (f^0 + \varepsilon)\|x\|\\
< & \|x\| \leq d, \qquad (11)
\end{aligned}$$

that is, condition (2) of Lemma 15 holds.

Finally, if $x \in K(\alpha, a, c)$ with $\|Hx\| > b$, by the definition of the cone K, we have

$$\begin{aligned}
\alpha(Hx) &= \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t)|_0\\
&\geq \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \xi\|Hx\| > \xi b = a,
\end{aligned}$$

which implies that condition (3) of Lemma 15 holds.

To sum up, all conditions in Lemma 15 hold. By Lemma 15, the operator H has at least three fixed point in $\overline{K}_c$. Therefore, system (1) has at least three positive ω-periodic solutions in shifts $\delta_\pm$, and

$$\begin{aligned}
&x_1 \in K_d, \ x_2 \in \{x \in K(\alpha, a, c), \alpha(x) > a\},\\
&x_3 \in \overline{K}_c \setminus \alpha(K(\alpha, a, c) \cup \overline{K}_d).
\end{aligned}$$

This completes the proof. □

Corollary 17. *Using the following*

(i^*) $f^0 = 0, f^\infty = 0;$

instead of (i) in Theorem 16, the conclusion of Theorem 16 remains true.

4 Numerical Examples

Consider the following system with feedback control on time scales

$$\begin{cases} x^\Delta(t) = A(t)x(t) + b(t)f(t, x(\tau(t)), u(t)),\\ u^\Delta(t) = -r(t)u(t) + g(t)x(t), \ t \in \mathbb{T}, \end{cases} \qquad (12)$$

then system (12) is equivalent to that of the following system

$$x^\Delta(t) = A(t)x(t) + b(t)f(t, x(\tau(t)), (\Phi x)(t)), \qquad (13)$$

where Φ is defined in (4).

Example 18. *Let*

$$\begin{aligned}
&A(t) = \begin{bmatrix} -1.5 & 1 \\ 1 & -1.5 \end{bmatrix},\\
&b(t) = \mathrm{diag}(1 - 0.5\sin 4\pi t, 1 - 0.5\sin 4\pi t),\\
&f(t, x(\tau(t)), (\Phi x)(t))\\
&= \begin{bmatrix} |x(t)|_0(0.05 - 0.03|\sin 2\pi t|) \\ (|x(t)|_0)^2 e^{-0.01|x(t)|_0} \end{bmatrix}
\end{aligned}$$

in system (13), where $|x(t)|_0 = |x_1(t)| + |x_2(t)|$. Then

$$\begin{aligned}
&e_A(t, t_0) = e_{-0.5}(t, t_0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\
&+ e_{-0.5}(t, t_0) \int_{t_0}^{t} \frac{1}{1 - 2.5\mu(s)} \Delta s \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.
\end{aligned}$$

Case I: $\mathbb{T} = \mathbb{R}$, and $\omega = 0.5$. Let $t_0 = 0$, then $\delta_+^{\omega}(t) = t + 0.5$. It is easy to verify $A(t)$, $b(t)$, $f(t, x, \Phi x)$ satisfy

$$A(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = A(t),\ b(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = b(t),\\ f(\delta_+^{\omega}(t), x, \Phi x) = f(t, x, \Phi x),\ \forall t \in \mathbb{T}^*,$$

and $A \in \mathcal{R}$. By a direct calculation, we can get

$$\begin{aligned} e_A(t,s) &= e^{-0.5(t-s)} \begin{bmatrix} 1-2(t-s) & (t-s) \\ (t-s) & 1-2(t-s) \end{bmatrix}, \\ G(t,s) &= (e_A(0,0.5) - I)^{-1} e_A(t,s) \\ &= e^{-0.5(t-s)} \begin{bmatrix} 0.7662 - 1.2187(t-s) & 0.3137 + 0.1388(t-s) \\ 0.3137 + 0.1388(t-s) & 0.7662 - 1.2187(t-s) \end{bmatrix}. \end{aligned}$$

Since $s \in [t, \delta_+^{\omega}(t)]_{\mathbb{T}} = [t, t+0.5]$, $t - s \in [-0.5, 0]$. Then

$$A_1 = 1.0779, B_1 = 2.0800, \xi = 0.5192, B^m = 0.5.$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$, then

(i) $f^0 = 0.16 < 0.4808 = \frac{1}{B_1}$, $f^{\infty} = 0.08 < 0.4808 = \frac{1}{B_1}$;

(ii) $B^m f_b = 12.8274 > 4.8080 = \frac{\xi b}{A_1}$ for $5.1920 \le |x|_0 \le 10, t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = \mathbb{R}$, system (12) exists at least three positive ω-periodic solutions in shifts $\delta_{\pm}$.

Case 2: $\mathbb{T} = \mathbb{Z}$, and $\omega = 0.5$. Let $t_0 = 0$, then $\delta_+^{\omega}(t) = t + 0.5$. It is easy to verify $A(t)$, $b(t)$, $f(t, x, \Phi x)$ satisfy

$$A(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = A(t),\ b(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = b(t),\\ f(\delta_+^{\omega}(t), x, \Phi x) = f(t, x, \Phi x),\ \forall t \in \mathbb{T}^*,$$

and $A \in \mathcal{R}$. By a direct calculation, we can get

$$\begin{aligned} e_A(t,s) &= \left(\frac{1}{2}\right)^{(t-s)} \begin{bmatrix} 1 - \frac{4(t-s)}{3} & \frac{2(t-s)}{3} \\ \frac{2(t-s)}{3} & 1 - \frac{4(t-s)}{3} \end{bmatrix}, \\ G(t,s) &= \big(e_A(0,\omega) - I\big)^{-1} e_A(t,s)(I+A)^{-1} \\ &= \left(\frac{1}{2}\right)^{(t-s)} \begin{bmatrix} 0.9468 - 0.3882(t-s) & 1.3114 - 1.1174(t-s) \\ 1.3114 - 1.1174(t-s) & 0.9468 - 0.3882(t-s) \end{bmatrix}, \end{aligned}$$

Since $s \in [t, \delta_+^{\omega}(t)]_{\mathbb{T}} = [t, t+0.5]$, $t - s \in [-0.5, 0]$. Then

$$A_1 = 2.2582, B_1 = 4.2582, \xi = 0.5303, B^m = 0.5.$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$, then

(i) $f^0 = 0.16 < 0.2348 = \frac{1}{B_1}$, $f^{\infty} = 0.08 < 0.2348 = \frac{1}{B_1}$;

(ii) $B^m f_b = 13.4673 > 2.3488 = \frac{\xi b}{A_1}$ for $5.3030 \le |x|_0 \le 10, t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = \mathbb{Z}$, system (12) exists at least three positive ω-periodic solutions in shifts $\delta_{\pm}$.

Example 19. *Let*

$$\begin{aligned} &A(t) = \begin{bmatrix} -\frac{1}{5t} & 0 \\ 0 & -\frac{1}{6t} \end{bmatrix},\ b(t) = \frac{1}{2t}, \\ &f(t, x(\tau(t)), (\Phi x)(t)) \\ &= \begin{bmatrix} |x(t)|_0(0.15 - 0.05|\sin 2\pi t|) \\ (|x(t)|_0)^2 e^{-0.01|x(t)|_0} \end{bmatrix} \end{aligned}$$

in system (13), where $|x(t)|_0 = |x_1(t)| + |x_2(t)|$.

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, $\omega = 4$, then $\delta_+^{\omega}(t) = 4t$. It is easy to verify $A(t)$, $b(t)$, $f(t, x, \Phi x)$ satisfy

$$A(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = A(t),\ b(\delta_+^{\omega}(t))\delta_+^{\Delta\omega}(t) = b(t),\\ f(\delta_+^{\omega}(t), x, \Phi x) = f(t, x, \Phi x),\ \forall t \in \mathbb{T}^*,$$

and $A \in \mathcal{R}^+$. By a direct calculation, we can get

$$\begin{aligned} e_A(t,s) &= \begin{bmatrix} e_{a_{11}}(t,s) & 0 \\ 0 & e_{a_{22}}(t,s), \end{bmatrix}, \\ a_{11}(t) &= -\frac{1}{5t},\ a_{22}(t) = -\frac{1}{6t}, \\ G(t,s) &= \big(e_A(1,4) - I\big)^{-1} e_A(t,s)(I + \mu(t)A)^{-1} \\ &= \begin{bmatrix} \frac{15}{13} e_{a_{11}}(t,s) & 0 \\ 0 & \frac{3}{2} e_{a_{22}}(t,s) \end{bmatrix}. \end{aligned}$$

Since $1 + \mu(t)a_{11}(t) = \frac{4}{5} > 0, 1 + \mu(t)a_{22}(t) = \frac{5}{6} > 0$, then $e_{a_{11}}(t,s) > 0, e_{a_{22}}(t,s) > 0, \forall s \in [t, \delta_+^{\omega}(t)]_{\mathbb{T}}$. Moreover, we have

$$A_1 = 2.4038, B_1 = 2.7, \xi = 0.8903, B^m = 1.$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$, then

(i) $f^0 = 0.3 < 0.3504 = \frac{1}{B_1}$, $f^\infty = 0.2 < 0.3704 = \frac{1}{B_1}$;

(ii) $B^m f_b = 36.4784 > 3.7036 = \frac{\xi b}{A_1}$ for $8.9030 \leq |x|_0 \leq 10, t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (12) exists at least three positive ω-periodic solutions in shifts $\delta_\pm$.

Remark 20. *From examples 18 and 19, we can see that the results obtained in this paper can be applied to systems on more general time scales, not only time scales are unbounded above and below.*

Remark 21. *In system (12), if $A(t)$ is a diagonal matrix, a similar calculation in example 2 shows that $G_{ij} = G_{ji} = 0$, $E_{ij} = E_{ji} = 0$, $i \neq j$, the condition (P_4) in [14] cannot be satisfied. So the main results in [24] cannot ensure the existence of positive periodic solution of system (12) with $A(t)$ is a diagonal matrix. Therefore, our main results improve and generalize the results in [24].*

5 Conclusion

This paper studied the existence and multiplicity of positive solutions in shifts $\delta_\pm$ for a class of higher-dimensional functional dynamic equations with feedback control on time scales using the cone theory techniques. The results obtained in this paper improve and generalize the results in [24]. Besides, the methods used in this paper can be applied to study many other dynamic systems.

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References:

[1] F. Chen, Permanence in nonautonomous multi-species predator-prey system with feedback controls, *Appl. Math. Comput.* 173, 2006, pp.694-709.

[2] X. Liao, S. Zhou, Y. Chen, Permanence and global stability in a discrete n-species competition system with feedback controls, *Nonlinear Anal. RWA* 9, 2008, pp.1661-1671.

[3] F. Chen, On the periodic solutions of periodic multi-species Kolmogorov type competitive system with delays and feedback controls, *Appl. Math. Comput.* 180, 2006, pp.366-373.

[4] Z. Liu, Persistence and periodic solution in two species competitive system with feedback controls, *J. Biomath.* 17, 2002, pp.251-255.

[5] H. Huo, W. Li, Positive periodic solutions of a class of delay differential system with feedback control, *Appl. Math. Comput.* 148, 2004, pp.35-46.

[6] Y. Li, L. Zhu, Positive periodic solutions for a class of higher-dimensional state-dependent delay functional differential equations with feedback control, *Appl. Math. Comput.* 159, 2004, pp.783-795.

[7] K. Gopalsamy, P. Weng, Feedback regulation of Logistic growth, *Int. J. Math. Math. Sci.* 16, 1993, pp.177-192.

[8] S. Ahmad, On nonautonomous Volterra-Lotka competition equations, *Proc. Am. Math. Soc.* 177, 1993, pp.199-204.

[9] M. Hu, H. Lv, Almost periodic solutions of a single-species system with feedback control on time scales, *Adv. Differ. Equ.* 2013, 2013:196.

[10] M. Hu, L. Wang, Dynamic inequalities on time scales with applications in permanence of predator-prey system, *Discrete Dyn. Nat. Soc.* Vol. 2012, Article ID 281052.

[11] M. Hu, L. Wang, Dynamic behaviors of n-species cooperation system with distributed delays and feedback controls on time scales, *WSEAS Trans. Syst. Contr.* 9, 2014, pp.291-301.

[12] M. Hu, L. Wang, Almost periodic solution of neutral-type neural networks with time delay in the leakage term on time scales, *WSEAS Trans. Syst.* 13, 2014, pp.231-241.

[13] S. Hilger, Ein Masskettenkalkl mit Anwendug auf Zentrumsmanningfaltigkeiten [Ph.D. thesis], Universität Würzburg, 1988.

[14] M. Fazly, M. Hesaaraki, Periodic solutions for predator-prey systems with Beddington-DeAngelis functional response on time scales, *Nonlinear Anal. RWA.* 9(3), 2008, pp.1224-1235.

[15] E. R. Kaufmann, Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, *J. Math. Anal. Appl.* 319(1), 2006, pp.315-325.

[16] X. Liu, W. Li, Periodic solutions for dynamic equations on time scales, *Nonlinear Anal. TMA.* 67(5), 2007, pp.1457-1463.

[17] D. R. Anderson, J. Hoffacker, Higher dimensional functional dynamic equations on periodic time scales, *J. Differ. Equ. Appl.* 14(1), 2008, pp.83-89.

[18] M. Adıvar, Function bounds for solutions of Volterra integro dynamic equations on the time scales, *Electron. J. Qual. Theo.* 7, 2010, pp.1-22.

[19] M. Adıvar, Y. N. Raffoul, Existence of resolvent for Volterra integral equations on time scales, *B. Aust. Math. Soc.* 82(1), 2010, pp.139-155.

[20] E. Çetin, Positive periodic solutions in shifts $\delta_{\pm}$ for a nonlinear first-order functional dynamic equation on time scales, *Adv. Differ. Equ.* 2014, 2014:76.

[21] E, Çetin, F. S. Topal, Periodic solutions in shifts $\delta_{\pm}$ for a nonlinear dynamic equation on time scales, *Abstr. Appl. Anal.* Vol. 2012, Article ID 707319.

[22] M. Hu, L. Wang, Multiple periodic solutions in shifts $\delta_{\pm}$ for an impulsive functional dynamic equation on time scales, *Adv. Differ. Equ.* 2014, 2014:152.

[23] M. Hu, L. Wang, Positive periodic solutions in shifts $\delta_{\pm}$ for a class of higher-dimensional functional dynamic equations with impulses on time scales, *Abstr. Appl. Anal.* Vol. 2014, Article ID 509052.

[24] M. Hu, P. Xie, Positive periodic solutions of delayed dynamic equations with feedback control on time scales, *WSEAS Trans. Math.* 7(12), 2013, pp.777-785.

[25] M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Boston: Birkhäuser, 2003.

[26] M. Adıvar, A new periodicity concept for time scales, *Math. Slovaca* 63(4), 2013, pp.817-828.

[27] R. W. Leggett, L. R. Williams, Multiple positive fixed points of nonlinear operator on ordered Banach spaces, *Indiana Univ. Math. J.* 28, 1979, pp.673-688.